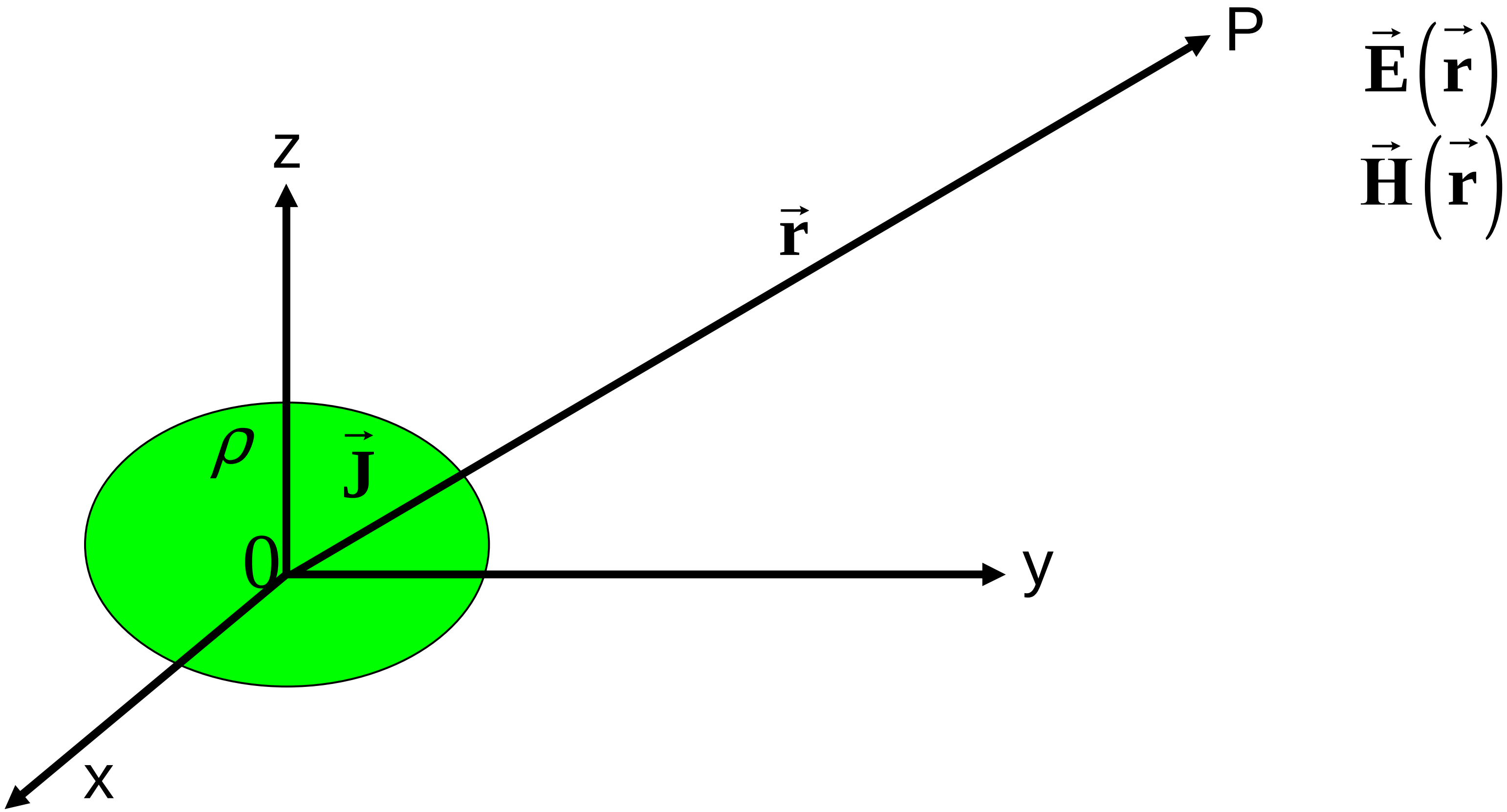
# Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

## Corso di Campi Elettromagnetici
## a.a. 2017-2018

# 7 Maggio 2018

# Summary of the past lecture

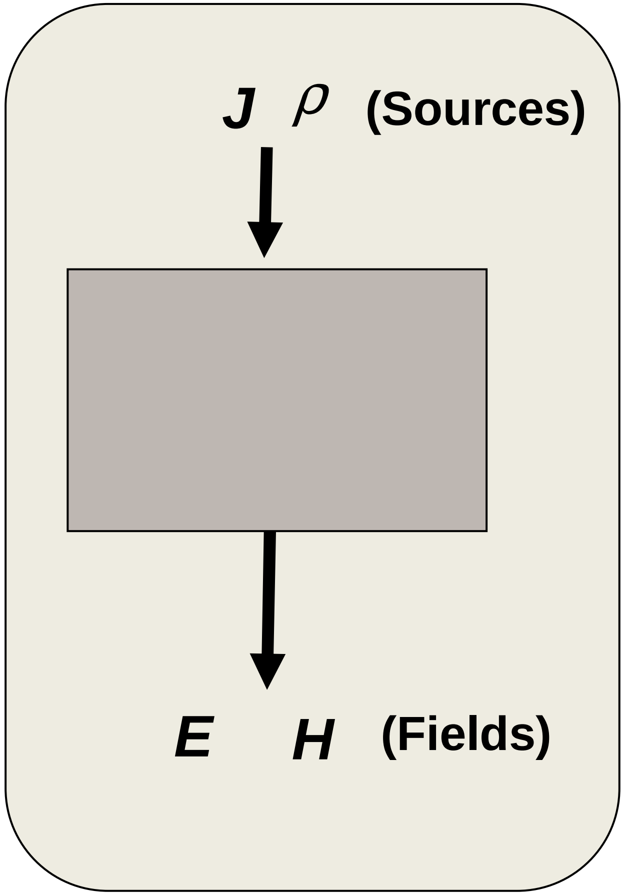
## Radiation problem

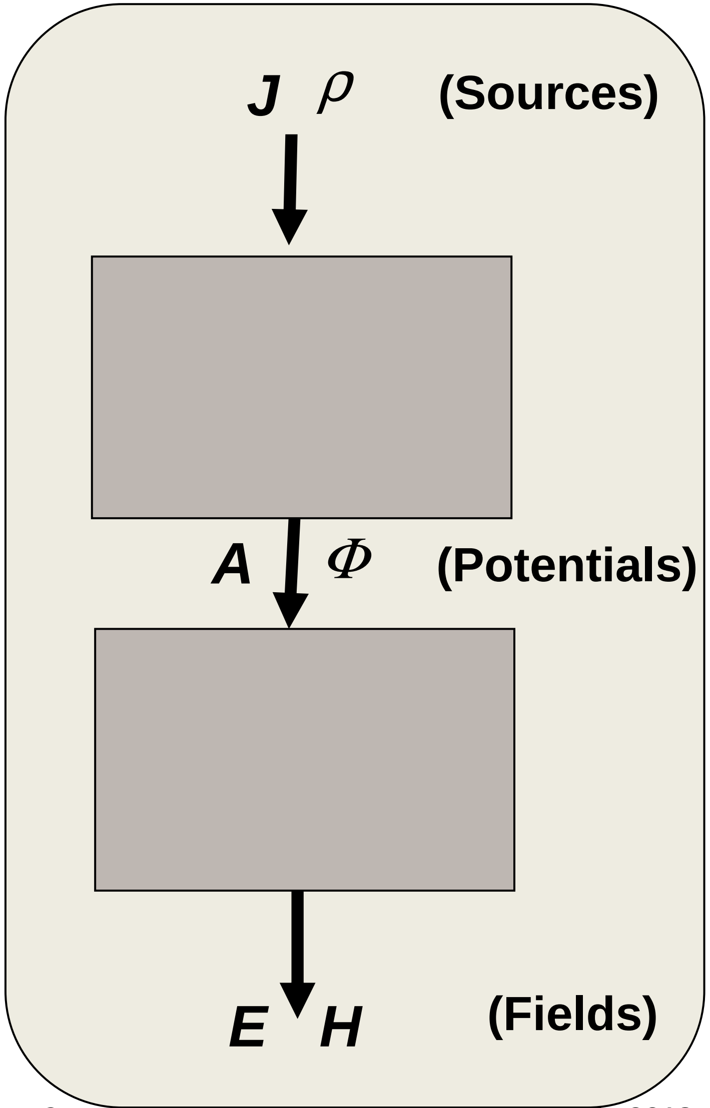


# Summary of the past lecture

## Potentials

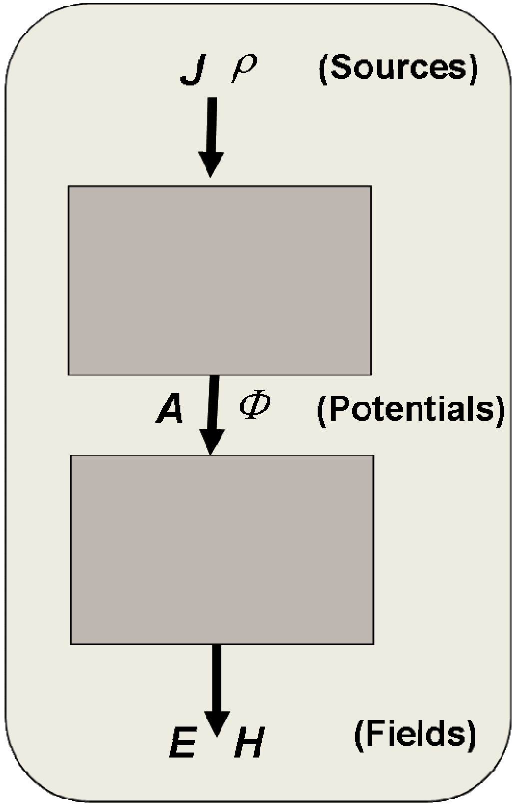
$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$



# Summary of the past lecture

## Potentials




$\mathbf{J}$ $\rho$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi\right)$$

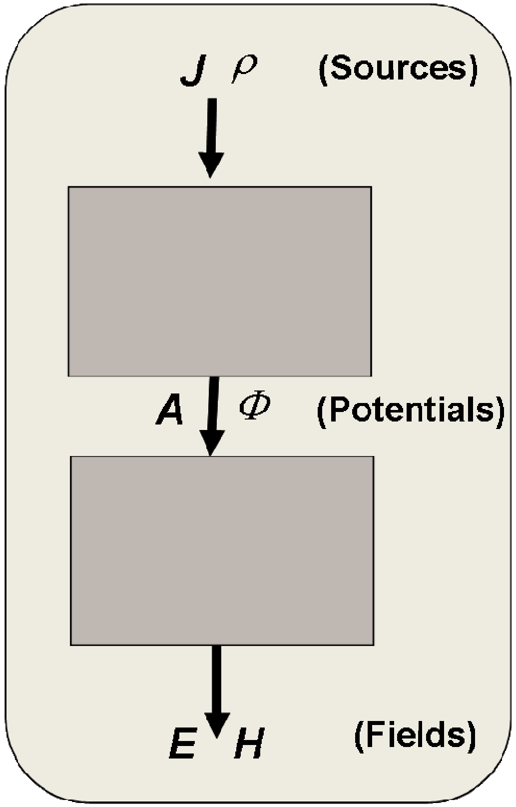
$\mathbf{A}$ $\Phi$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

$\mathbf{E}$ $\mathbf{H}$

# Summary of the past lecture

## Potentials




**J**

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

**A**

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla \nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

**E H**

# Potentials

$\boldsymbol{J}$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

**Vector Helmholtz equation**

$\boldsymbol{A}$

# … mathematical tools …

$$\mathbf{C} = \mathrm{C_x}(x,y,z)\hat{i}_x + \mathrm{C_y}(x,y,z)\hat{i}_y + \mathrm{C_z}(x,y,z)\hat{i}_z$$

$$\Phi = \Phi(x,y,z)$$

$$\nabla^2 \Phi = \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\nabla^2 \mathbf{C} = \nabla^2 \mathrm{C}_x \, \hat{i}_x + \nabla^2 \mathrm{C}_y \, \hat{i}_y + \nabla^2 \mathrm{C}_z \, \hat{i}_z$$

# Potentials

***J***

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

**Vector Helmholtz equation**

***A***

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

# Potentials

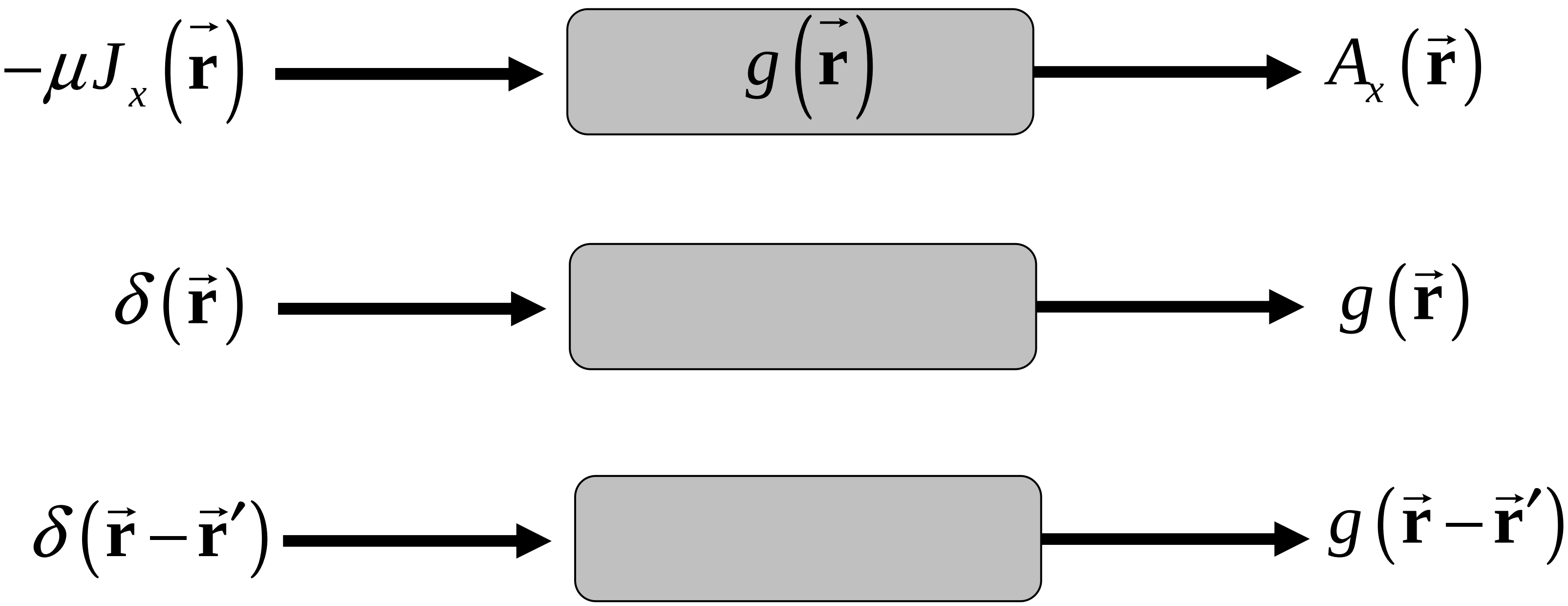
$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

Let us address the solution of the following scalar Helmholtz equation

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

# Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

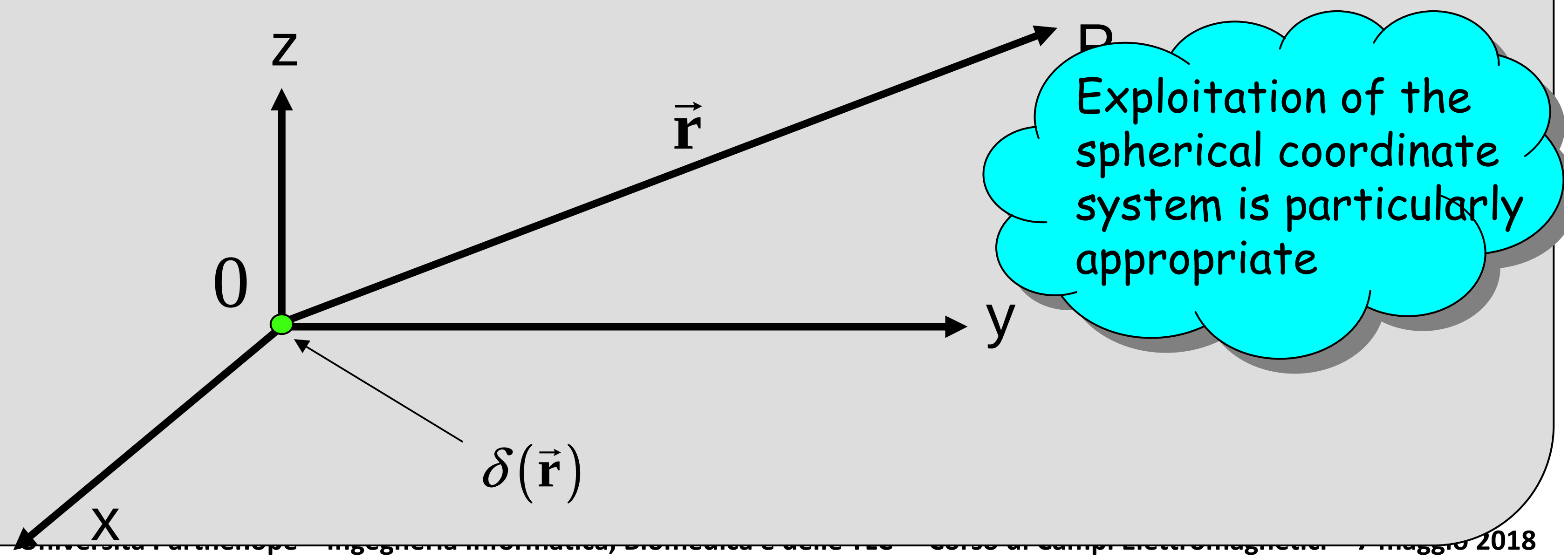



$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\vec{\mathbf{r}}'$$

# Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

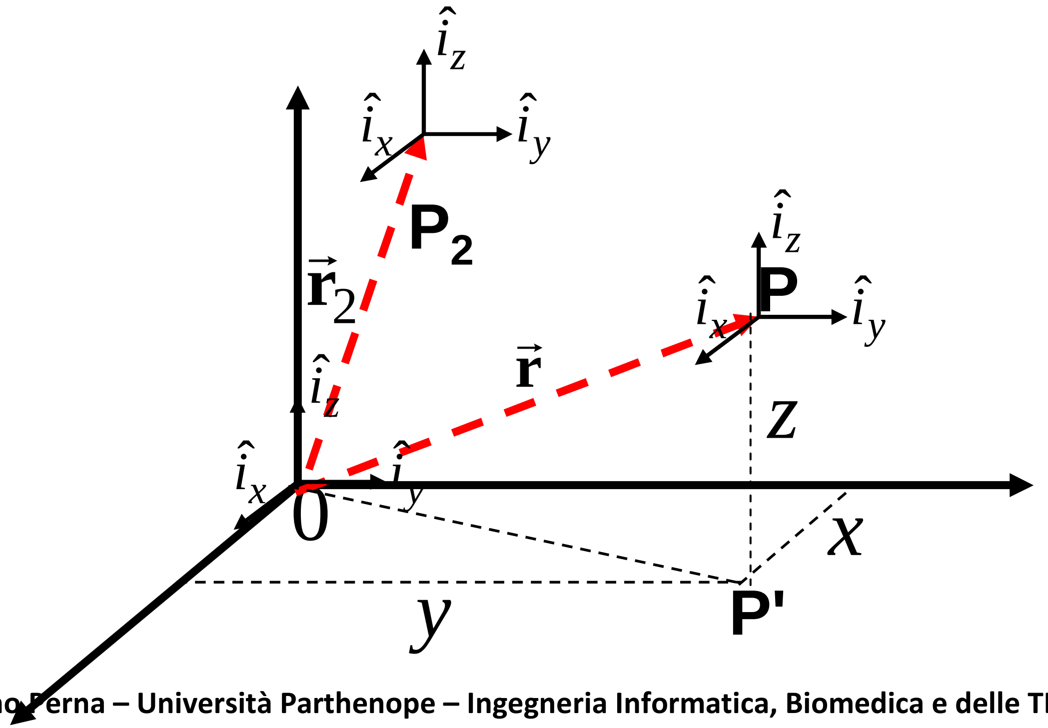

$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$



# Reference systems
# Cartesian coordinate system

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E_x}(\vec{\mathbf{r}})\hat{i}_x + \mathrm{E_y}(\vec{\mathbf{r}})\hat{i}_y + \mathrm{E_z}(\vec{\mathbf{r}})\hat{i}_z$$

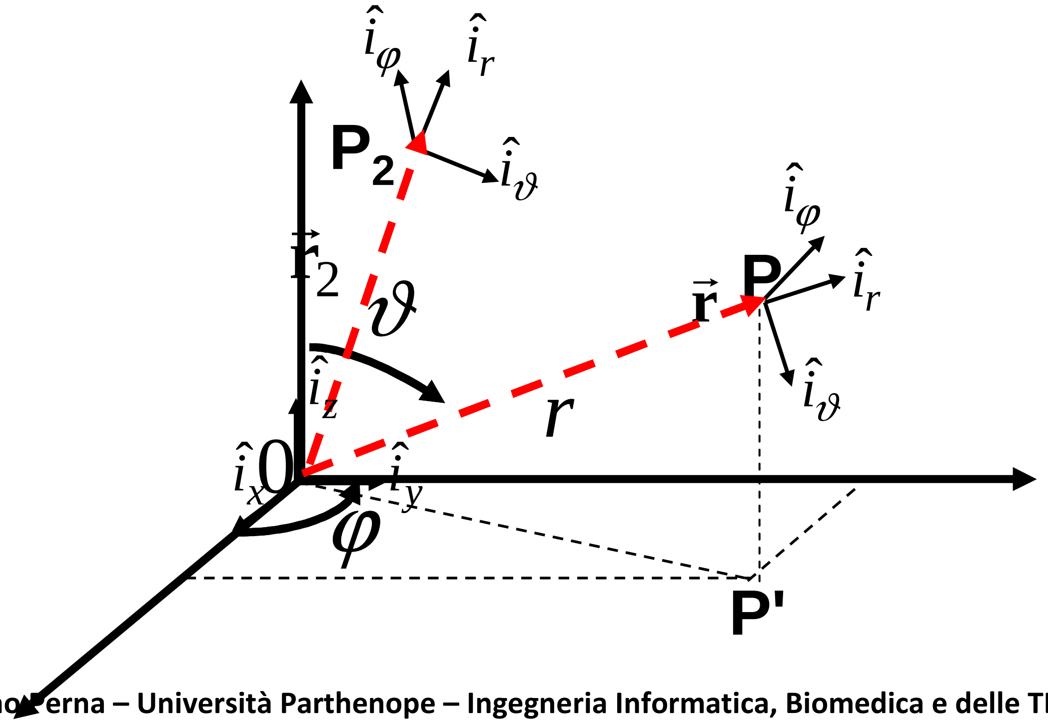
$$\vec{\mathbf{r}} = (x, y, z)$$



# Reference systems
# Spherical coordinate system

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_r(\vec{\mathbf{r}})\hat{i}_r + \mathrm{E}_\varphi(\vec{\mathbf{r}})\hat{i}_\varphi + \mathrm{E}_\vartheta(\vec{\mathbf{r}})\hat{i}_\vartheta$$

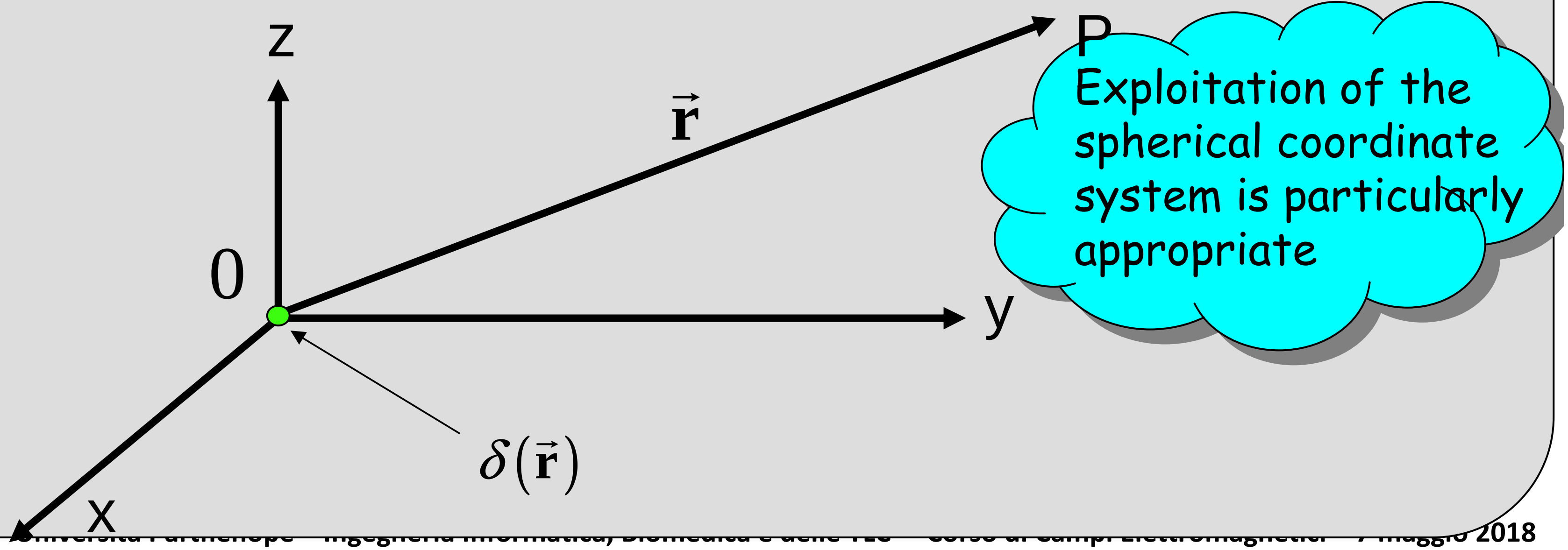
$$\vec{\mathbf{r}} = (r, \vartheta, \varphi)$$



# Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$


$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$



# Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$


$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$

where, in principle, $\quad g(\vec{\mathbf{r}}) = g(r, \vartheta, \varphi)$

However, due to symmetry considerations, the function $A_x(r,\vartheta,\varphi)$ turns out to be independent of $\vartheta$ and $\varphi$, that is,

$$g(\vec{\mathbf{r}}) = g(r)$$

Accordingly, in the whole three dimensional space the solution of the Helmholtz equation is:

$$g(r) = -\frac{1}{4\pi}\frac{e^{-jkr}}{r}$$

# Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$-\mu J_x(\vec{\mathbf{r}}) \longrightarrow \boxed{\; g(\vec{\mathbf{r}}) \;} \longrightarrow A_x(\vec{\mathbf{r}})$$

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{\quad\quad} \longrightarrow g(\vec{\mathbf{r}}) = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}|}}{|\vec{\mathbf{r}}|} = -\frac{1}{4\pi}\frac{e^{-jkr}}{r}$$

$$\delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \longrightarrow \boxed{\quad\quad} \longrightarrow = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')$$

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\vec{\mathbf{r}}' = \frac{\mu}{4\pi}\int J_x(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}\, d\vec{\mathbf{r}}'$$

# Potentials

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} \quad \Rightarrow \quad \mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int \mathbf{J}(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \quad \Rightarrow \quad A_x(\vec{\mathbf{r}}) = \dfrac{\mu}{4\pi} \displaystyle\int J_x(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}' \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \quad \Rightarrow \quad A_y(\vec{\mathbf{r}}) = \dfrac{\mu}{4\pi} \displaystyle\int J_y(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}' \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \quad \Rightarrow \quad A_z(\vec{\mathbf{r}}) = \dfrac{\mu}{4\pi} \displaystyle\int J_z(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}' \end{cases}$$

# Potentials

$\downarrow$ $\boldsymbol{J}(\boldsymbol{r})$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int \mathbf{J}(\mathbf{r}')\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

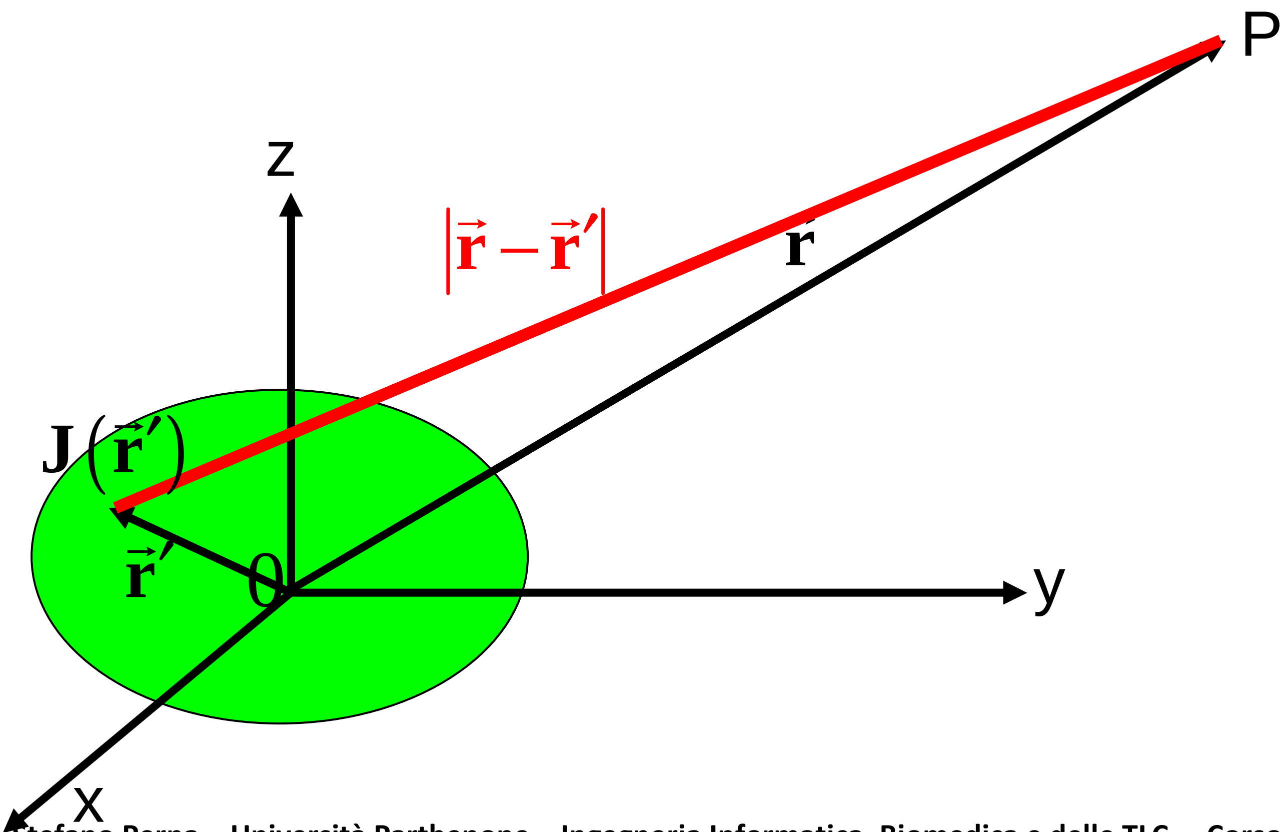
$$\boldsymbol{H} = \frac{1}{\mu}\nabla\times\boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu}$$

$\downarrow$ $\boldsymbol{E}(\boldsymbol{r})$

$\boldsymbol{H}(\boldsymbol{r})$

# Potentials

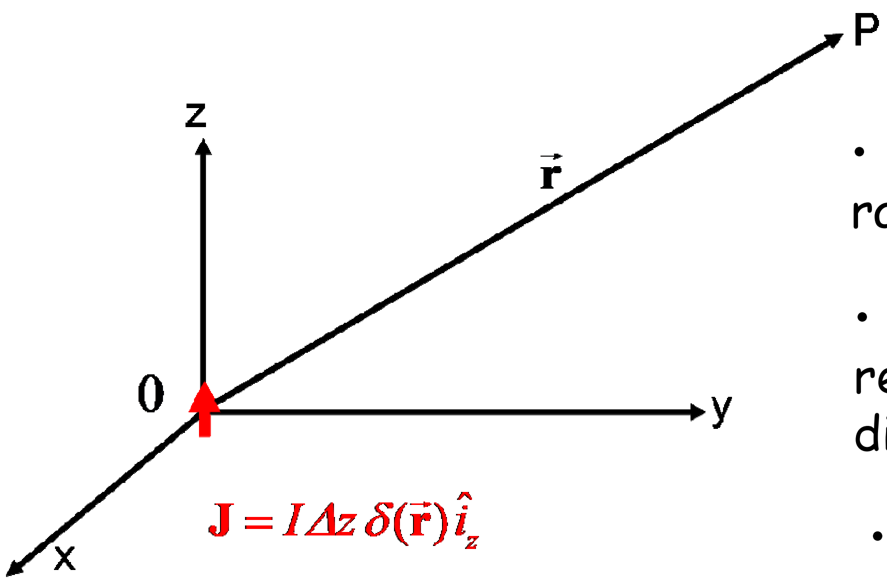
$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\vec{\mathbf{r}}'$$



# Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$




- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?