Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

Corso di Campi Elettromagnetici
a.a. 2018-2019

15 Aprile 2019

Reciprocity theorem

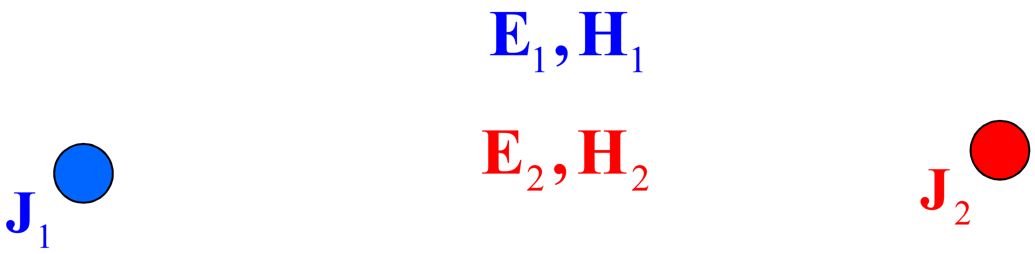
$\mathbf{E}_1, \mathbf{H}_1$

Consider a source distribution $\mathbf{J}_1$ with its associated electromagnetic field $(\mathbf{E}_1,\mathbf{H}_1)$

Reciprocity theorem

Consider a source distribution $\mathbf{J}_1$ with its associated electromagnetic field **($\mathbf{E}_1$,$\mathbf{H}_1$)**

Reciprocity theorem



Consider a source distribution $\mathbf{J}_1$ with its associated electromagnetic field **($\mathbf{E}_1$,$\mathbf{H}_1$)**

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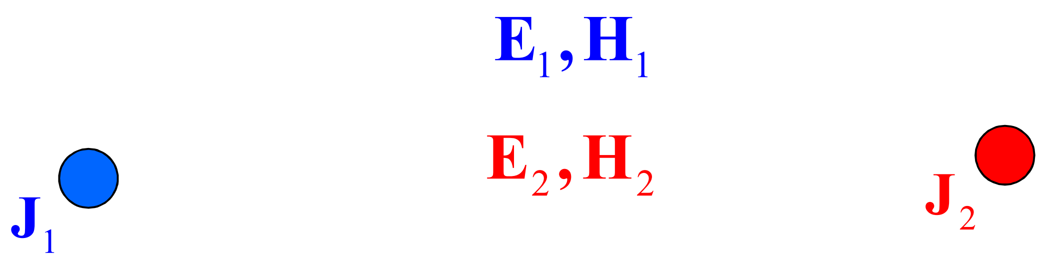
Reciprocity theorem



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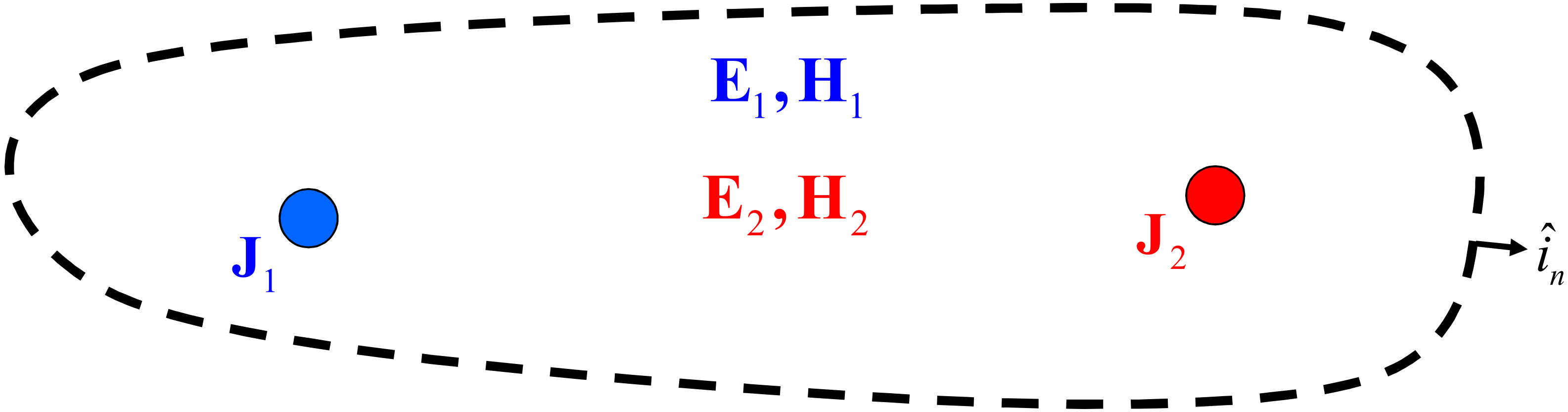
We define the mixed Poynting-like vector $\mathbf{S}_{12}$

$$\mathbf{S}_{12} = \mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1$$

Reciprocity theorem



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The reciprocity theorem sates that

$$\oiint_A dA\, \mathbf{S}_{12} \cdot \hat{i}_n = \iiint_V dV \left[\mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{J}_2 \cdot \mathbf{E}_1 \right]$$

Reciprocity theorem

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Two interesting cases

1) The surface material is a PEC ➡ $\oiint_A dA\, \mathbf{S}_{12} \cdot \hat{i}_n = 0$

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Two interesting cases

2) The volume encompasses all the space ➡ $\oiint_A dA\, \mathbf{S}_{12} \cdot \hat{i}_n = 0$

$$\iiint_V dV\, \mathbf{J}_1 \cdot \mathbf{E}_2 = \iiint_V dV\, \mathbf{J}_2 \cdot \mathbf{E}_1$$

Reciprocity theorem

$$\mathbf{S}_{12} = \mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1$$

$$\oiint_A dA\, \mathbf{S}_{12} \cdot \hat{i}_n = \iiint_V dV \left[\mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{J}_2 \cdot \mathbf{E}_1 \right]$$

Two interesting cases

When the surface material is a PEC or the volume encompasses all the space, the reciprocity theorem simplifies as:

$$\iiint_V dV\, \mathbf{J}_1 \cdot \mathbf{E}_2 = \iiint_V dV\, \mathbf{J}_2 \cdot \mathbf{E}_1$$

Maxwell Equations

James Clerk Maxwell 1831-1879

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \\ \nabla \cdot \varepsilon\mathbf{E} = \rho \\ \nabla \cdot \mu\mathbf{H} = 0 \end{cases}$$

Magnetic Sources

James Clerk Maxwell 1831-1879

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{J}_{\mathrm{m}} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \\ \nabla \cdot \varepsilon\mathbf{E} = \rho \\ \nabla \cdot \mu\mathbf{H} = \rho_{\mathrm{m}} \end{cases}$$

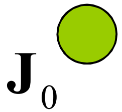
Equivalence theorem

$$\mathbf{E}_0, \mathbf{H}_0$$

Consider a source distribution $\mathbf{J}_0$ with its associated electromagnetic field **($\mathbf{E}_0$,$\mathbf{H}_0$)**

Equivalence theorem

$\mathbf{E}_0, \mathbf{H}_0$



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Equivalence theorem

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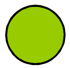
$$
\begin{cases}
\nabla \times \mathbf{E}_0 = -j\omega\mu\mathbf{H}_0 \\
\nabla \times \mathbf{H}_0 = j\omega\varepsilon\mathbf{E}_0 + \mathbf{J}_0 \\
\nabla \cdot \varepsilon\mathbf{E}_0 = \rho_0 \\
\nabla \cdot \mu\mathbf{H} = 0
\end{cases}
$$

Equivalence theorem

$\mathbf{E}_0, \mathbf{H}_0$

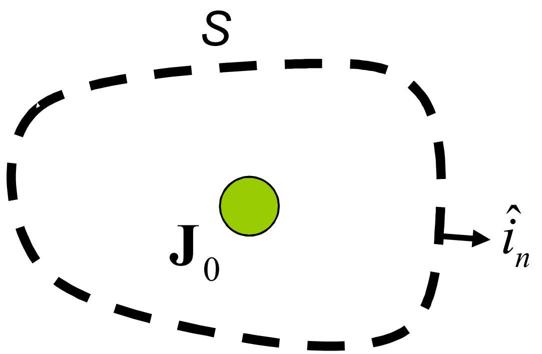


$\mathbf{J}_0$

Consider a source distribution $\mathbf{J_0}$ with its associated electromagnetic field **($\mathbf{E_0}$,$\mathbf{H_0}$)**

$$\mathbf{J_0} \rightarrow \left(\mathbf{E}_0, \mathbf{H}_0\right)$$

Equivalence theorem



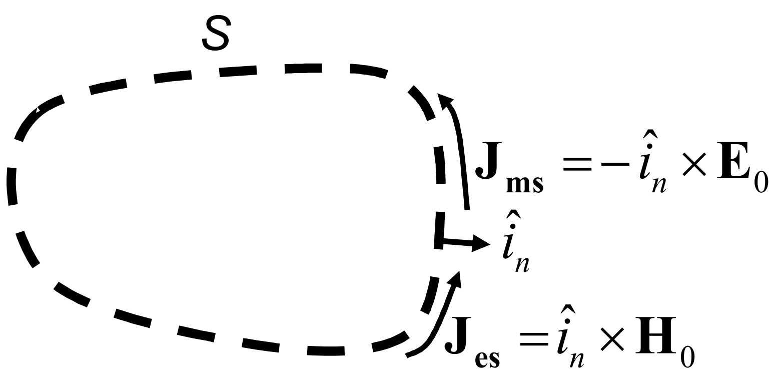
$\mathbf{E}_0, \mathbf{H}_0$

Consider a source distribution $\mathbf{J}_0$ with its associated electromagnetic field **($\mathbf{E}_0$,$\mathbf{H}_0$)**

Consider a (smooth) surface *S* with an everywhere defined unit normal $\hat{i}_n$

$$\mathbf{J}_0 \rightarrow (\mathbf{E}_0, \mathbf{H}_0)$$

Equivalence theorem



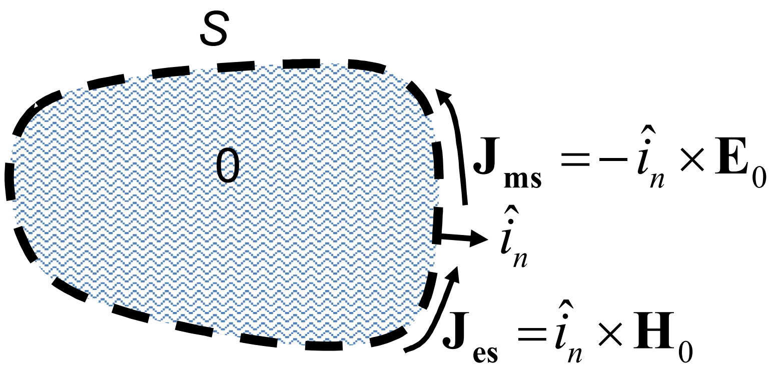
Consider a source distribution $\mathbf{J_0}$ with its associated electromagnetic field **($\mathbf{E_0}$,$\mathbf{H_0}$)**

Consider a (smooth) surface *S* with an everywhere defined unit normal $\hat{i}_n$

The original sources $\mathbf{J_0}$ enclosed in *S* can be removed and substituted by equivalent sources, i.e., electric $\mathbf{J}_{\mathbf{es}} = \hat{i}_n \times \mathbf{H}_0$ and magnetic $\mathbf{J}_{\mathbf{ms}} = -\hat{i}_n \times \mathbf{E}_0$ current densities distributed over the surface *S*.

$$\mathbf{J_0} \to \left(\mathbf{E}_0, \mathbf{H}_0\right)$$

Equivalence theorem

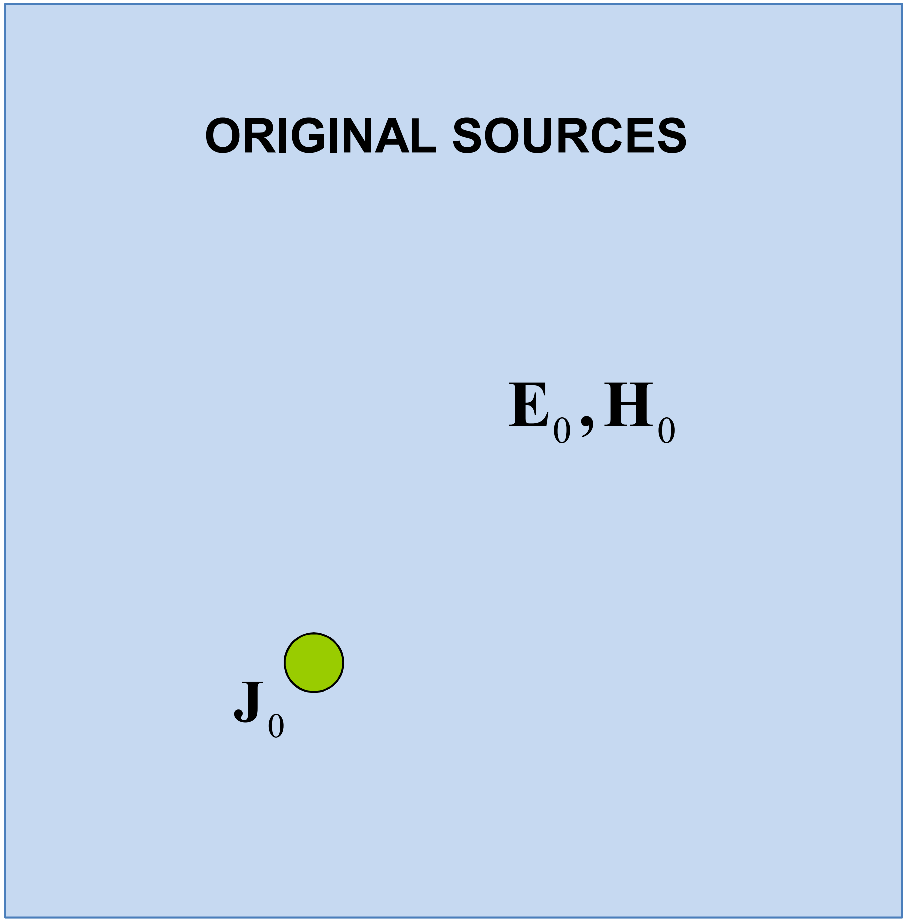


$\mathbf{E}_0, \mathbf{H}_0$

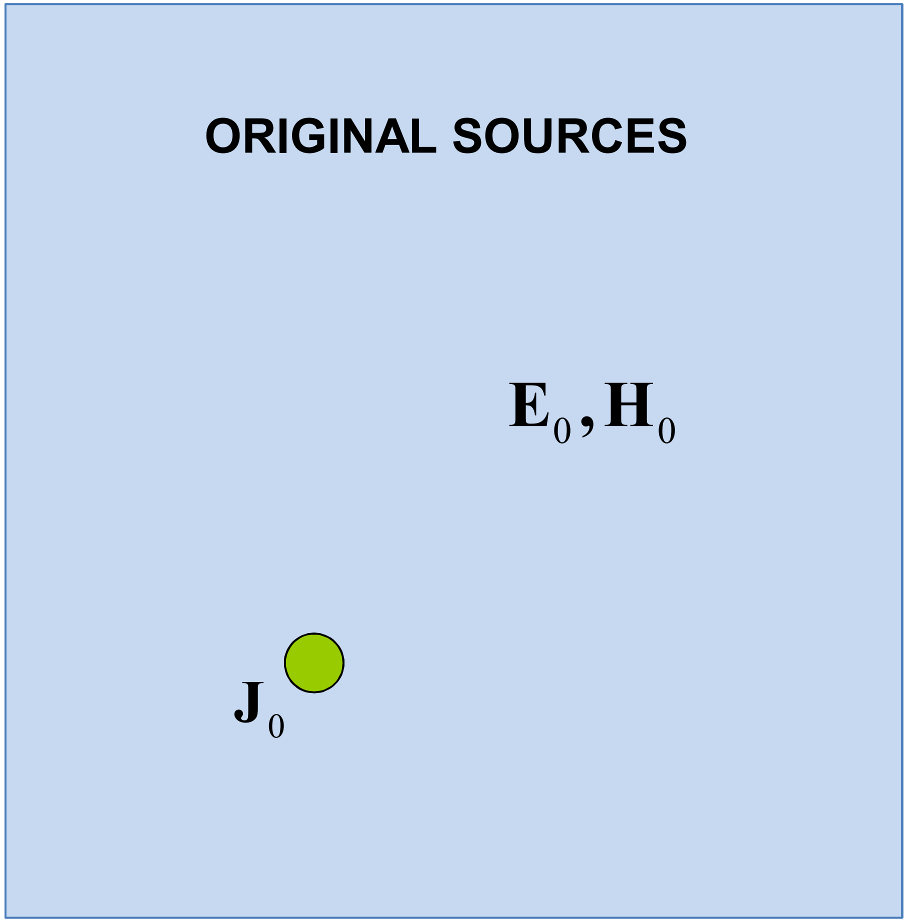
The Equivalence Theorem states that the equivalent sources $\mathbf{J}_{\mathbf{es}}$ and $\mathbf{J}_{\mathbf{ms}}$ generate a field **(E',H')** coincident with **($\mathbf{E}_0$,$\mathbf{H}_0$)** outside *S* and identically equal to zero inside

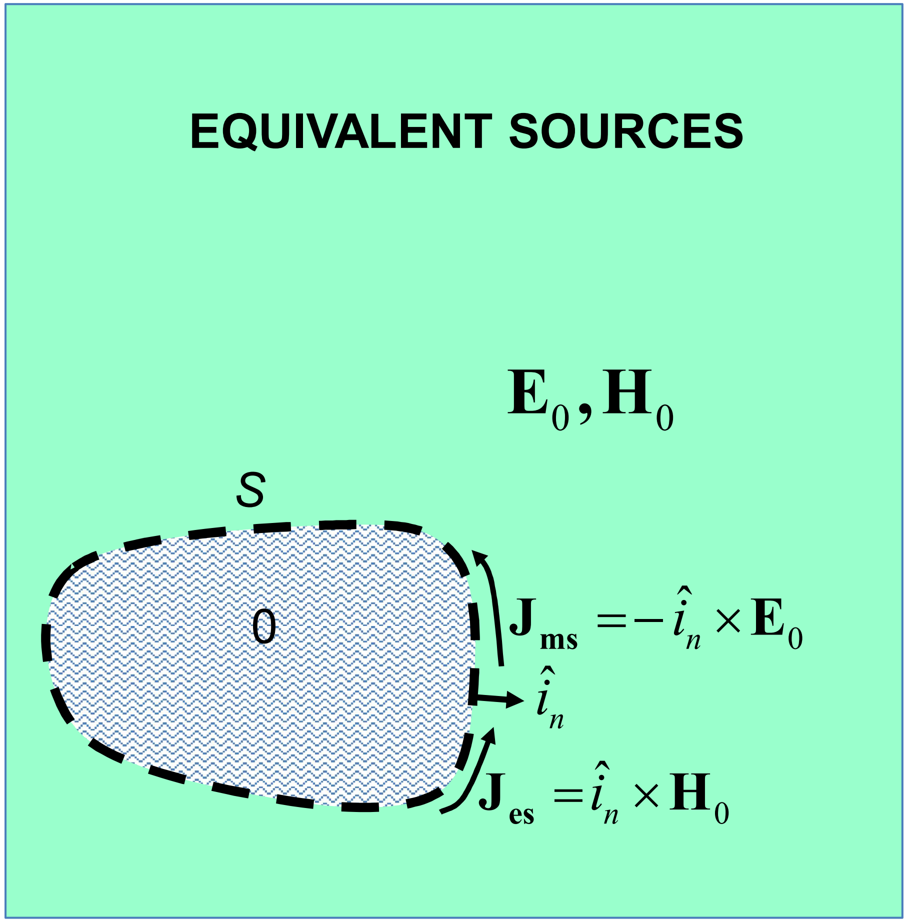
$$\mathbf{J}_0 \rightarrow (\mathbf{E}_0, \mathbf{H}_0)$$

Equivalence theorem



Equivalence theorem

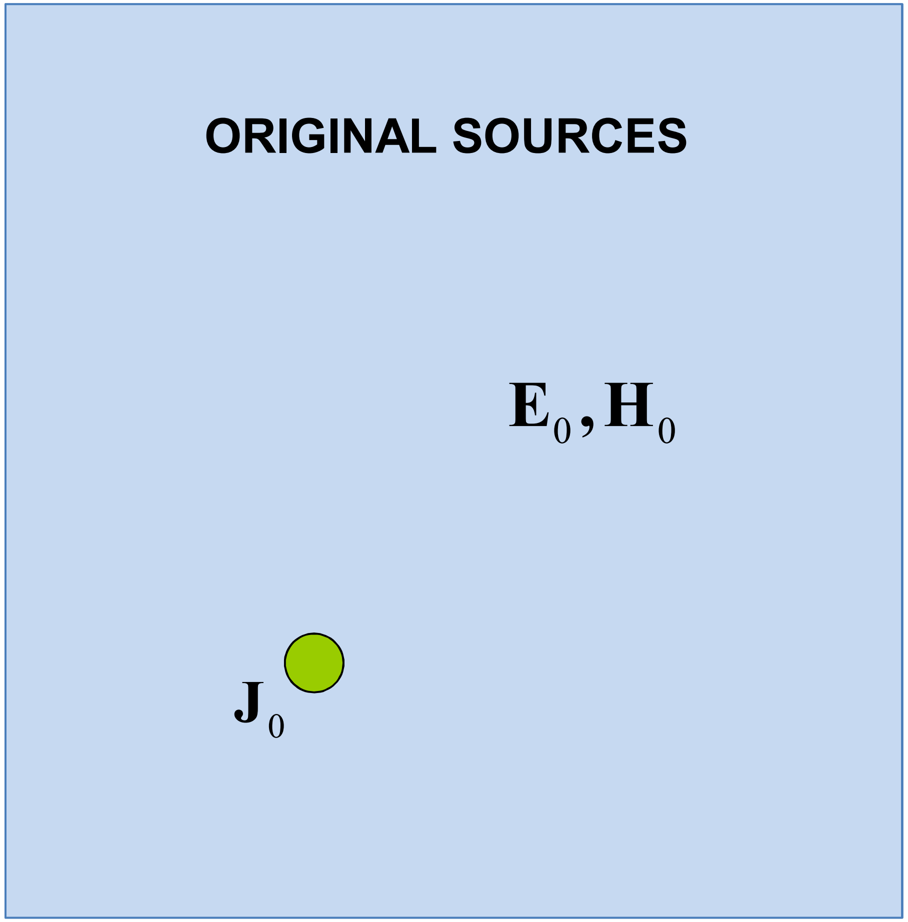


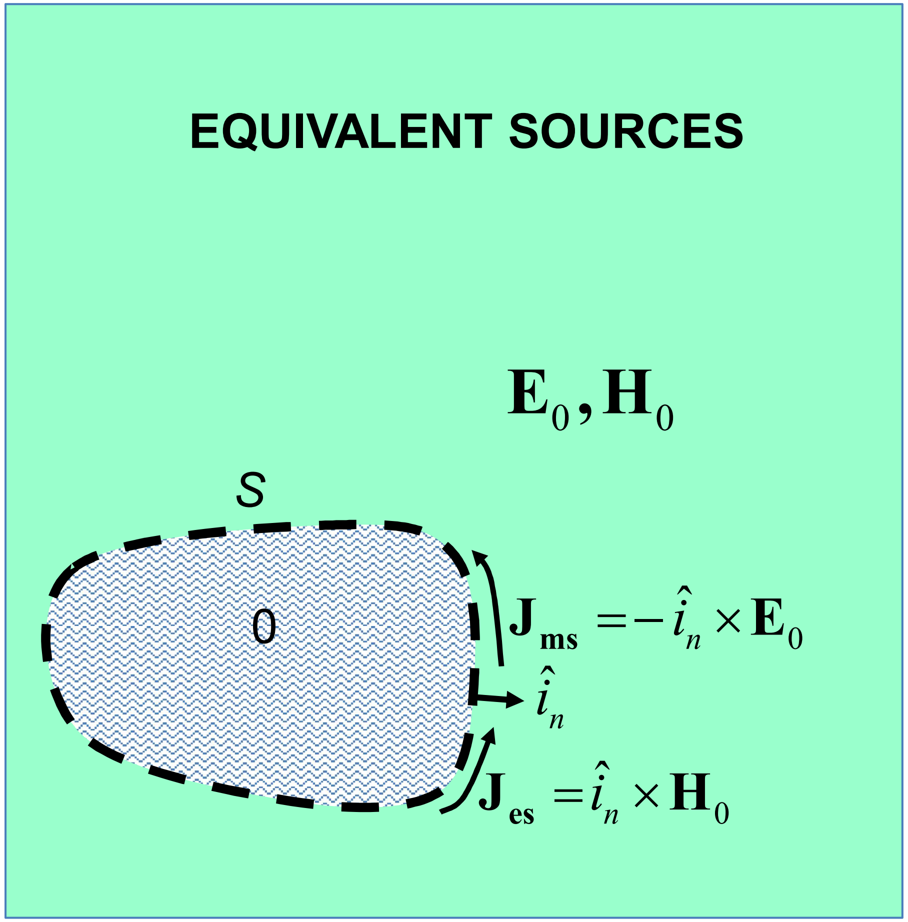


Equivalence theorem

It's a powerful theorem that allows calculating the e.m. field in **all** the space, starting from the knowledge of its value **just on a surface**.

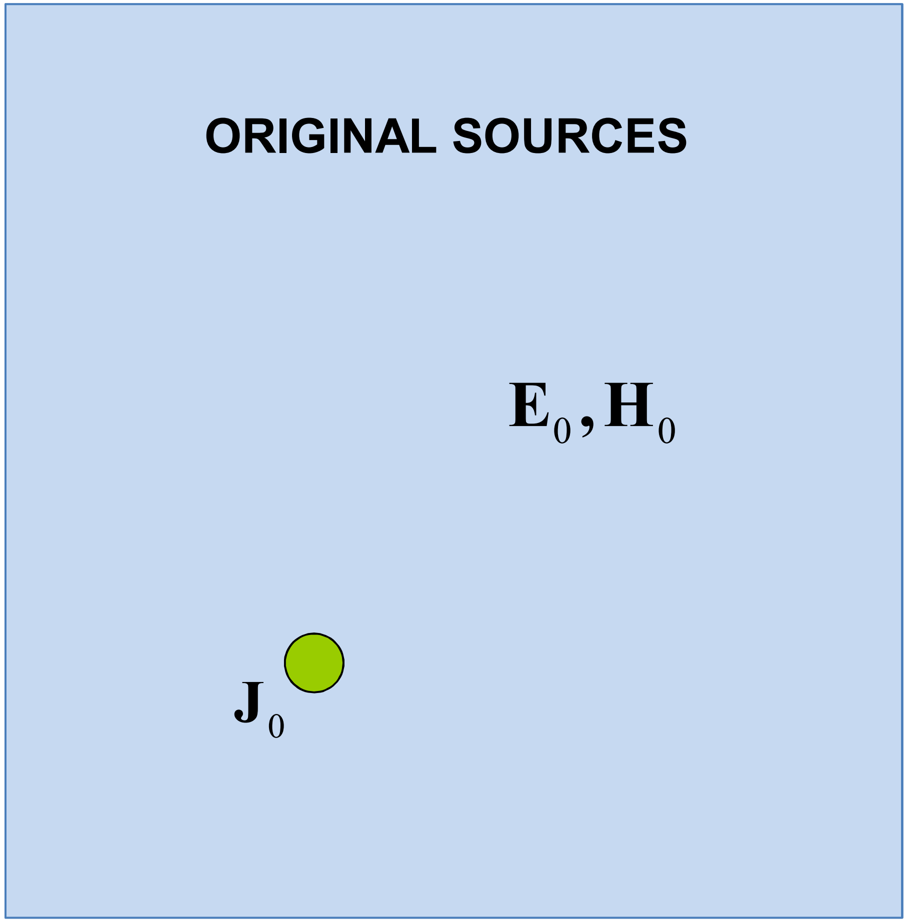
Equivalence theorem

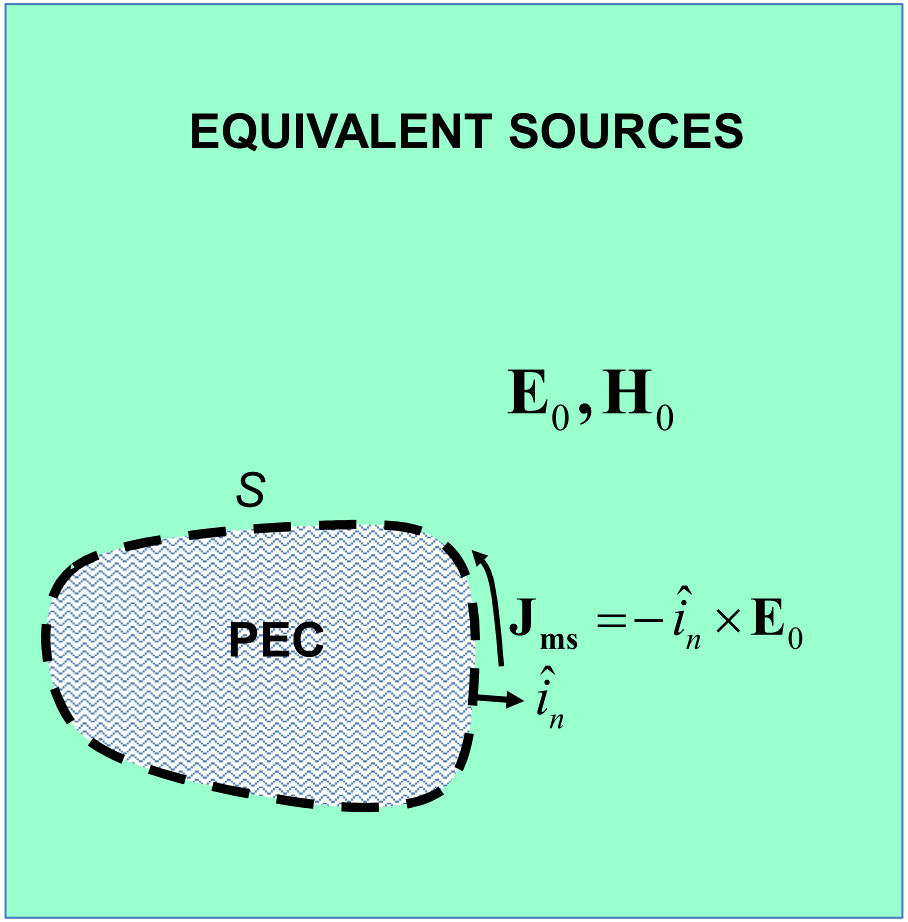




Equivalence theorem

Alternative formulation





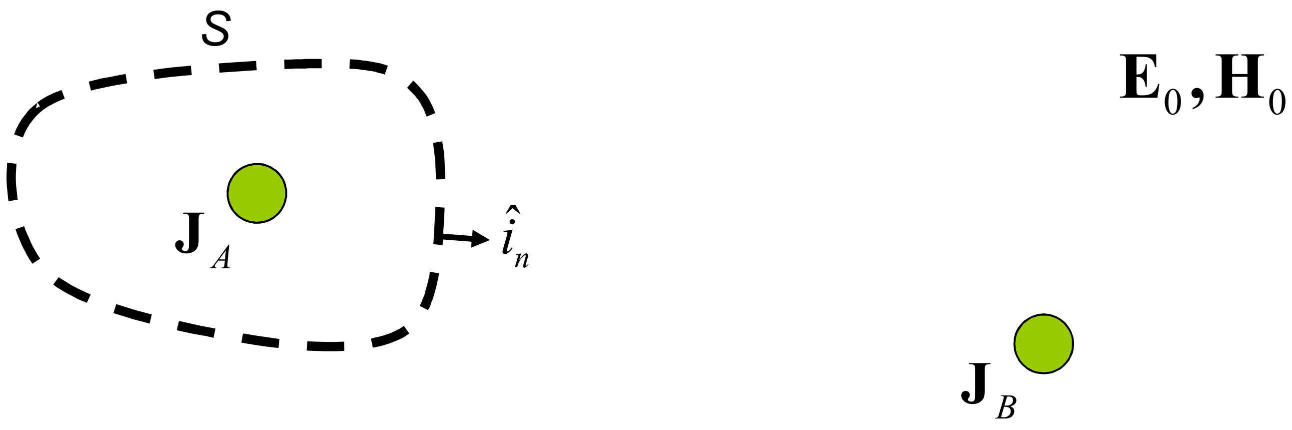
Equivalence theorem

More general formulation

$$\mathbf{J}_A + \mathbf{J}_B \rightarrow \left(\mathbf{E}_0, \mathbf{H}_0\right)$$

Equivalence theorem

More general formulation



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Equivalence theorem

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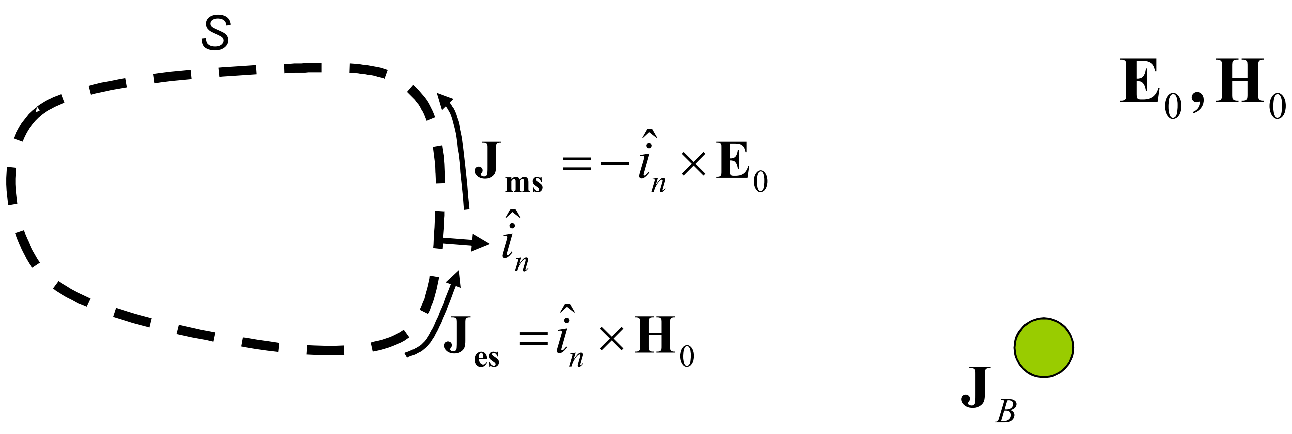

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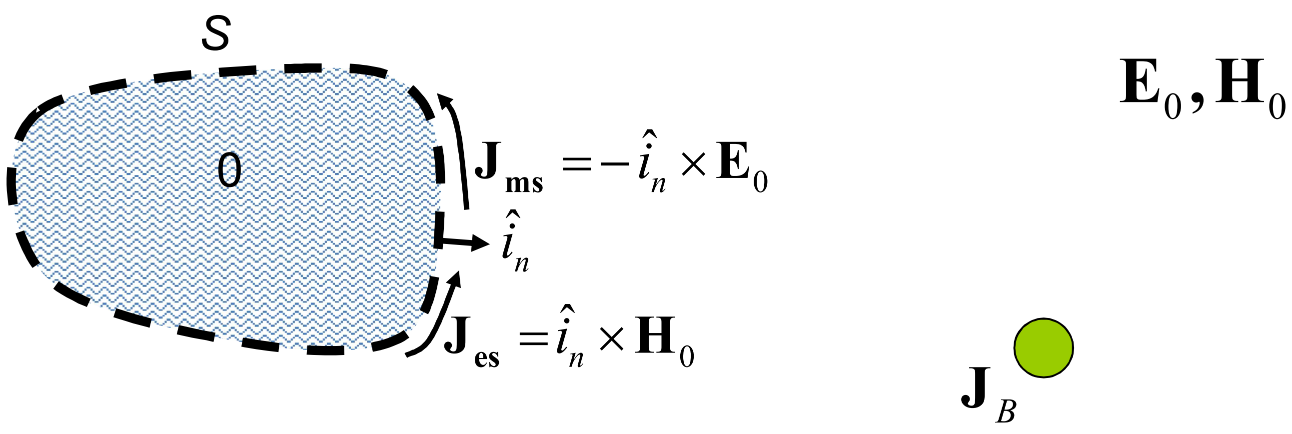

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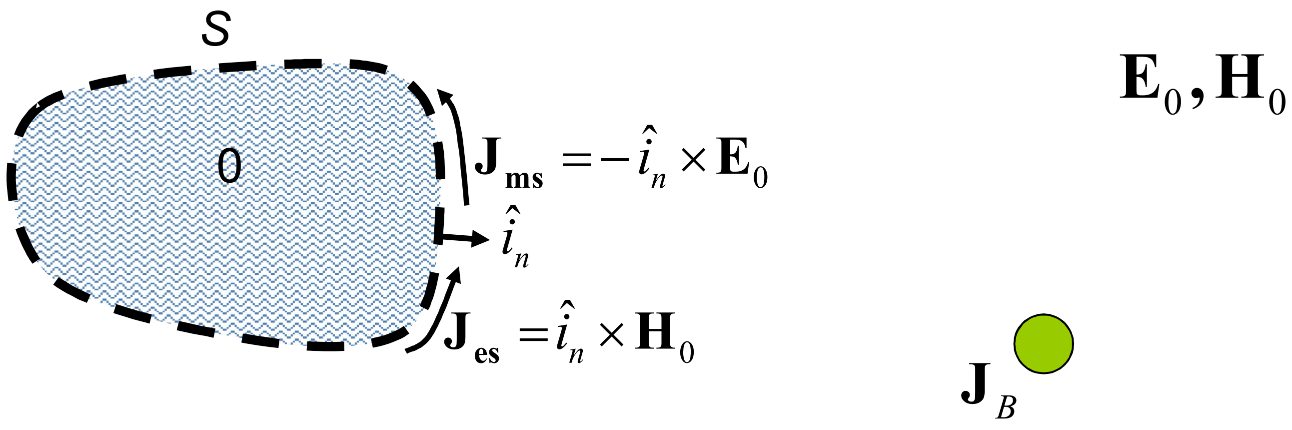

… exercises

$$\mathbf{J}_A + \mathbf{J}_B \rightarrow (\mathbf{E}_0, \mathbf{H}_0)$$