Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

Corso di Campi Elettromagnetici
a.a. 2017-2018

4 Maggio 2018

Outline

- Radiation problem
- Potentials

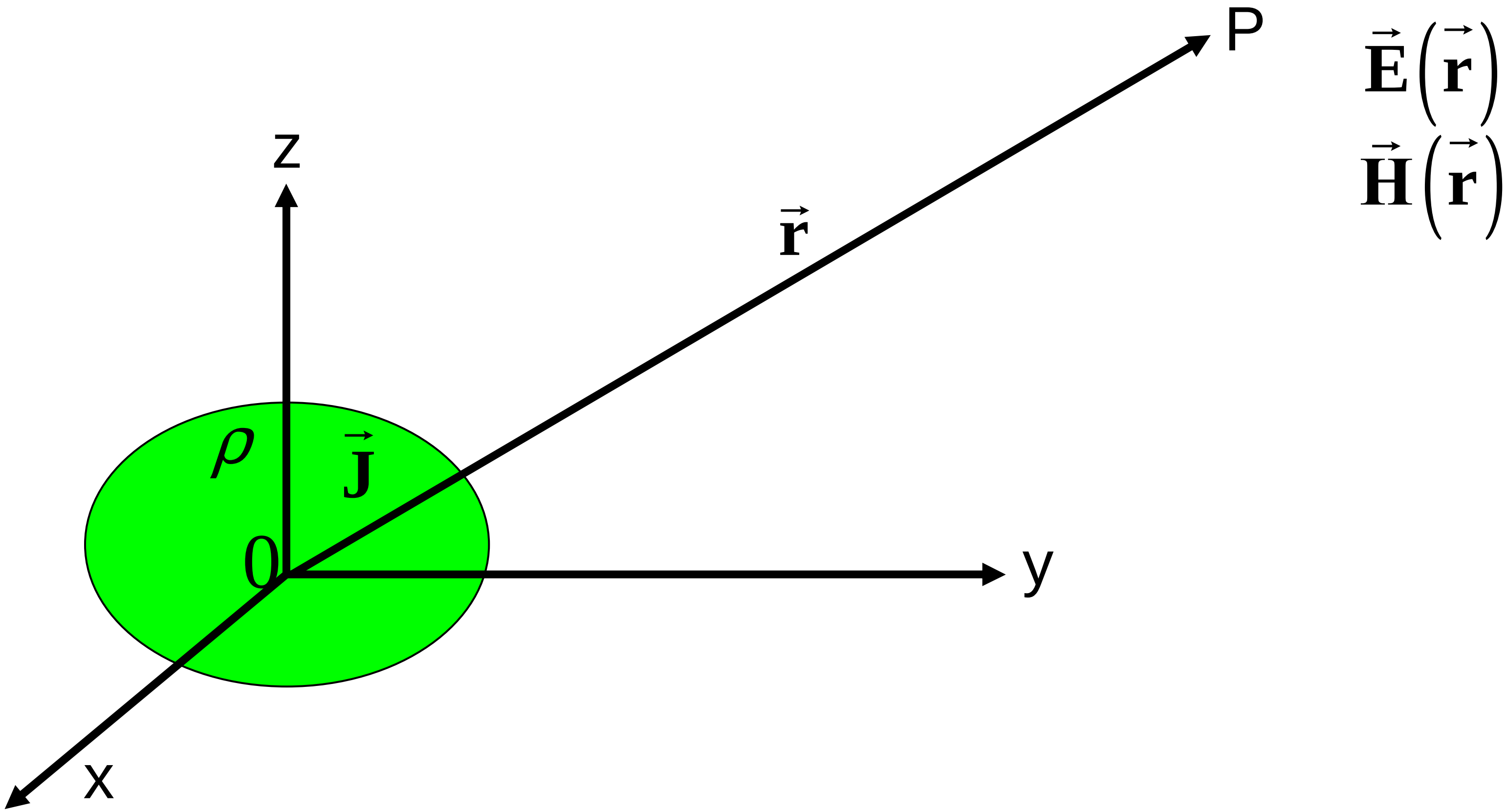
Radiation problem

- We start from the following Maxwell's equations:

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$

- The media is assumed to be linear, isotropic, homogeneous, stationary, non-dispersive in space and time.

Radiation problem



Symbols and notations

$\vec{\mathbf{r}}$ $\mathbf{r}$

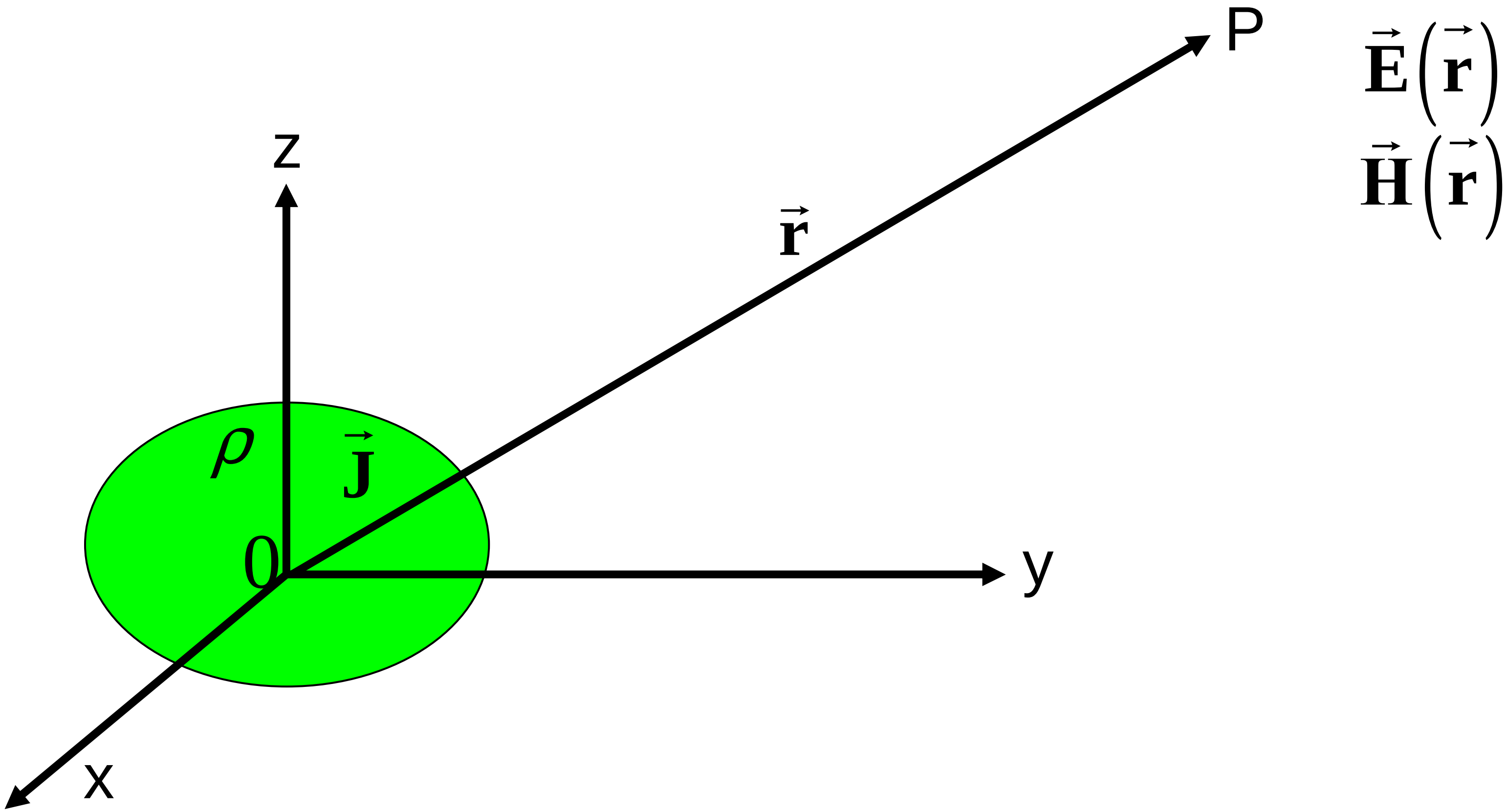
$\vec{\mathbf{E}}$ $\mathbf{E}$

vectors

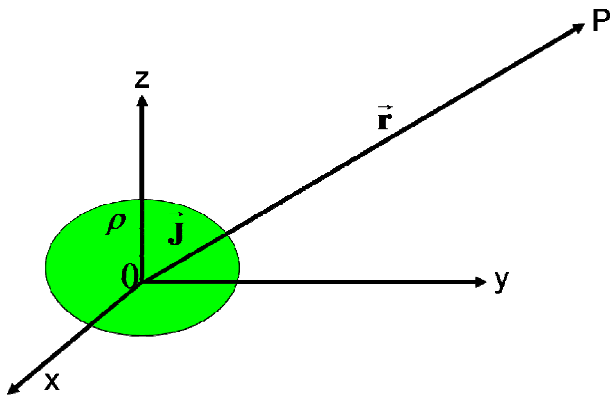
$\vec{\mathbf{E}}(\vec{\mathbf{r}})$ $\vec{\mathbf{E}}(\mathbf{r})$ $\mathbf{E}(\mathbf{r})$ Vector fields

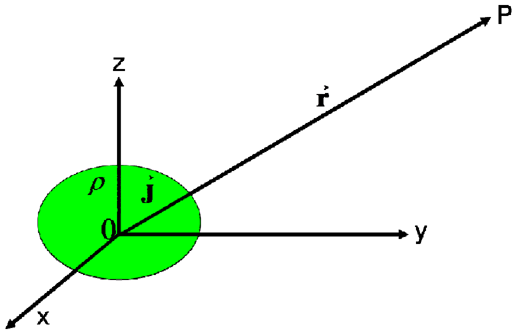
$\phi(\vec{\mathbf{r}})$ $\phi(\mathbf{r})$ Scalar fields

Radiation problem



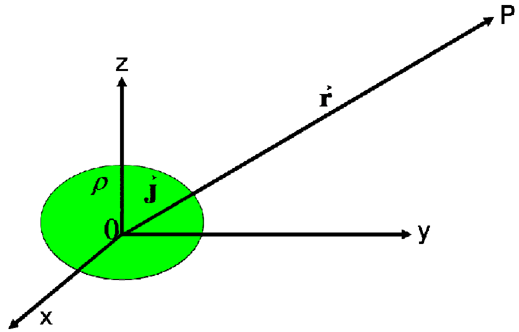
An antenna (or aerial) is an electrical device which converts electric power into radio waves, and vice versa.





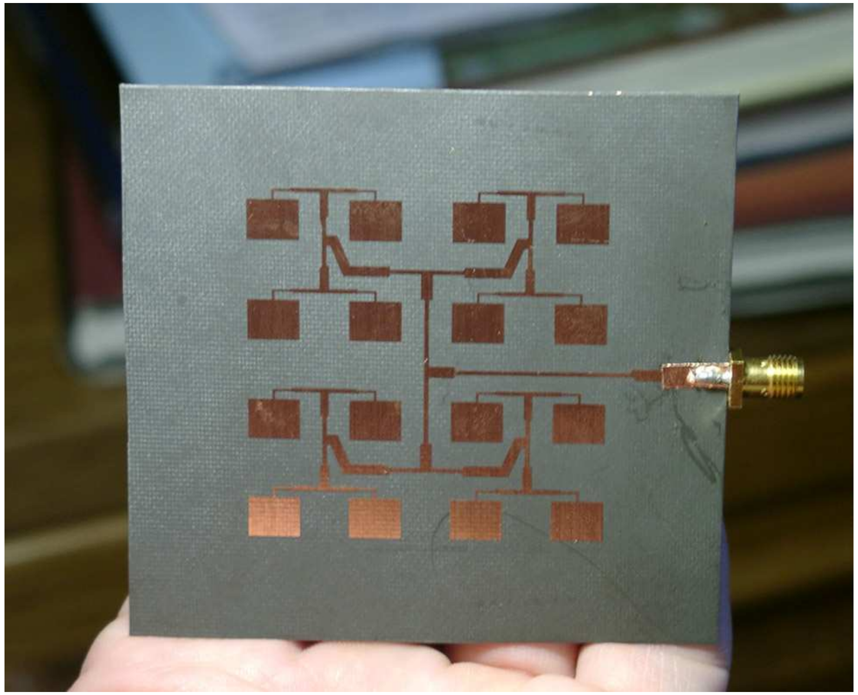
Horn antenna

Dipole antenna



Helix or helical antenna

Microstrip antenna

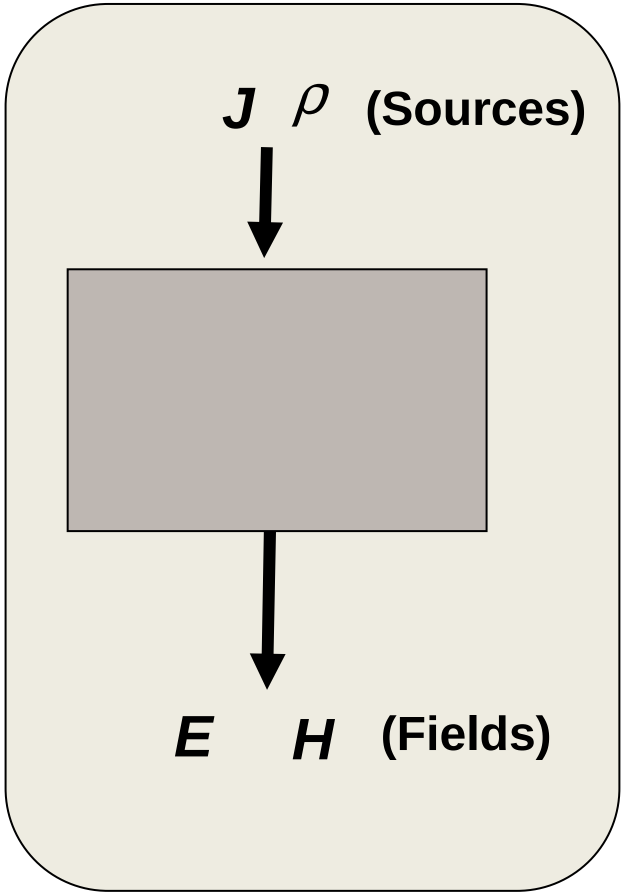


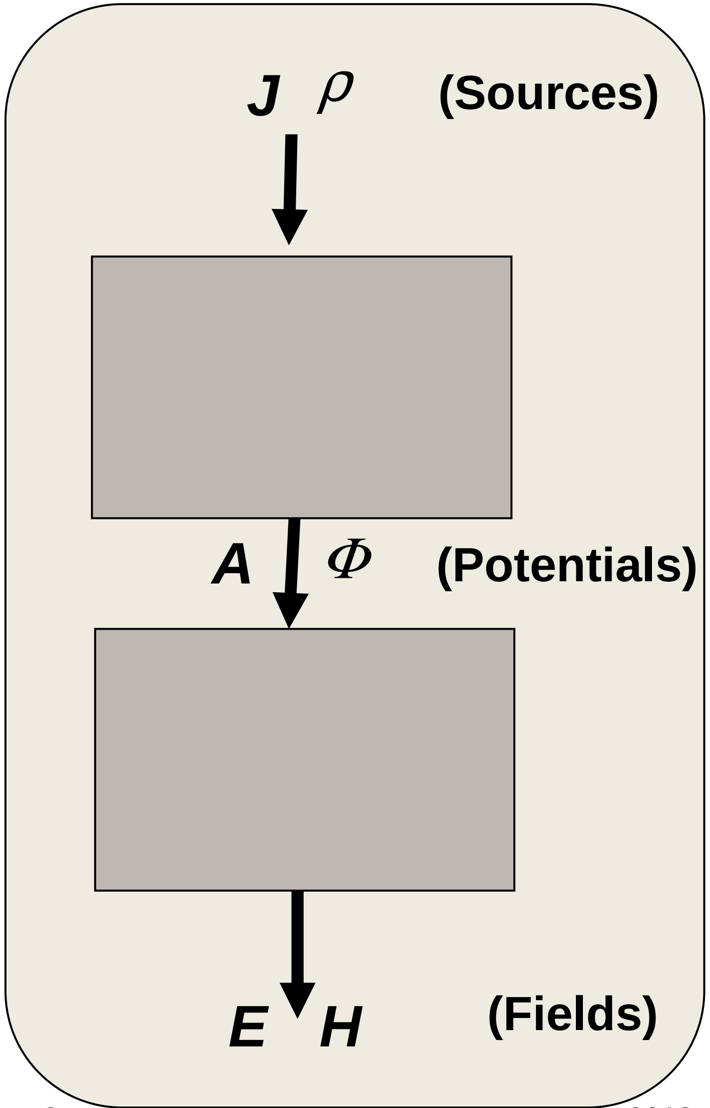
Radiation problem

- We approach the radiation problem of an antenna assuming that the sources are known and the media in which propagation occurs is simple.
- Mathematically we exploit the potentials solution.

Radiation problem

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$





… mathematical tools that we will exploit today…

I) $\nabla \times \vec{\mathbf{C}} = 0 \quad \Rightarrow \quad \exists\, \phi \quad : \quad \vec{\mathbf{C}} = \nabla \phi$

II) $\nabla \cdot \vec{\mathbf{C}} = 0 \quad \Rightarrow \quad \exists\, \vec{\mathbf{A}} \quad : \quad \vec{\mathbf{C}} = \nabla \times \vec{\mathbf{A}}$

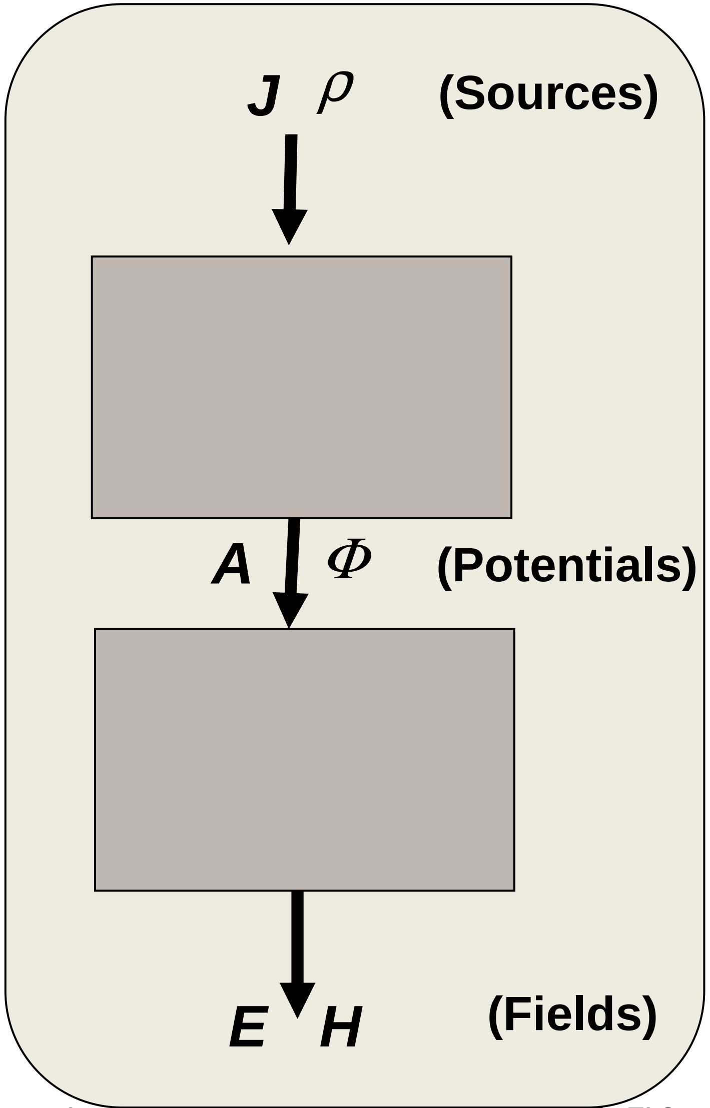
Radiation problem & potentials

- Since $\nabla \cdot \boldsymbol{B} = \nabla \cdot \mu \boldsymbol{H} = 0 \Rightarrow \boldsymbol{B} = \mu \boldsymbol{H} = \nabla \times \boldsymbol{A}$
- **A** is defined but for an $\mathbf{A}_o$ that is curl free.
- Therefore using the first Maxwell equation we get:

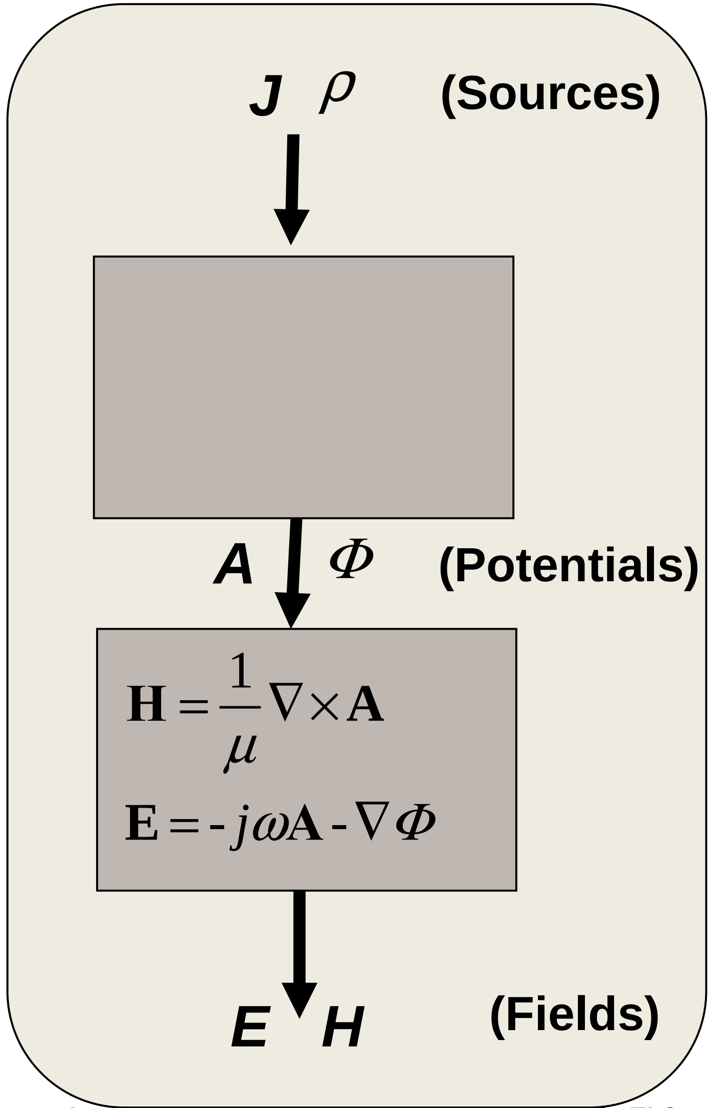
$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} = -j\omega \nabla \times \boldsymbol{A} \Rightarrow \nabla \times \left(\boldsymbol{E} + j\omega \boldsymbol{A} \right) = 0$$

- Hence $\boldsymbol{E} + j\omega \boldsymbol{A} = -\nabla \Phi$
- Φ is defined but for a Φ_o that has zero gradient.

Radiation problem & potentials



Radiation problem & potentials



Potentials

- Let us now exploit the other Maxwell curl equation:

$$\nabla \times \boldsymbol{H} = \frac{1}{\mu} \nabla \times \nabla \times \boldsymbol{A} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} = -j\omega\varepsilon\left(\nabla\Phi + j\omega \boldsymbol{A}\right) + \boldsymbol{J}$$

- Since:

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla\nabla \cdot \boldsymbol{A} - \nabla^2 \boldsymbol{A}$$

- We have:

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

Potentials

- Considering the other divergence Maxwell equation we get:

$$\nabla \cdot \boldsymbol{D} = \nabla \cdot \varepsilon \boldsymbol{E} = \varepsilon \nabla \cdot \left(-\nabla \Phi - j\omega \boldsymbol{A} \right) = \rho$$

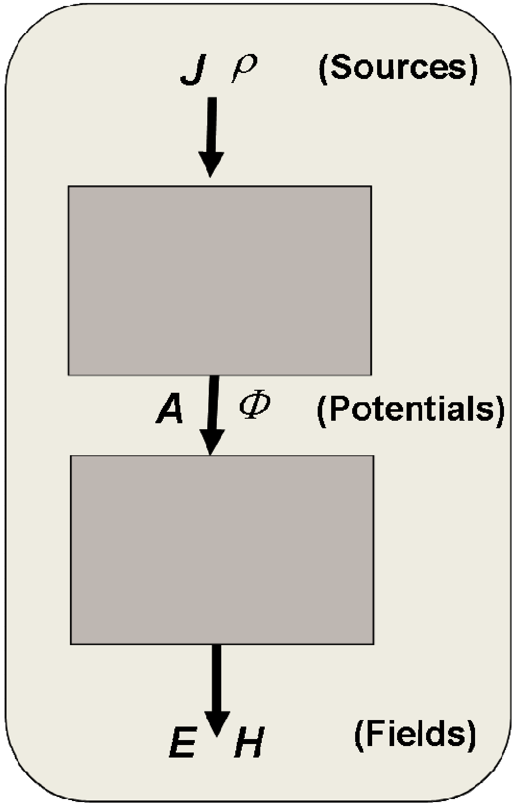
But since

$$\nabla \cdot \nabla \Phi = \nabla^2 \Phi$$

We have

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega (\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

Potentials



J ρ

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

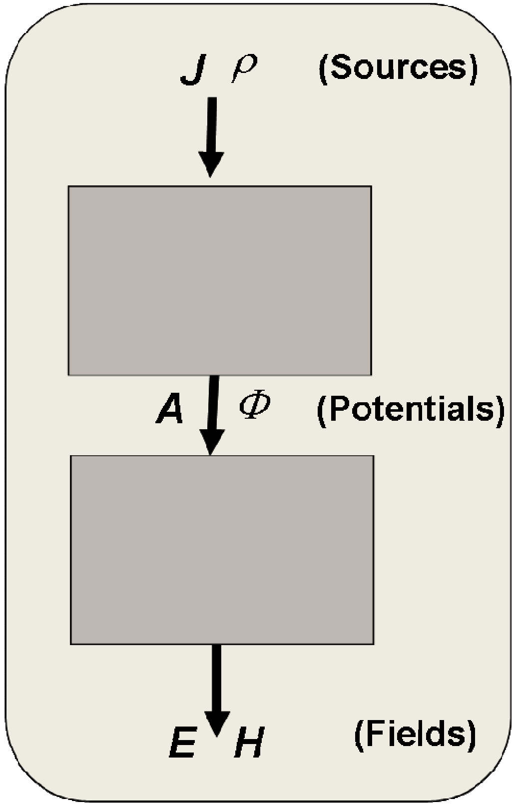
$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

A Φ

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

E **H**

Potentials



$\mathbf{J}$ ρ

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi\right)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi\right)$$

$\mathbf{A}$ Φ

… mathematical tools …

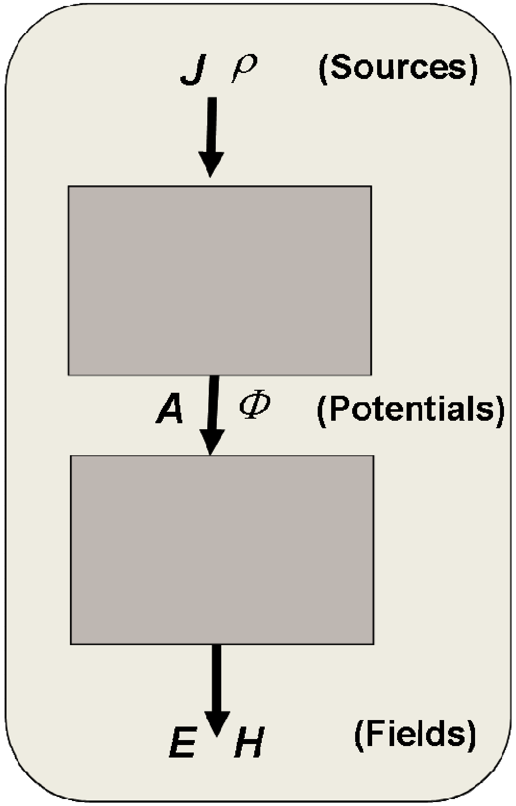
$$\mathbf{C} = \mathrm{C_x}(x,y,z)\hat{i}_x + \mathrm{C_y}(x,y,z)\hat{i}_y + \mathrm{C_z}(x,y,z)\hat{i}_z$$

$$\Phi = \Phi(x,y,z)$$

$$\nabla^2 \Phi = \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\nabla^2 \mathbf{C} = \nabla^2 \mathrm{C}_x \, \hat{i}_x + \nabla^2 \mathrm{C}_y \, \hat{i}_y + \nabla^2 \mathrm{C}_z \, \hat{i}_z$$

Potentials



$\boldsymbol{J} \quad \rho$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi\right)$$

$\boldsymbol{A} \quad \Phi$

… mathematical tools …

$$\mathbf{C} = \mathrm{C_x}(x,y,z)\hat{i}_x + \mathrm{C_y}(x,y,z)\hat{i}_y + \mathrm{C_z}(x,y,z)\hat{i}_z$$

$$\mathbf{A} = \mathrm{A_x}(x,y,z)\hat{i}_x + \mathrm{A_y}(x,y,z)\hat{i}_y + \mathrm{A_z}(x,y,z)\hat{i}_z$$

$$\Phi = \Phi(x,y,z)$$

I) $\nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$

II) $\nabla \times \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$

… mathematical tools …

I) $\nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists \, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$

Let us suppose that a vector $\mathbf{A}_0$ exists such that $\nabla \times \mathbf{A}_0 = \mathbf{0}$

$$\nabla \times (\mathbf{A} + \mathbf{A}_0) = \nabla \times \mathbf{A} + \nabla \times \mathbf{A}_0 = \nabla \times \mathbf{A}$$

$$\mathrm{I}) \quad \Rightarrow \quad \mathbf{C} = \nabla \times (\mathbf{A} + \mathbf{A}_0)$$

where $\nabla \times \mathbf{A}_0 = \mathbf{0}$

A is defined but for a vector $\mathbf{A}_o$ that is curl free.

… mathematical tools …

II) $\nabla \times \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$

Let us suppose that a scalar Φ_0 exists such that $\nabla \Phi_0 = \mathbf{0}$

$$\nabla\left(\Phi + \Phi_0\right) = \nabla \Phi + \nabla \Phi_0 = \nabla \Phi$$

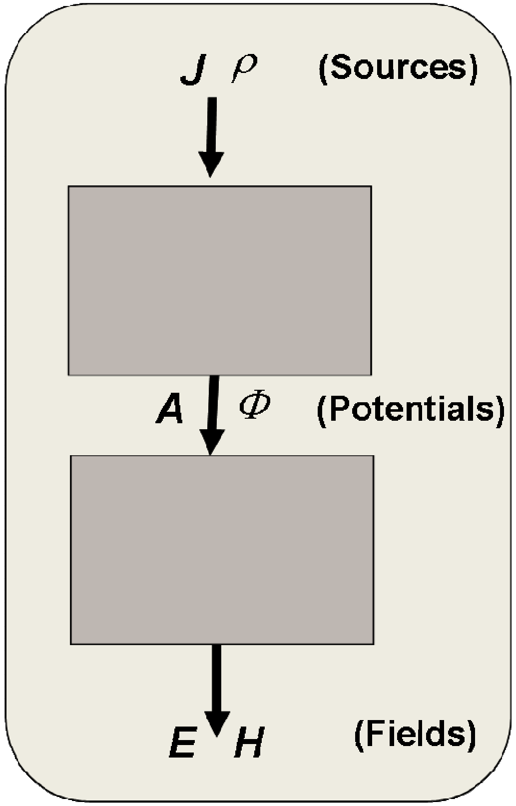
II) $\Rightarrow \quad \mathbf{C} = \nabla\left(\Phi + \Phi_0\right)$

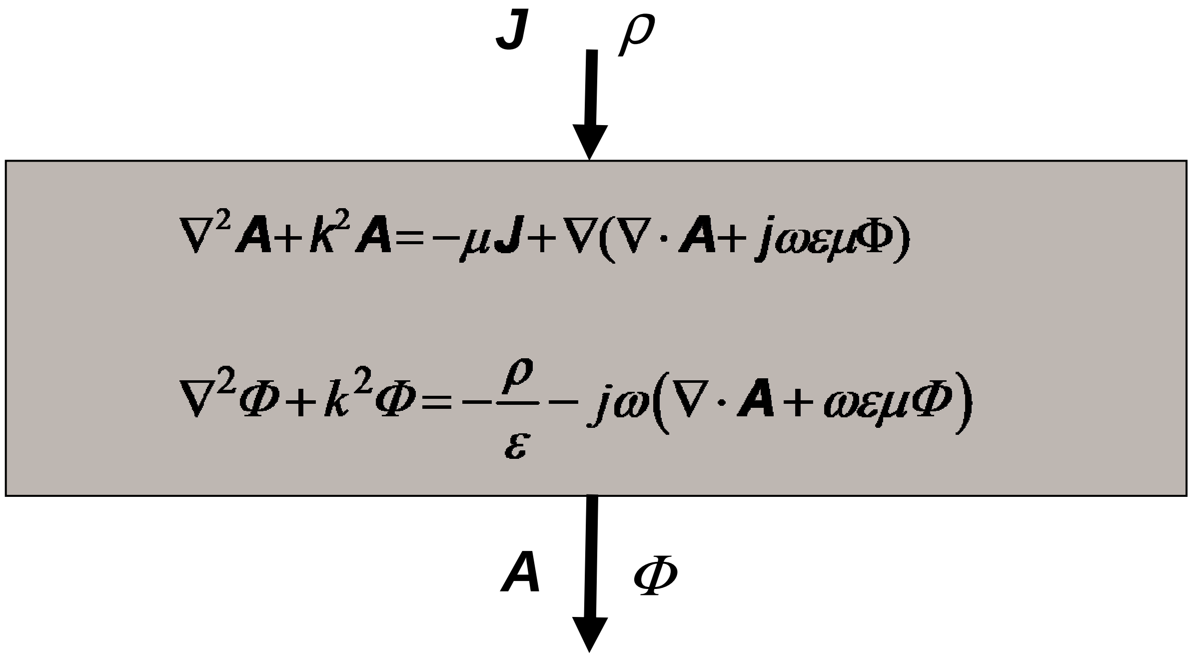
where $\nabla \Phi_0 = \mathbf{0}$

Φ is defined but for a scalar Φ_o that is gradient free.

Potentials

Amongst the infinite couples of potentials, is it possible to find a couple such that
$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0 \quad ?$$



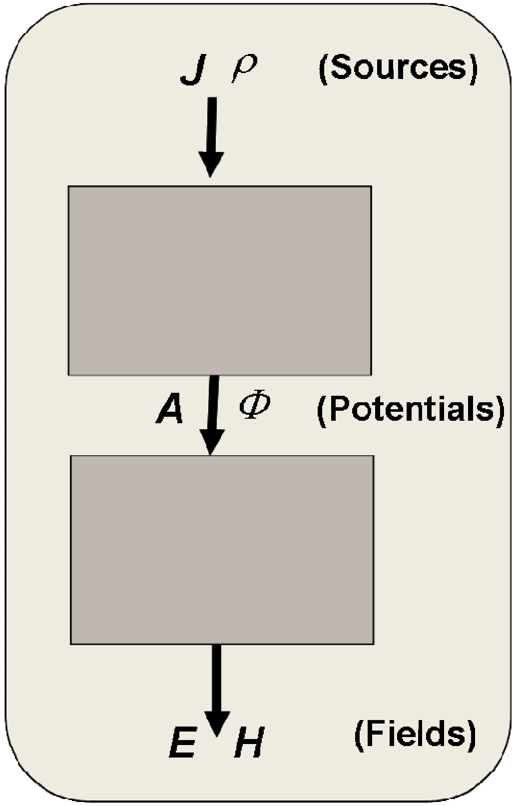


$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega(\nabla \cdot \boldsymbol{A} + \omega\varepsilon\mu\Phi)$$

Potentials

$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$$ *Lorentz gauge*



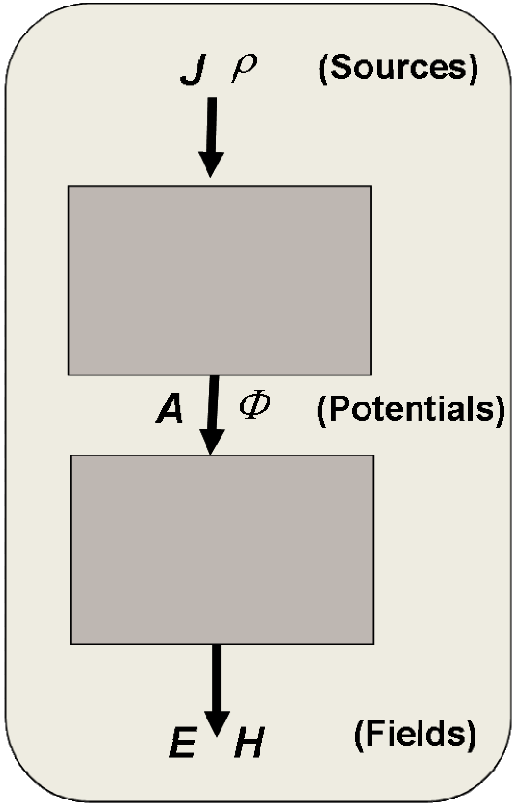
$\boldsymbol{J}$ ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \cancel{\nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)}$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - \cancel{j\omega(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)}$$

$\boldsymbol{A}$ Φ

Potentials



J ρ

$$\begin{cases} \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} \\ \nabla^2 \Phi + k^2 \Phi = -\dfrac{\rho}{\varepsilon} \end{cases}$$

Vector Helmholtz equation

Scalar Helmholtz equation

A Φ

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega \mathbf{A} - \nabla \Phi$$

E **H**

Potentials

$$\nabla\cdot\boldsymbol{A}+j\omega\varepsilon\mu\Phi=0$$

Lorentz gauge

$\boldsymbol{J}$ ρ

$$\begin{cases}\nabla^2\boldsymbol{A}+k^2\boldsymbol{A}=-\mu\boldsymbol{J} & \textbf{Vector Helmholtz equation}\\ \nabla^2\Phi+k^2\Phi=-\dfrac{\rho}{\varepsilon} & \textbf{Scalar Helmholtz equation}\end{cases}$$

$\boldsymbol{A}$ Φ

$$\mathbf{H}=\frac{1}{\mu}\nabla\times\mathbf{A} \qquad \mathbf{E}=-j\omega\mathbf{A}-\nabla\Phi$$

$$\nabla\left(\frac{\nabla\cdot\mathbf{A}}{j\omega\varepsilon\mu}\right)$$

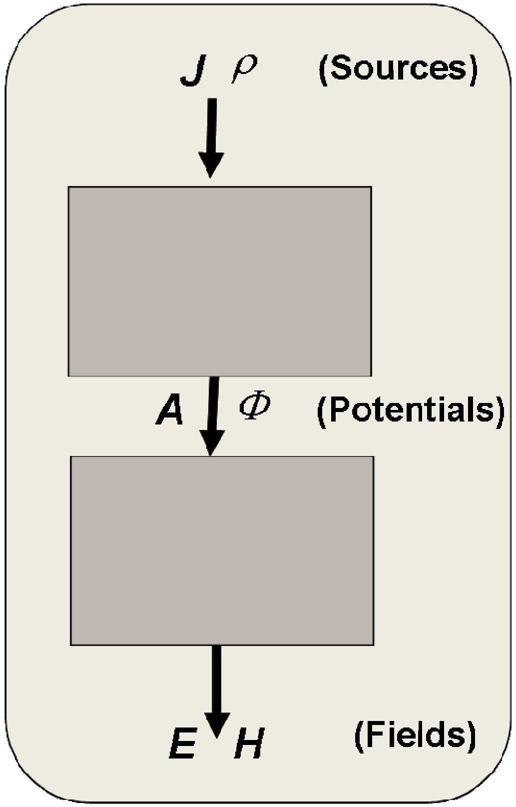
$\boldsymbol{E}$ $\boldsymbol{H}$

Note that once ***A*** is calculated by solving the (vector) Helmholtz equation involving ***A*** and ***J***, subsequent calculation of Φ can be straightforwardly achieved by means of the *Lorentz gauge*

$$\Phi=-\frac{\nabla\cdot\mathbf{A}}{j\omega\varepsilon\mu}$$

thus rendering unnecessary the solution of the (scalar) Helmholtz equation relevant to Φ

Potentials



J

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$ **Vector Helmholtz equation**

A

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -j\omega \mathbf{A} + \frac{\nabla\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}$$

E **H**