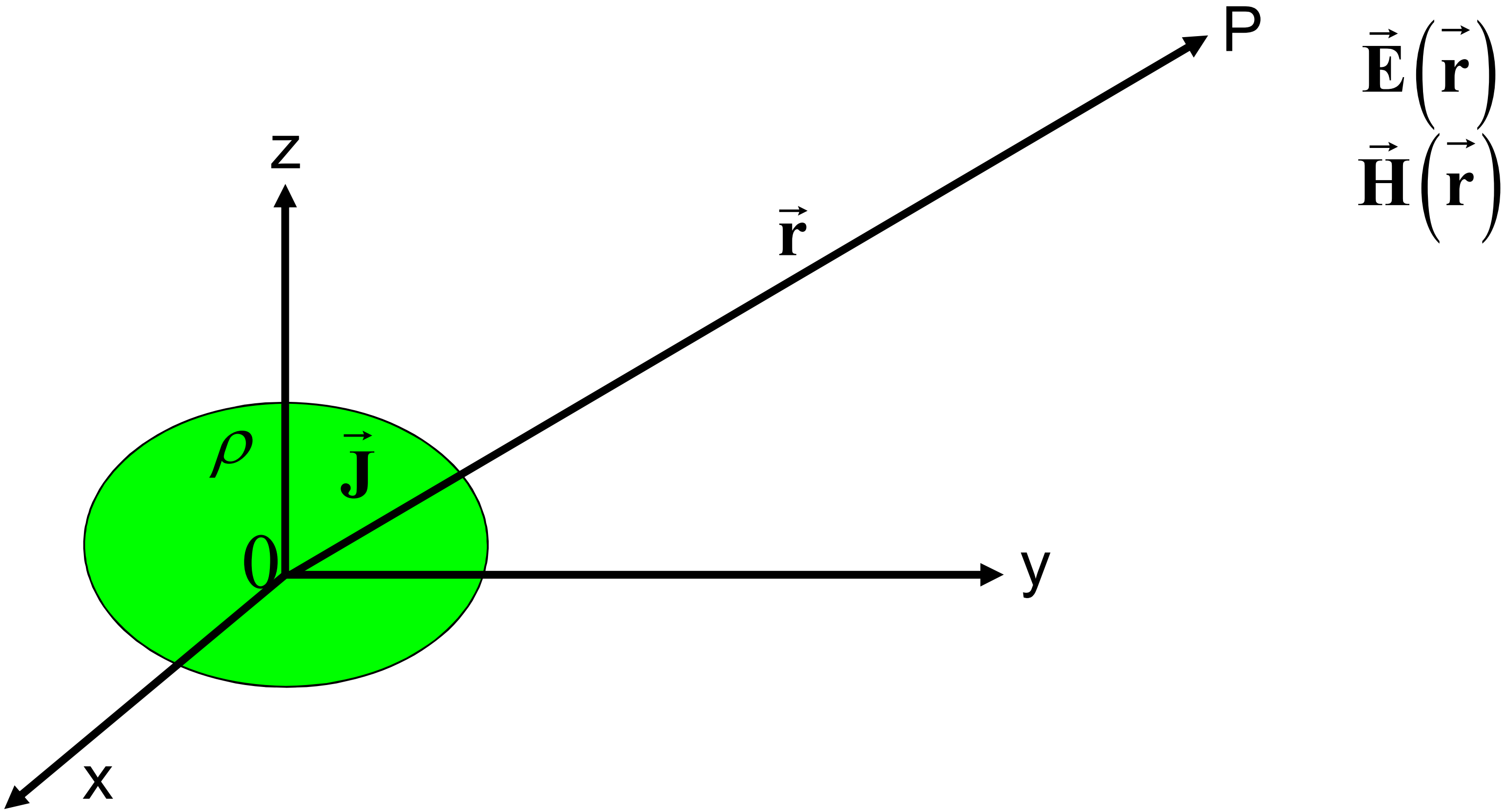
Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

Corso di Campi Elettromagnetici
a.a. 2018-2019

13 Maggio 2019

Summary of the past lecture

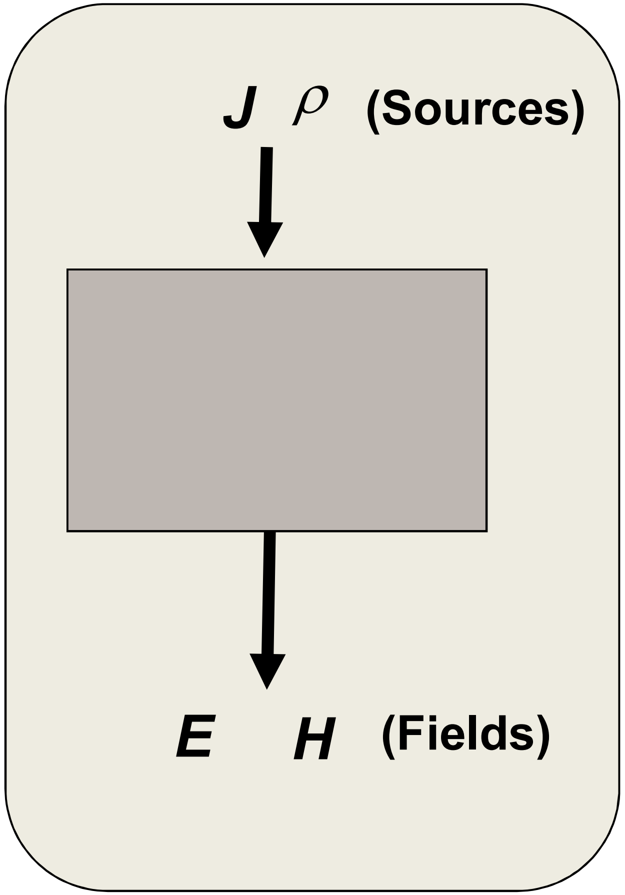
Radiation problem

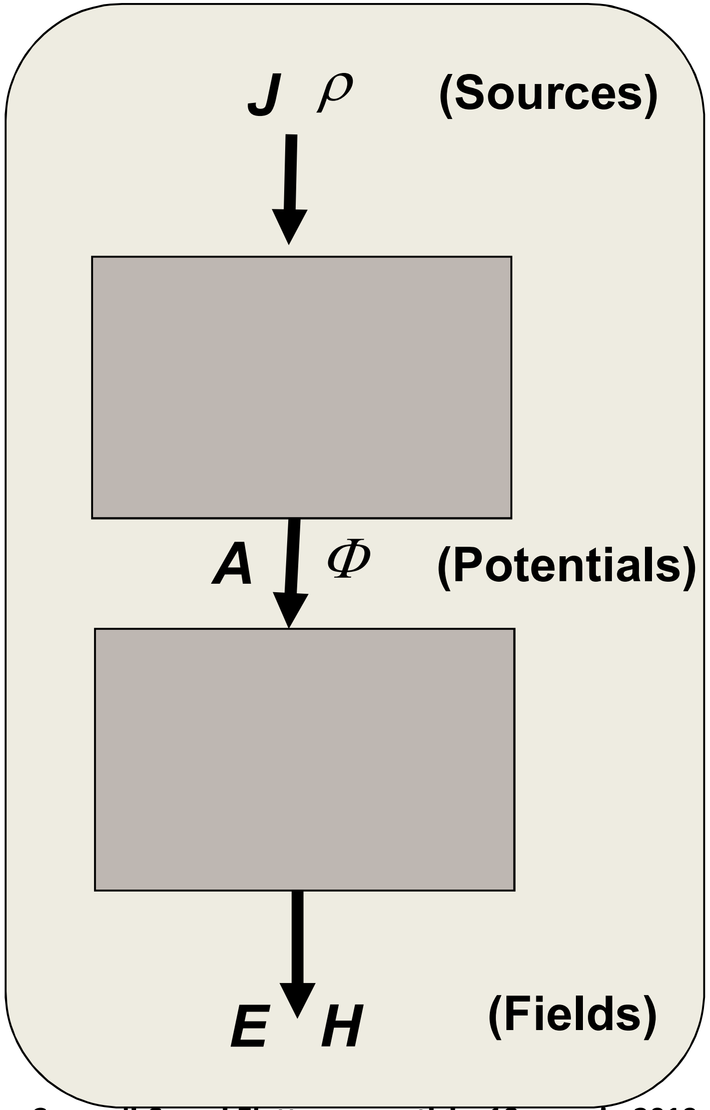


Summary of the past lecture

Potentials

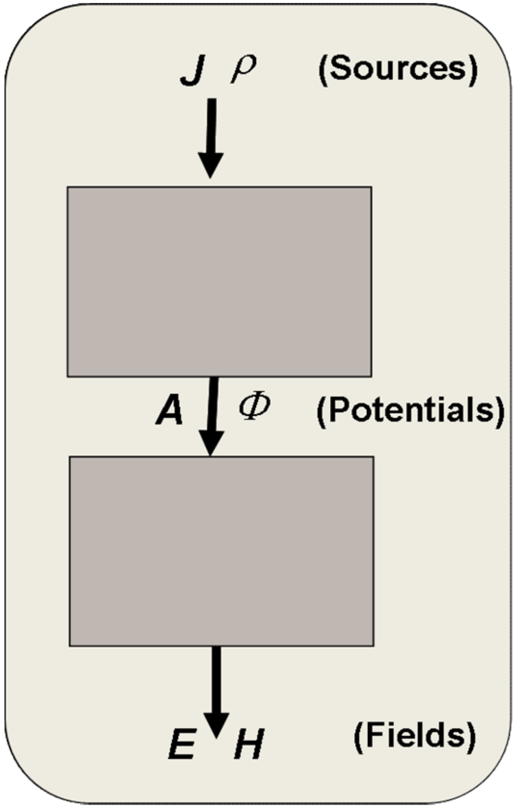
$$
\begin{cases}
\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\
\nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\
\nabla \cdot \varepsilon \boldsymbol{E} = \rho \\
\nabla \cdot \mu \boldsymbol{H} = 0
\end{cases}
$$





Summary of the past lecture

Potentials



J ρ

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi\right)$$

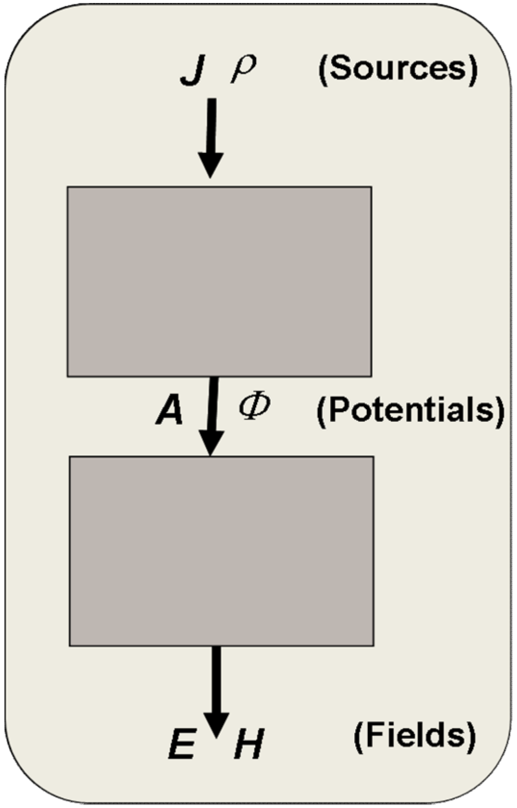
A Φ

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

E **H**

Summary of the past lecture

Potentials



J

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$ **Vector Helmholtz equation**

A

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -j\omega \mathbf{A} + \frac{\nabla\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}$$

E **H**

Potentials

$\boldsymbol{J}$

↓

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

↓

$\boldsymbol{A}$

… mathematical tools …

$$\mathbf{C} = \mathrm{C_x}(x,y,z)\hat{i}_x + \mathrm{C_y}(x,y,z)\hat{i}_y + \mathrm{C_z}(x,y,z)\hat{i}_z$$

$$\Phi = \Phi(x,y,z)$$

$$\nabla^2 \Phi = \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\nabla^2 \mathbf{C} = \nabla^2 \mathrm{C}_x \hat{i}_x + \nabla^2 \mathrm{C}_y \hat{i}_y + \nabla^2 \mathrm{C}_z \hat{i}_z$$

Potentials

J

↓

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

Vector Helmholtz equation

↓

A

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

Potentials

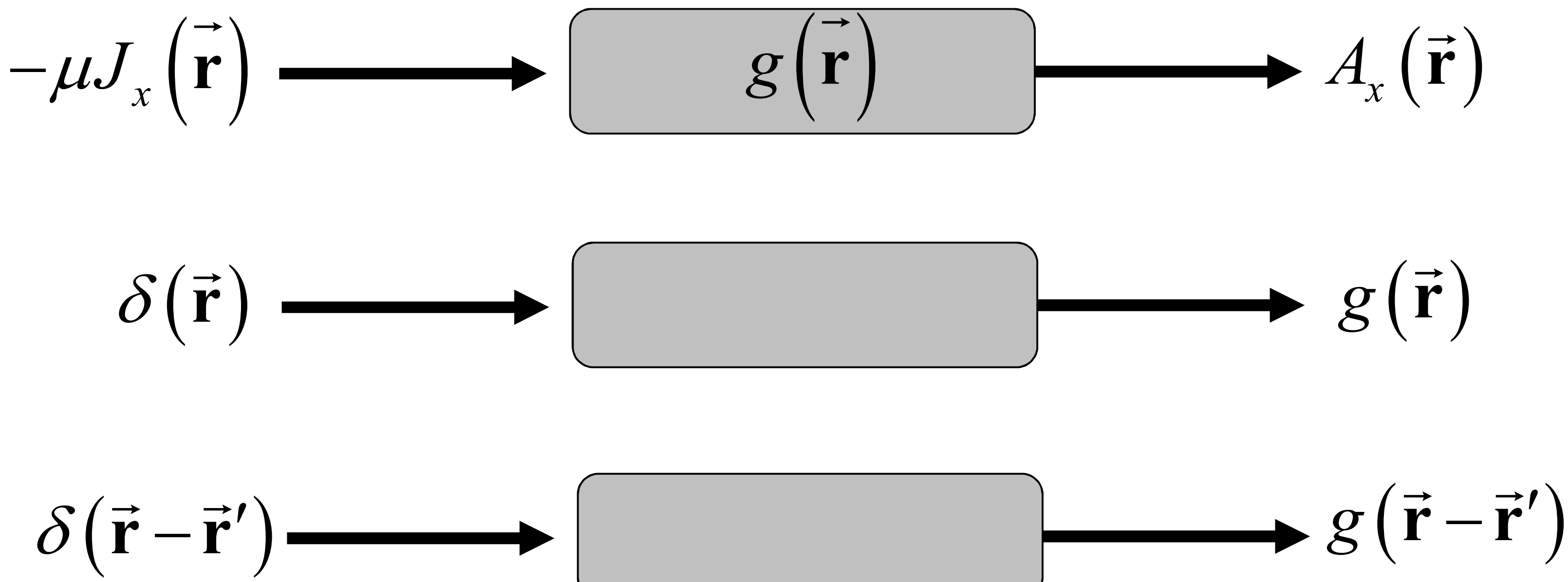
$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

Let us address the solution of the following scalar Helmholtz equation

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

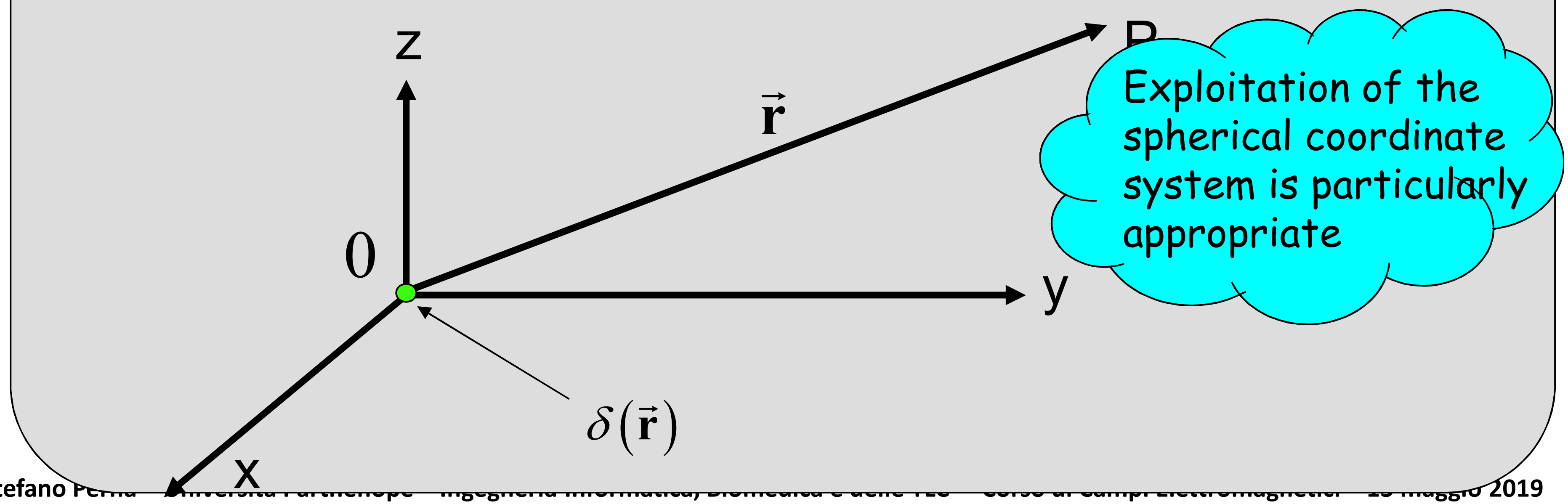


$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\,\vec{\mathbf{r}}'$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

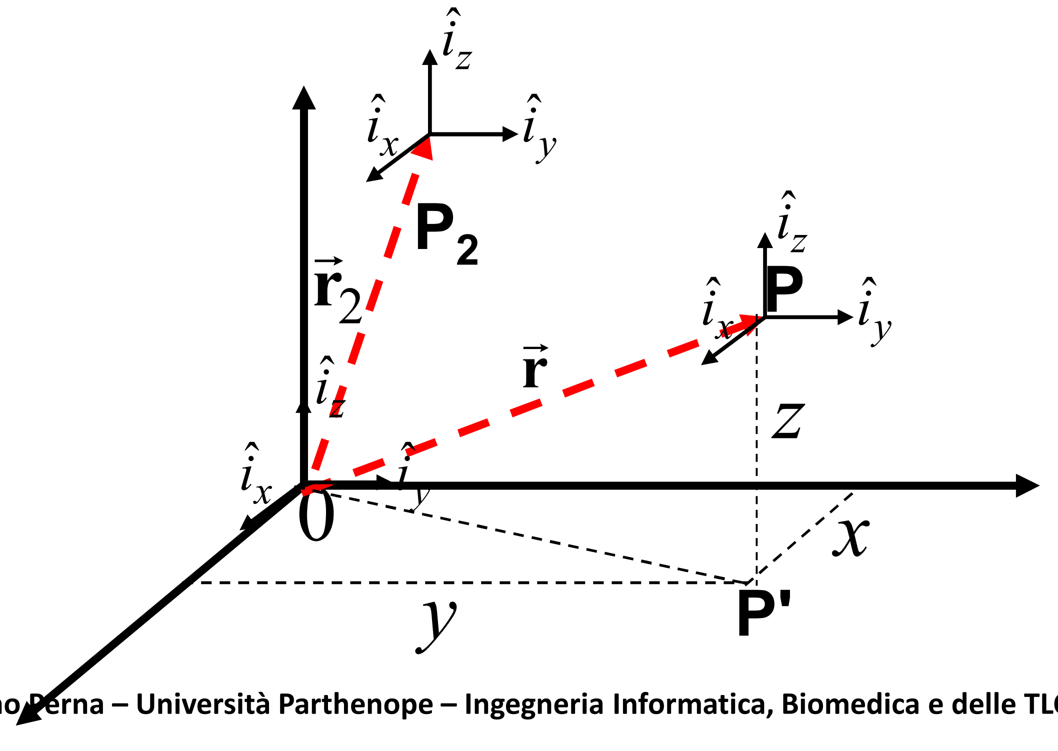
$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$



Reference systems
Cartesian coordinate system

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E_x}(\vec{\mathbf{r}})\hat{i}_x + \mathrm{E_y}(\vec{\mathbf{r}})\hat{i}_y + \mathrm{E_z}(\vec{\mathbf{r}})\hat{i}_z$$

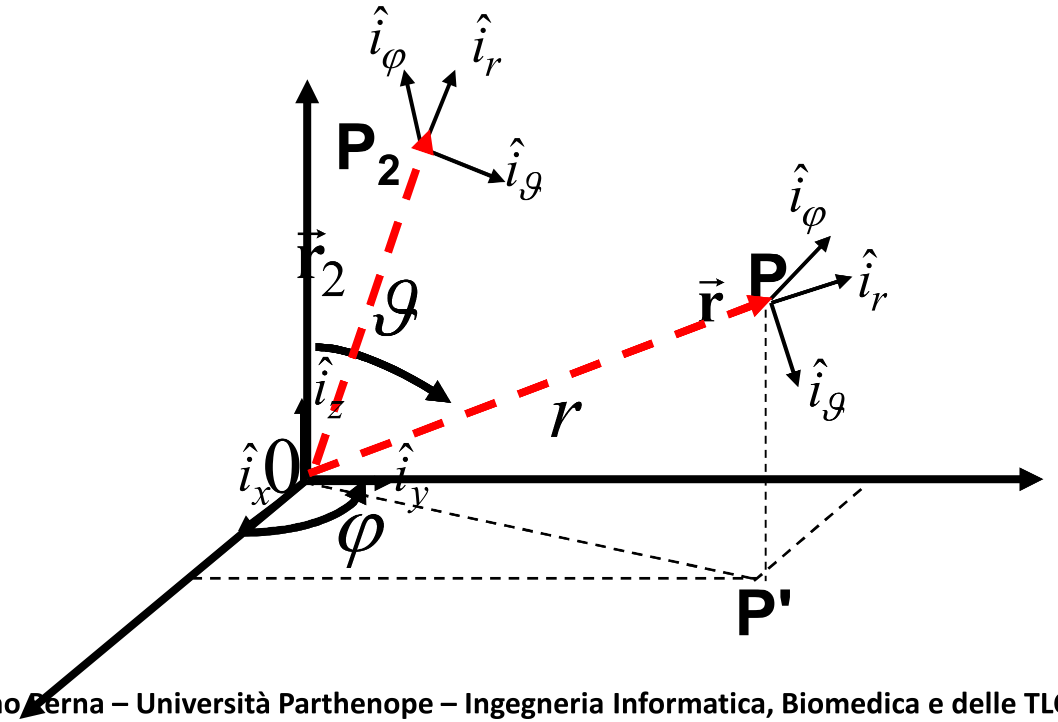
$$\vec{\mathbf{r}} = (x, y, z)$$



Reference systems
Spherical coordinate system

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_r(\vec{\mathbf{r}})\hat{i}_r + \mathrm{E}_\varphi(\vec{\mathbf{r}})\hat{i}_\varphi + \mathrm{E}_\vartheta(\vec{\mathbf{r}})\hat{i}_\vartheta$$

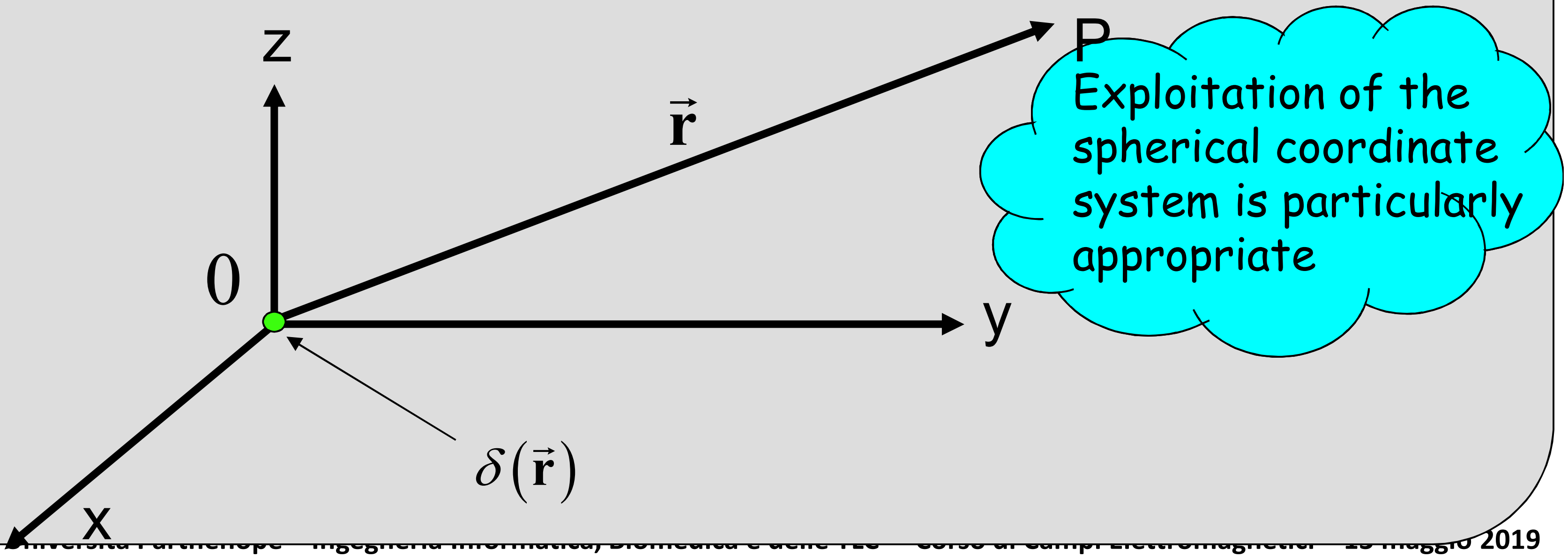
$$\vec{\mathbf{r}} = (r, \vartheta, \varphi)$$



Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$



Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$

where, in principle, $g(\vec{\mathbf{r}}) = g(r, \vartheta, \varphi)$

However, due to symmetry considerations, the function $A_x(r,\vartheta,\varphi)$ turns out to be independent of ϑ and φ, that is,

$$g(\vec{\mathbf{r}}) = g(r)$$

Accordingly, in the whole three dimensional space the solution of the Helmholtz equation is:

$$g(r) = -\frac{1}{4\pi}\frac{e^{-jkr}}{r}$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$-\mu J_x(\vec{\mathbf{r}}) \longrightarrow \boxed{g(\vec{\mathbf{r}})} \longrightarrow A_x(\vec{\mathbf{r}})$$

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow g(\vec{\mathbf{r}}) = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}|}}{|\vec{\mathbf{r}}|} = -\frac{1}{4\pi}\frac{e^{-jkr}}{r}$$

$$\delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \longrightarrow \boxed{} \longrightarrow = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')$$

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\,\vec{\mathbf{r}}' = \frac{\mu}{4\pi}\int J_x(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}\, d\,\vec{\mathbf{r}}'$$

Potentials

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} \quad \Longrightarrow \quad \mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int \mathbf{J}(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \quad \Longrightarrow \quad A_x(\vec{\mathbf{r}}) = \dfrac{\mu}{4\pi} \displaystyle\int J_x(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\vec{\mathbf{r}}' \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \quad \Longrightarrow \quad A_y(\vec{\mathbf{r}}) = \dfrac{\mu}{4\pi} \displaystyle\int J_y(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\vec{\mathbf{r}}' \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \quad \Longrightarrow \quad A_z(\vec{\mathbf{r}}) = \dfrac{\mu}{4\pi} \displaystyle\int J_z(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\vec{\mathbf{r}}' \end{cases}$$

Potentials

$\boldsymbol{J}(\boldsymbol{r})$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

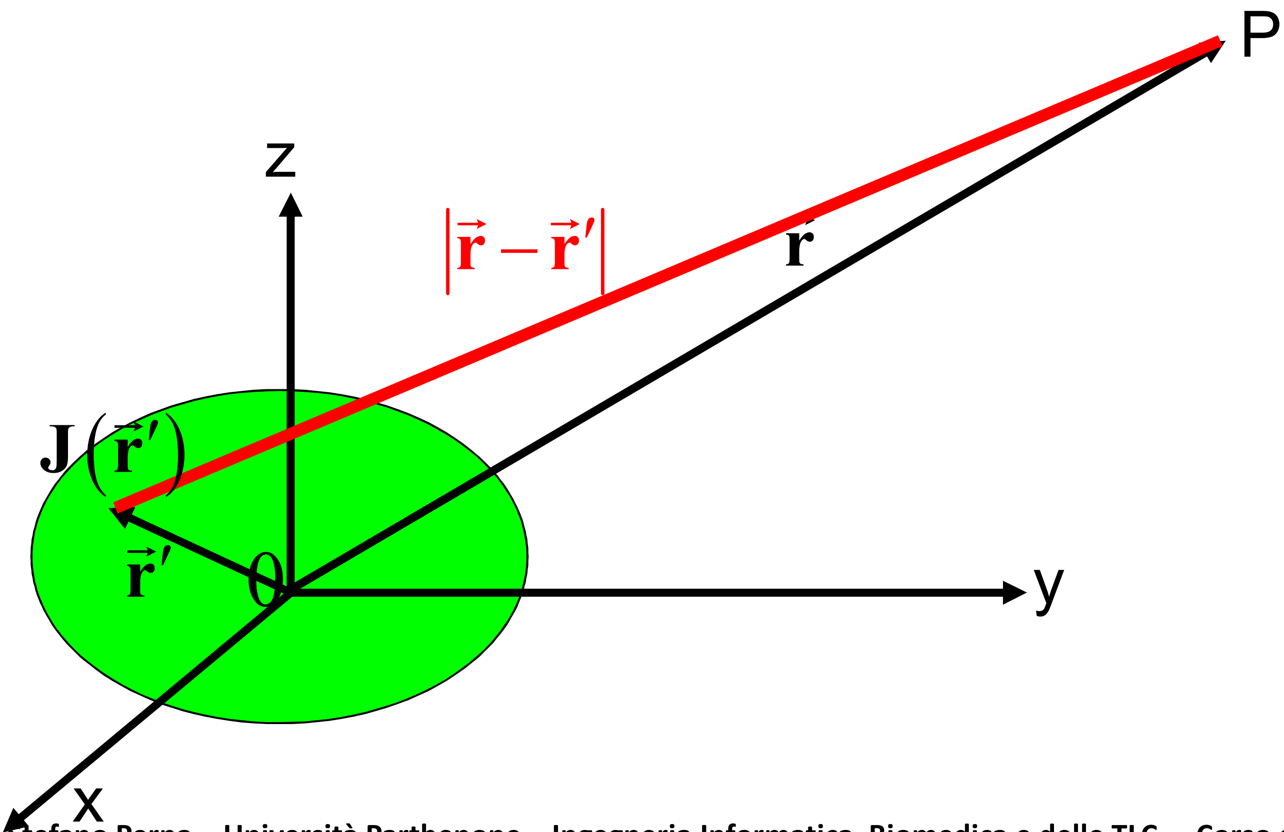
$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

$\boldsymbol{E}(\boldsymbol{r})$

$\boldsymbol{H}(\boldsymbol{r})$

Potentials

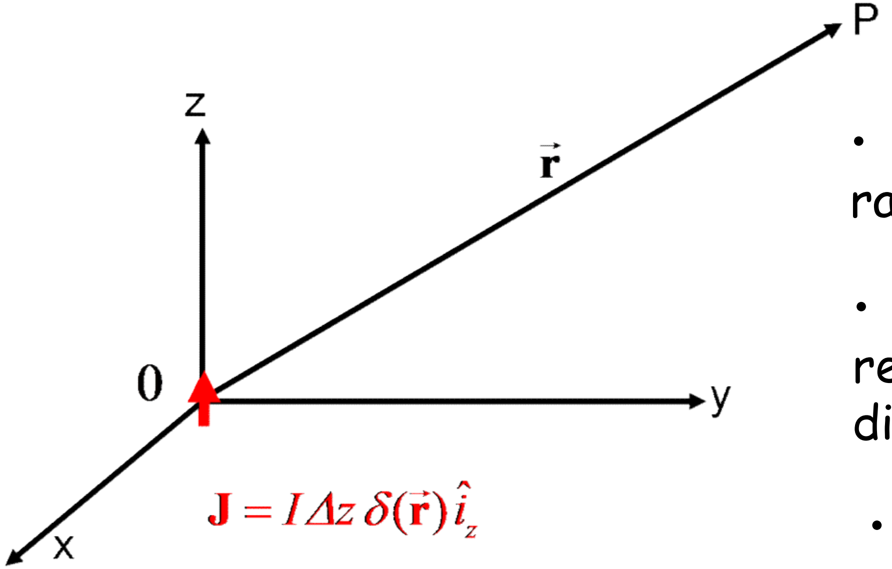
$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\,\vec{\mathbf{r}}'$$



Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

Elementary electrical dipole

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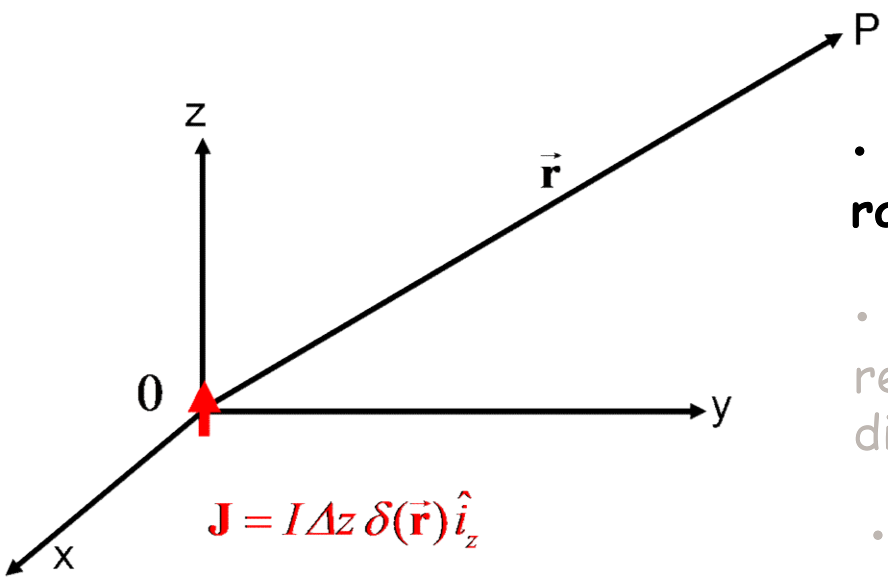


- **Why are we interested in such a radiating element?**
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

… memo …

$$\mathbf{A}(\vec{\mathbf{r}}) = \int -\mu \mathbf{J}(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\, d\vec{\mathbf{r}}'$$

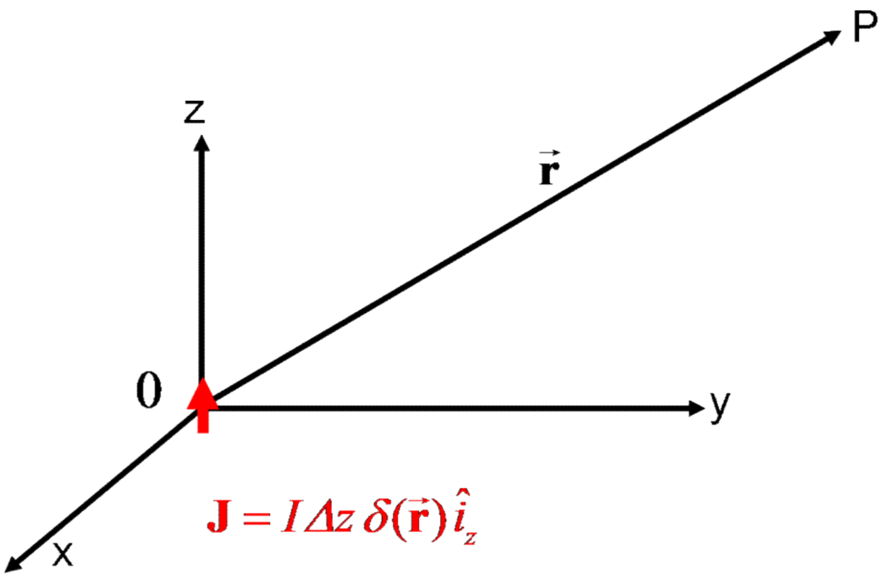
$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$

Mathematically, a δ-source radiating element is related to the radiation of any antenna!!

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- **Why is such a radiating element referred to as elementary electrical dipole?**
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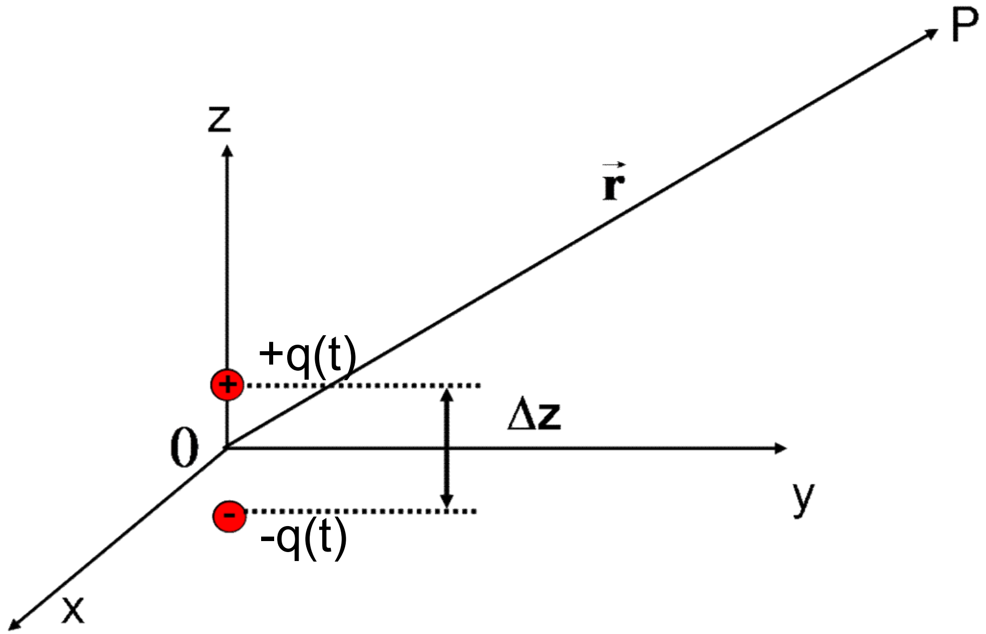
Elementary electrical dipole

$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\,\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



It can be easily shown that this electric current density is the same as that of an electrical dipole such that:

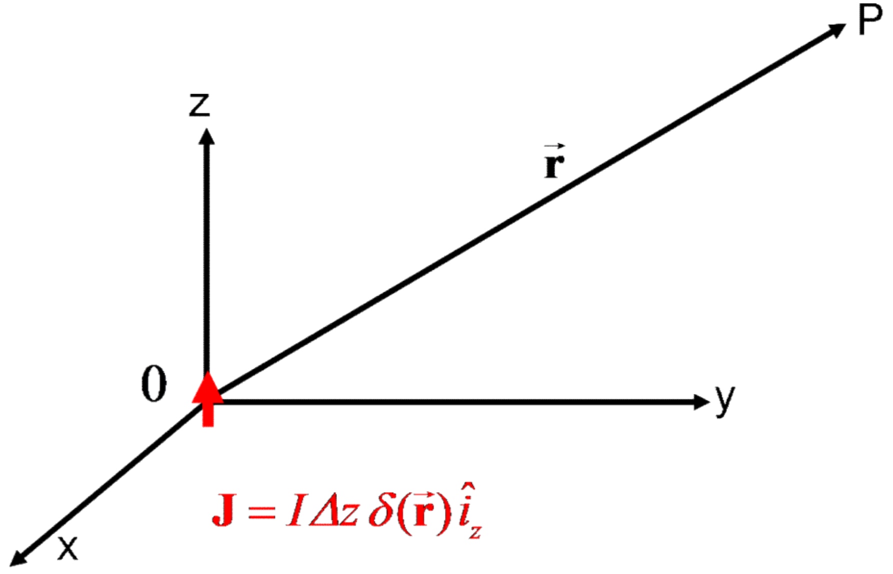
1) the two charges, of opposite sign, have equal time variation;

2) in the spectral domain, the relation between *I* and the time-varying charge Q is:

$$j\omega Q = I$$

Elementary electrical dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- **How can we physically approximate an elementary electrical dipole?**

Hertzian dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

- Of course in real life one cannot physically build up a δ-source radiating element but only an approximation.
- An approximation of the elementary dipole was used by Hertz in his experiments, in fact the elementary dipole is often called as Hertzian dipole.
- Note however that an Hertzian dipole is a dipole characterized by:

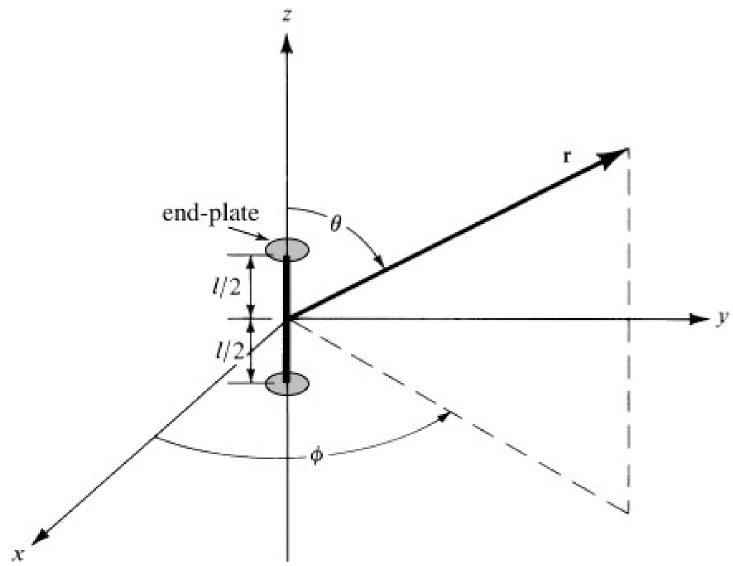
$$\mathbf{J} = I\delta(x)\delta(y)\,\mathrm{rect}\left[\frac{z}{\Delta z}\right]\hat{i}_z$$

when $\Delta z \to 0$ *then*

$$\mathrm{rect}\left[\frac{z}{\Delta z}\right] \to \Delta z\,\delta(z)$$

Hertzian dipole

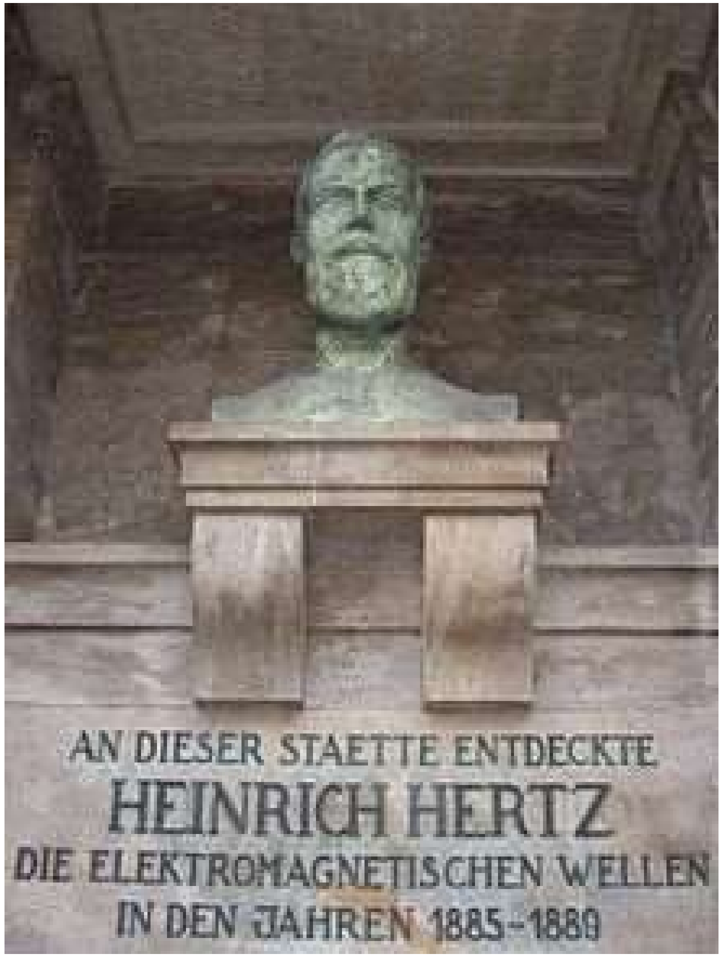
- The creation of the constant current distribution can be made by two large charge “tanks” at the two edges.
- Note that in practical case this model is meant to be suitable for electrical dipole smaller than $\lambda/50$.



Infinitesimal dipole

Heinrich Rudolf Hertz

- **Heinrich Rudolf Hertz** (German; 22 February 1857 – 1 January 1894) was a German physicist who first conclusively proved the existence of electromagnetic waves theorized by James Clerk Maxwell.
- Hertz proved the theory by engineering instruments to transmit and receive radio pulses using experimental procedures.
- The scientific unit of frequency – cycles per second – was named the ”Hertz" in his honor.



Elementary electrical dipole

$\boldsymbol{J}$

↓

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ **Vector Helmholtz equation**

↓

$\boldsymbol{A}$

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

$$\mathbf{J} = I \Delta z \, \delta(\vec{\mathbf{r}}) \hat{i}_z = J_z(\vec{\mathbf{r}}) \hat{i}_z$$

Potentials

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

$$\mathbf{J} = I \Delta z\, \delta(\vec{\mathbf{r}})\, \hat{i}_z = J_z(\vec{\mathbf{r}})\, \hat{i}_z$$

$$-\mu J_z(\vec{\mathbf{r}}) = -\mu I \Delta z \delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow \frac{\mu}{4\pi} I\, \Delta z \frac{e^{-jkr}}{r}$$

… memo …

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow g(\vec{\mathbf{r}}) = -\frac{1}{4\pi} \frac{e^{-jkr}}{r}$$

Potentials

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = J_z(\vec{\mathbf{r}})\,\hat{i}_z$$

$$-\mu J_z(\vec{\mathbf{r}}) = -\mu I \Delta z \delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow \frac{\mu}{4\pi} I\,\Delta z \frac{e^{-jkr}}{r}$$

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z \quad \Longrightarrow \quad \mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} I\Delta z \frac{e^{-jkr}}{r}\hat{i}_z$$

Elementary electrical dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

↓ $\boldsymbol{J}$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ Vector Helmholtz equation

↓ $$\mathbf{A} = \frac{\mu}{4\pi} I\Delta z \frac{e^{-jkr}}{r}\hat{i}_z$$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu}$$

↓ $\boldsymbol{E}$, $\boldsymbol{H}$