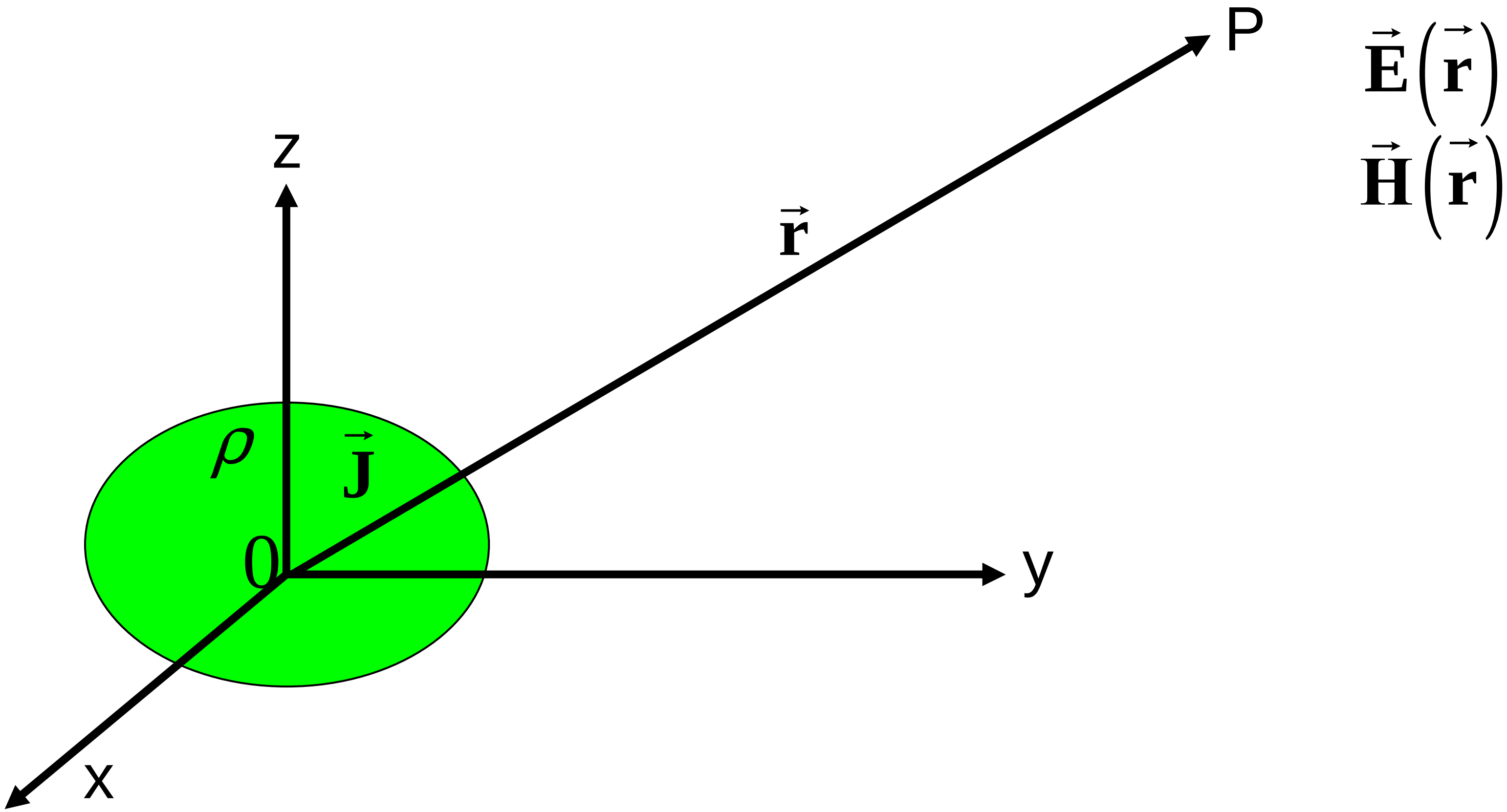
Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

Corso di Campi Elettromagnetici
a.a. 2017-2018

10 Maggio 2018

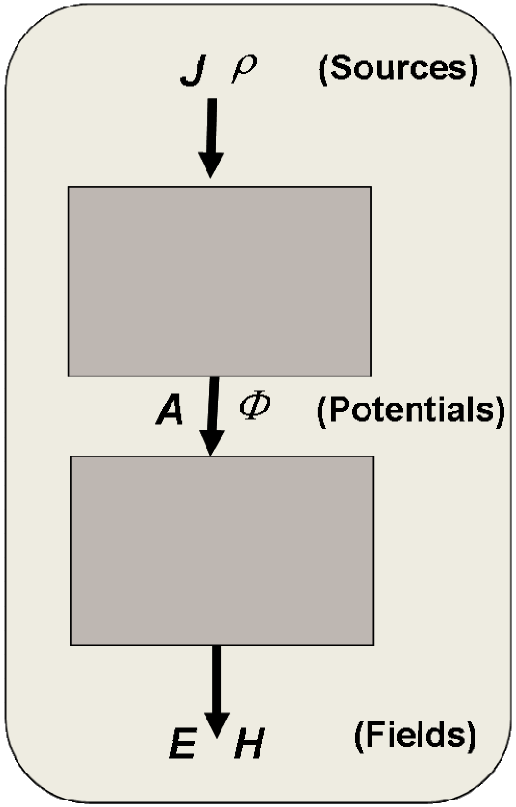
Summary of the past lecture

Radiation problem



Summary of the past lecture

Potentials



$\boldsymbol{J}$ ↓

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

↓ $\boldsymbol{A}$

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla \nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

↓

$\boldsymbol{E}$ $\boldsymbol{H}$

Summary of the past lecture

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$-\mu J_x(\vec{\mathbf{r}}) \longrightarrow \boxed{g(\vec{\mathbf{r}})} \longrightarrow A_x(\vec{\mathbf{r}})$$

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow g(\vec{\mathbf{r}}) = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}|}}{|\vec{\mathbf{r}}|} = -\frac{1}{4\pi}\frac{e^{-jkr}}{r}$$

$$\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}') \longrightarrow \boxed{} \longrightarrow = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = g(\vec{\mathbf{r}} - \vec{\mathbf{r}}')$$

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\, d\,\vec{\mathbf{r}}' = \frac{\mu}{4\pi}\int J_x(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}\, d\,\vec{\mathbf{r}}'$$

Summary of the past lecture

Potentials

$\boldsymbol{J}(\boldsymbol{r})$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla \nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

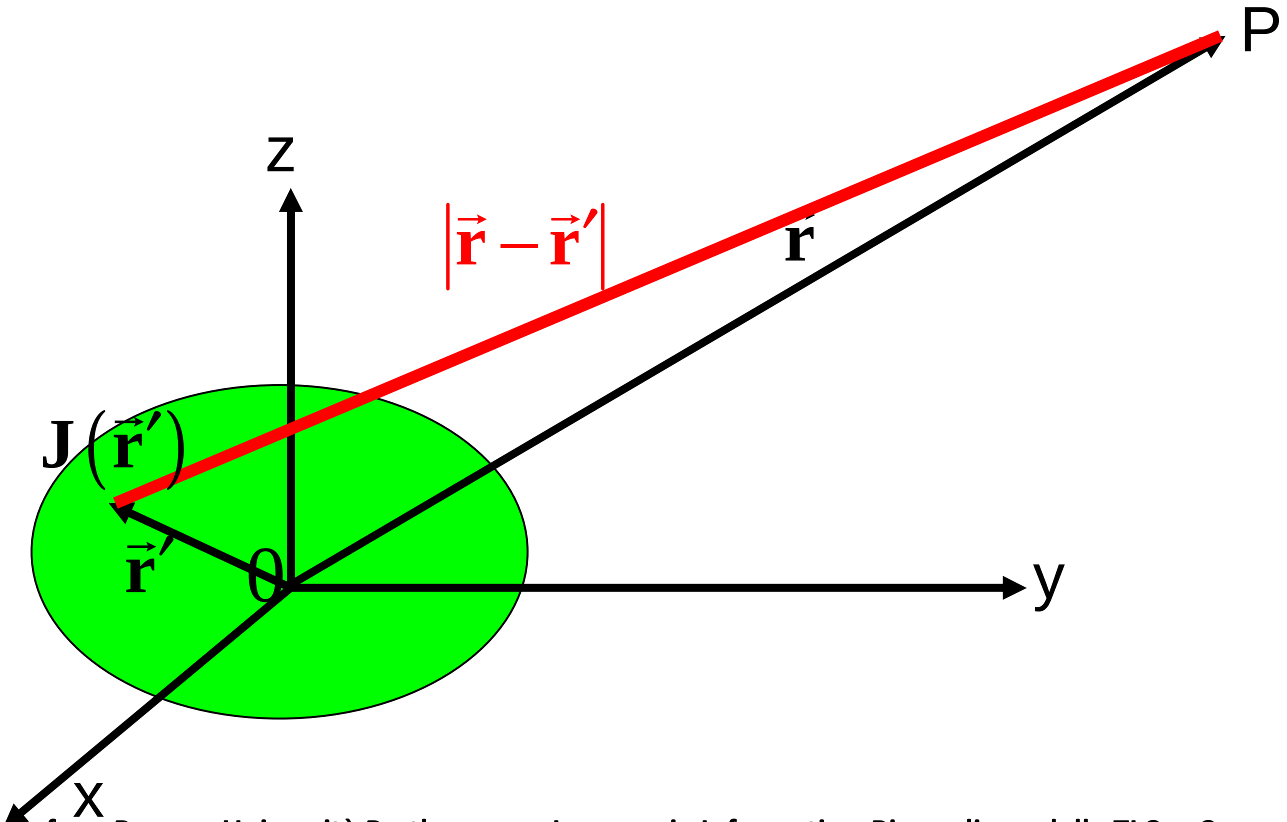
$\boldsymbol{E}(\boldsymbol{r})$

$\boldsymbol{H}(\boldsymbol{r})$

Summary of the past lecture

Potentials

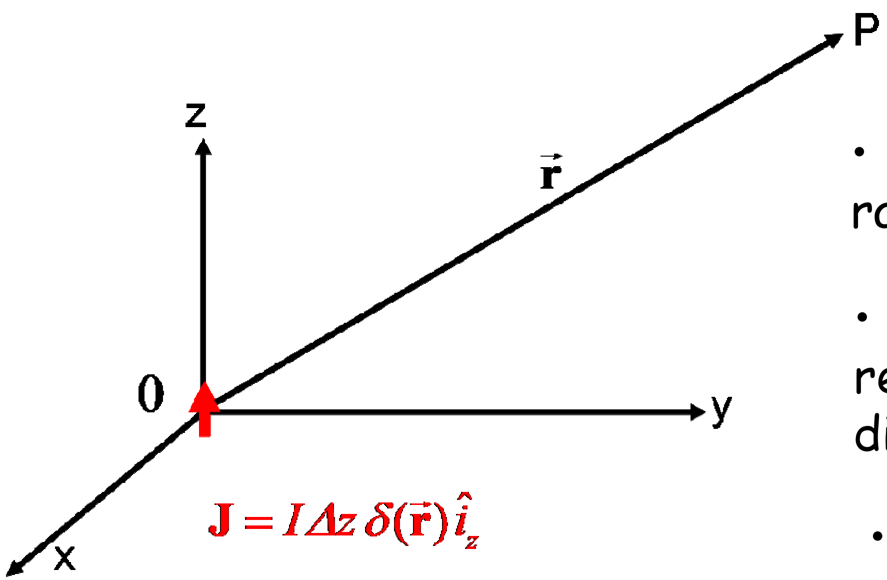
$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int \mathbf{J}(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$



Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

Elementary electrical dipole

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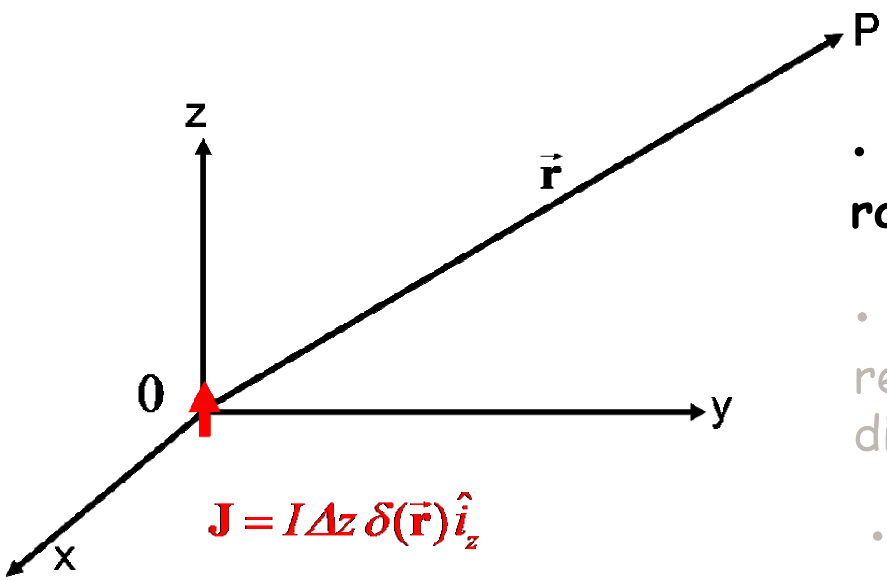


- **Why are we interested in such a radiating element?**
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

… memo …

$$\mathbf{A}(\vec{\mathbf{r}}) = \int -\mu \mathbf{J}(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\vec{\mathbf{r}}'$$

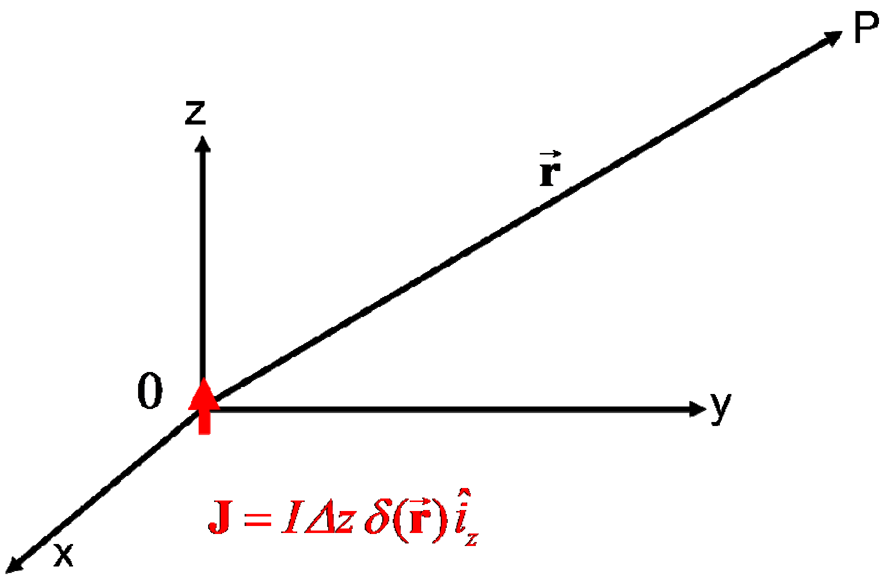
$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$

Mathematically, a δ-source radiating element is related to the radiation of any antenna!!

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



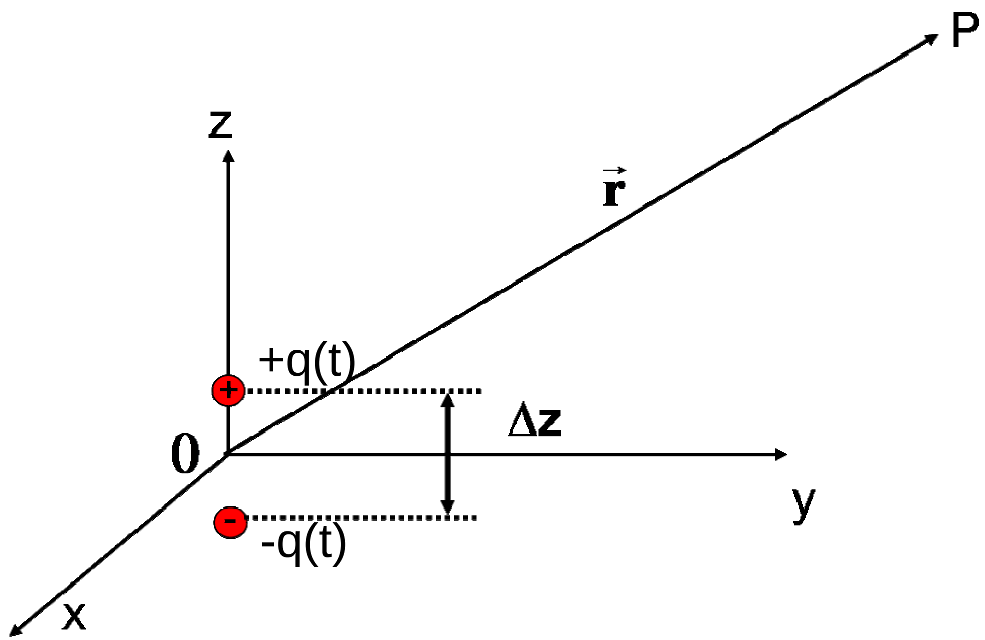
- Why are we interested in such a radiating element?

- **Why is such a radiating element referred to as elementary electrical dipole?**

- How can we physically approximate an elementary electrical dipole?

Elementary electrical dipole

$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\,\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



It can be easily shown that this electric current density is the same as that of an electrical dipole such that:

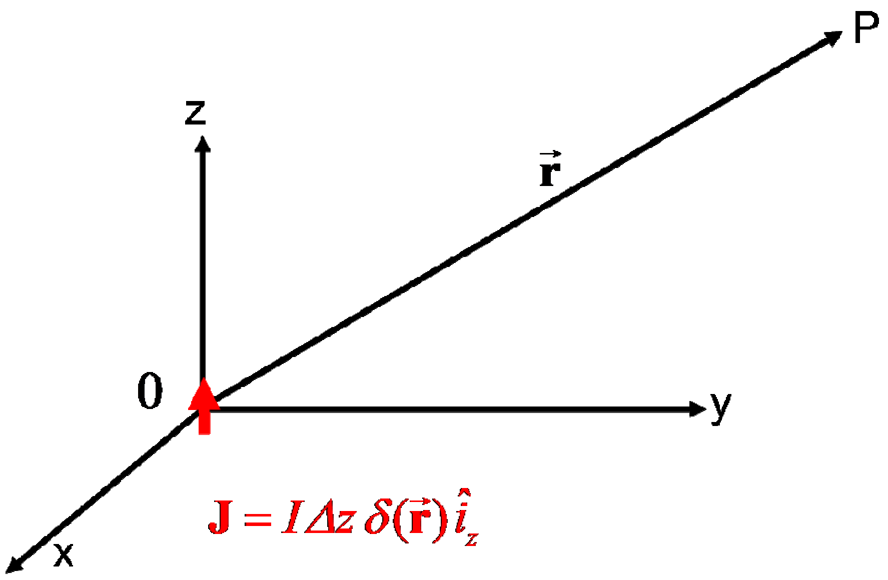
1) the two charges, of opposite sign, have equal time variation;

2) in the spectral domain, the relation between I and the time-varying charge Q is:

$$j\omega Q = I$$

Elementary electrical dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- **How can we physically approximate an elementary electrical dipole?**

Hertzian dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

- Of course in real life one cannot physically build up a δ-source radiating element but only an approximation.
- An approximation of the elementary dipole was used by Hertz in his experiments, in fact the elementary dipole is often called as Hertzian dipole.
- Note however that an Hertzian dipole is a dipole characterized by:

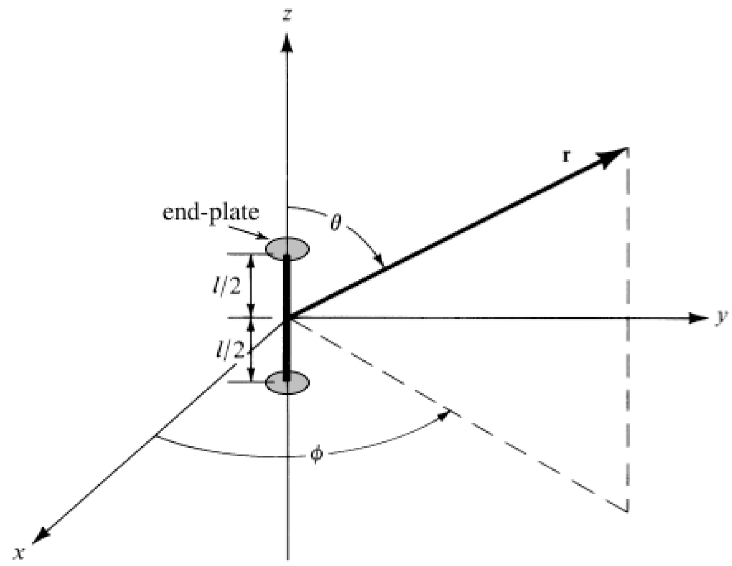
$$\mathbf{J} = I\,\delta(x)\delta(y)\,\mathrm{rect}\left[\frac{z}{\Delta z}\right]\hat{i}_z$$

$$when\ \ \Delta z \to 0 \quad then$$

$$\mathrm{rect}\left[\frac{z}{\Delta z}\right] \to \Delta z\,\delta(z)$$

Hertzian dipole

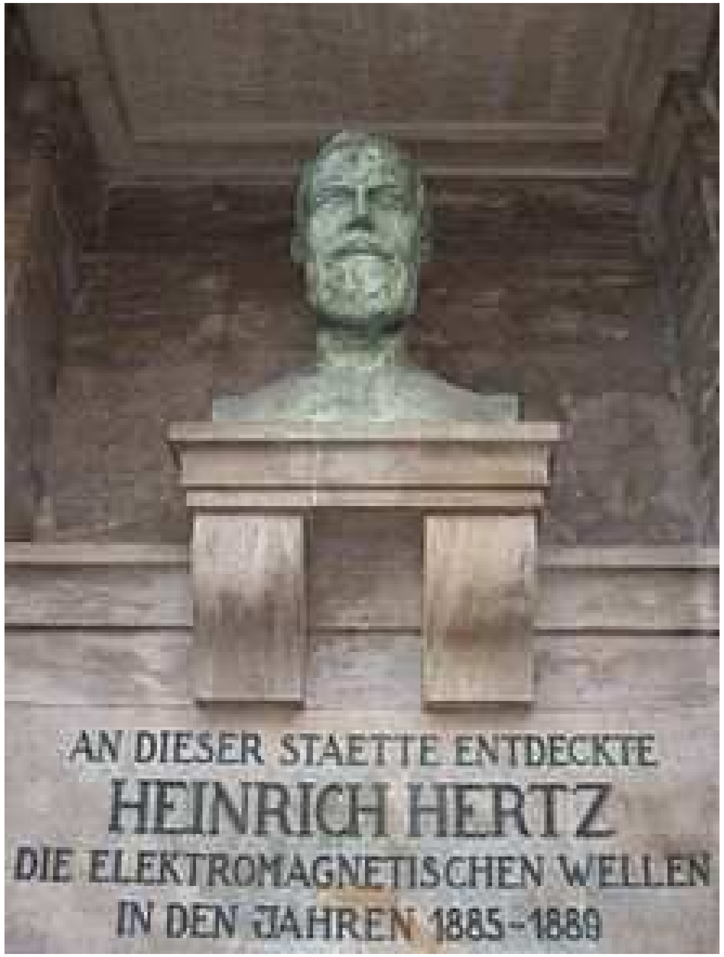
- The creation of the constant current distribution can be made by two large charge "tanks" at the two edges.
- Note that in practical case this model is meant to be suitable for electrical dipole smaller than λ/50.



Infinitesimal dipole

Heinrich Rudolf Hertz

- **Heinrich Rudolf Hertz** (German; 22 February 1857 – 1 January 1894) was a German physicist who first conclusively proved the existence of electromagnetic waves theorized by James Clerk Maxwell.
- Hertz proved the theory by engineering instruments to transmit and receive radio pulses using experimental procedures.
- The scientific unit of frequency – cycles per second – was named the ”Hertz" in his honor.



Elementary electrical dipole

J

↓

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ **Vector Helmholtz equation**

↓

A

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

$$\mathbf{J} = I \Delta z \, \delta(\vec{\mathbf{r}}) \hat{i}_z = J_z(\vec{\mathbf{r}}) \hat{i}_z$$

Potentials

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = J_z(\vec{\mathbf{r}})\,\hat{i}_z$$

$$-\mu J_z(\vec{\mathbf{r}}) = -\mu I\,\Delta z\,\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow \frac{\mu}{4\pi} I\,\Delta z \frac{e^{-jkr}}{r}$$

... memo ...

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow g(\vec{\mathbf{r}}) = -\frac{1}{4\pi}\frac{e^{-jkr}}{r}$$

Potentials

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = J_z(\vec{\mathbf{r}})\,\hat{i}_z$$

$$-\mu J_z(\vec{\mathbf{r}}) = -\mu I\,\Delta z\,\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{\quad} \longrightarrow \frac{\mu}{4\pi} I\,\Delta z \frac{e^{-jkr}}{r}$$

$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z \Longrightarrow \mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} I\,\Delta z \frac{e^{-jkr}}{r}\,\hat{i}_z$$

Elementary electrical dipole

$$\mathbf{J} = I\Delta z\, \delta(\vec{\mathbf{r}})\, \hat{i}_z = I\Delta z\, \delta(x)\delta(y)\delta(z)\, \hat{i}_z$$

↓ $\boldsymbol{J}$

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ Vector Helmholtz equation

↓ $\mathbf{A} = \dfrac{\mu}{4\pi} I\Delta z \dfrac{e^{-jkr}}{r} \hat{i}_z$

$\boldsymbol{H} = \dfrac{1}{\mu} \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -j\omega \boldsymbol{A} + \dfrac{\nabla\nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$

↓ $\boldsymbol{E}$ $\boldsymbol{H}$

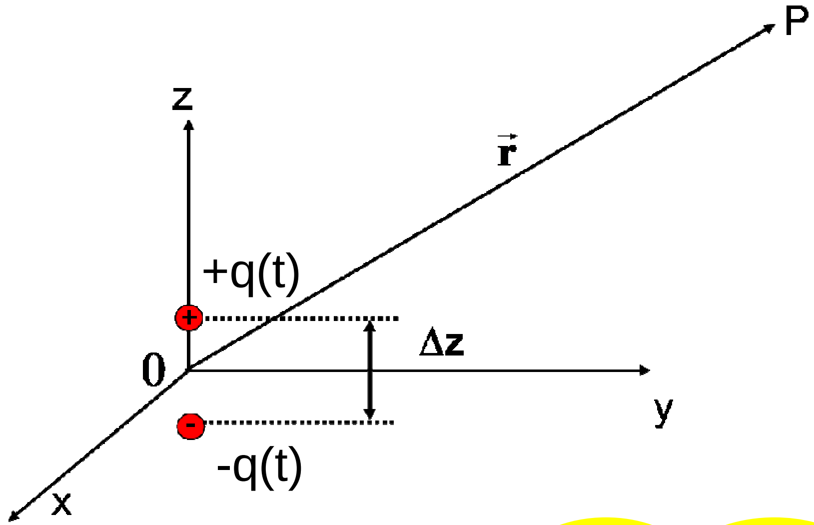
Elementary electrical dipole

The E.M. field radiated by the elementary electrical dipole

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

…. Memo

$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\,\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



It can be easily shown that this electric current density is the same as that of an electrical dipole such that:

1) the two charges, of opposite sign, have equal time variation;

2) in the spectral domain, the relation between I and the time-varying charge Q is:

$$j\omega Q = I$$

All the quantities, included the expressions of the fields, can be provided in terms of dipole moment U

$$I\,\Delta z = j\omega Q\,\Delta z = j\omega U$$

Elementary electrical dipole

The E.M. field radiated by the elementary electrical dipole

$$
\begin{cases}
E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\
E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\
H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r)
\end{cases}
$$

… for $\omega = 0$ simplifies as

$$
\begin{cases}
E_r = \dfrac{Q\Delta z}{2\pi}\dfrac{1}{\varepsilon r^3}\cos\vartheta \\
E_\vartheta = \dfrac{Q\Delta z}{4\pi}\dfrac{1}{\varepsilon r^3}\sin\vartheta \\
H_\varphi = 0
\end{cases}
$$

Elementary electrical dipole

…the E.M. field radiated by the elementary electrical dipole..

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

… for $r \gg \lambda$ simplifies as

Elementary electrical dipole

- Because of the problem symmetry there is no dependence on the azimuth angle φ.
- The distance-dependent terms can be written as follows:

$$\left(1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right)$$

- When $\beta r=2\pi r/\lambda>>1$ only the first term can be considered, i.e. $r>>\lambda$.
- The remaining term is known as far-field component while the neglected ones are near-field ones.

Elementary electrical dipole

…the E.M. field radiated by the elementary electrical dipole..

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

… for r>>λ simplifies as

$$\begin{cases} E_r = 0 \\ E_\vartheta = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_\varphi = j\dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) = \dfrac{E_\vartheta}{\zeta} \end{cases}$$

Elementary electrical dipole

- In the far-field case the elementary electrical dipole behaves as follows:

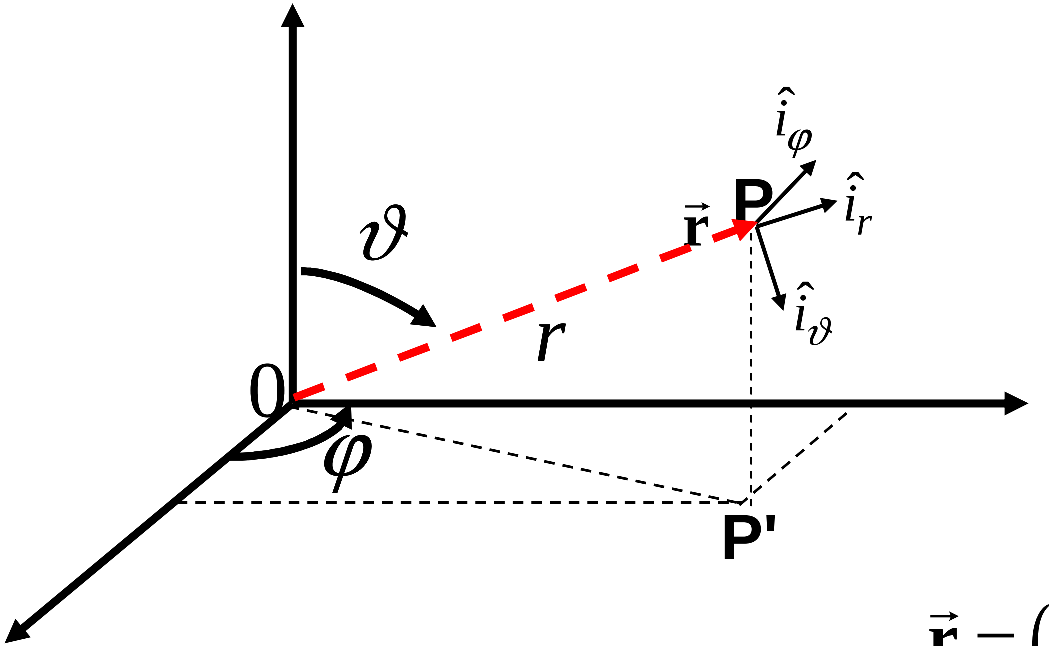
$$\begin{cases} E_r = 0 \\ E_\vartheta = j\zeta \dfrac{I\Delta z}{2\lambda r} \sin\vartheta \exp(-j\beta r) \\ H_\varphi = j\dfrac{I\Delta z}{2\lambda r} \sin\vartheta \exp(-j\beta r) = \dfrac{E_\vartheta}{\zeta} \end{cases}$$

- Note the far-field relationship between ***E*** and ***H***

…memo….

spherical coordinate system

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_r(\vec{\mathbf{r}})\hat{i}_r + \mathrm{E}_\varphi(\vec{\mathbf{r}})\hat{i}_\varphi + \mathrm{E}_\vartheta(\vec{\mathbf{r}})\hat{i}_\vartheta$$



$$\hat{i}_\varphi = \hat{i}_r \times \hat{i}_\vartheta$$

$$\hat{i}_\vartheta = \hat{i}_\varphi \times \hat{i}_r$$

$$\hat{i}_r = \hat{i}_\vartheta \times \hat{i}_\varphi$$

$$\vec{\mathbf{r}} = (r, \vartheta, \varphi)$$

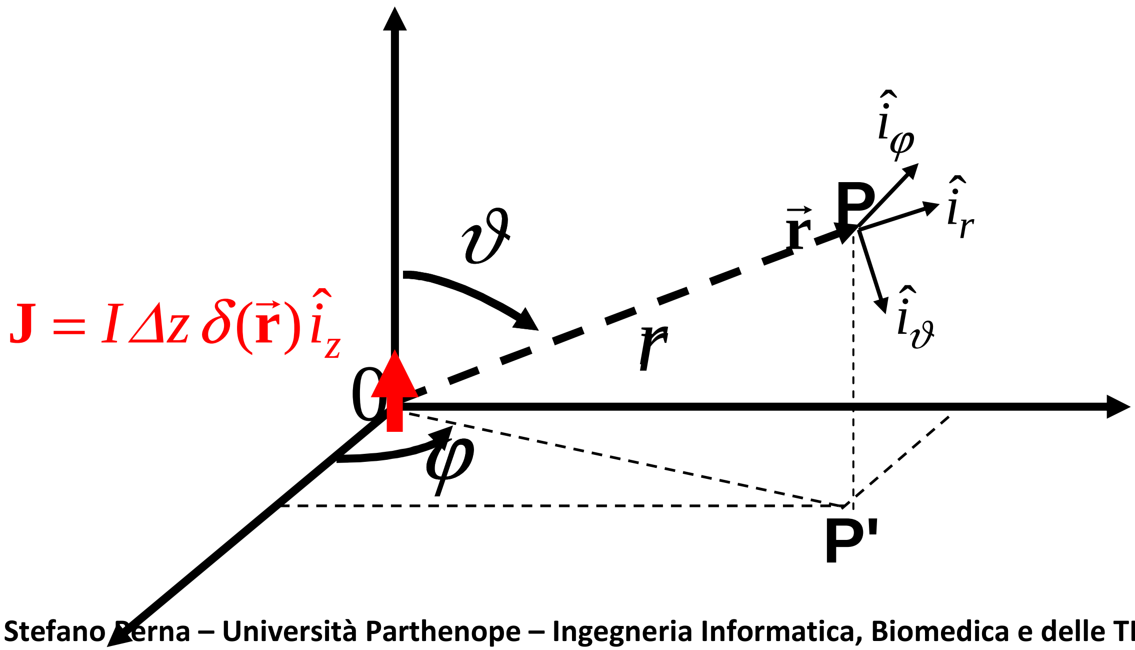
Elementary electrical dipole

for $r >> \lambda$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_{\vartheta}(r,\vartheta)\hat{i}_{\vartheta}$$

$$\vec{\mathbf{H}} = \vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_{\varphi}(r,\vartheta)\hat{i}_{\varphi}$$

$$\begin{cases} E_{\vartheta} = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_{\varphi} = j\dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) = \dfrac{E_{\vartheta}}{\zeta} \end{cases}$$



- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction

Elementary electrical dipole

In the far-field region, we have

$$\begin{cases} E_r = 0 \\ E_\vartheta = j\zeta \dfrac{I\Delta z}{2\lambda r} \sin\vartheta \exp(-j\beta r) \\ H_\varphi = j\dfrac{I\Delta z}{2\lambda r} \sin\vartheta \exp(-j\beta r) = \dfrac{E_\vartheta}{\zeta} \end{cases} \Rightarrow \zeta \mathbf{H} = \hat{i}_r \times \mathbf{E}$$

$$\hat{i}_\varphi = \hat{i}_r \times \hat{i}_\vartheta$$

$$\hat{i}_\vartheta = \hat{i}_\varphi \times \hat{i}_r$$

$$\hat{i}_r = \hat{i}_\vartheta \times \hat{i}_\varphi$$

and the Poynting vector:

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\frac{|E_\vartheta|^2}{\zeta}\hat{i}_r = \frac{1}{2}\frac{|\mathbf{E}|^2}{\zeta}\hat{i}_r$$

Elementary electrical dipole

- In order to further characterize its behavior one can evaluate the Poynting vector and the associated power for the overall e.m. field (over a sphere centered in the origin):

$$P = \frac{1}{2} \oiint_S \left[\mathbf{E} \times \mathbf{H}^* \right] \cdot \hat{i}_r dS = \frac{1}{2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta E_\vartheta H_\varphi^*$$

I) $dS = r^2 \sin\vartheta d\vartheta d\varphi$

II) $\left[\mathbf{E} \times \mathbf{H}^* \right] \cdot \hat{i}_r = \left[\left(E_\vartheta \hat{i}_\vartheta + E_r \hat{i}_r \right) \times \left(H_\varphi^* \hat{i}_\varphi \right) \right] \cdot \hat{i}_r = E_\vartheta H_\varphi^*$

$$P=\frac{1}{2}\oiint_S\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{i}_r dS=\frac{1}{2}\int_0^{2\pi}d\varphi\int_0^{\pi}d\vartheta r^2\sin\vartheta E_\vartheta H_\varphi^*=\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2\left[1-j\frac{1}{(\beta r)^3}\right]|I|^2$$

$$\begin{cases} E_r=\zeta\dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta\exp(-j\beta r) \\ E_\vartheta=\zeta\dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta\exp(-j\beta r) \\ H_\varphi=\dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$

$$P=P_1+jP_2$$

$$P_1=\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2|I|^2$$

$$P_2=-\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2\frac{1}{(\beta r)^3}|I|^2$$

$$\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{-j\beta}{r}+\frac{1}{r^2}\right)=\left(\frac{\beta}{r}\right)^2-j\frac{1}{\beta r^5}$$

- *Note that in the far-field case only the first active power term exists and it does not depend on r.*

Elementary electrical dipole

- Note that the real part of the power, in lossless medium, is independent of *r*, therefore if one consider two different spherical surfaces one gets the same result. Only the so-called radiative terms contribute.
- The reactive part depends on *r*. Its sign is negative showing that there is an excess of stored **electric** energy in the neighbor of the electrical dipole (see Poynting's theorem)