Intelligent Signal Processing

DFT and DCT

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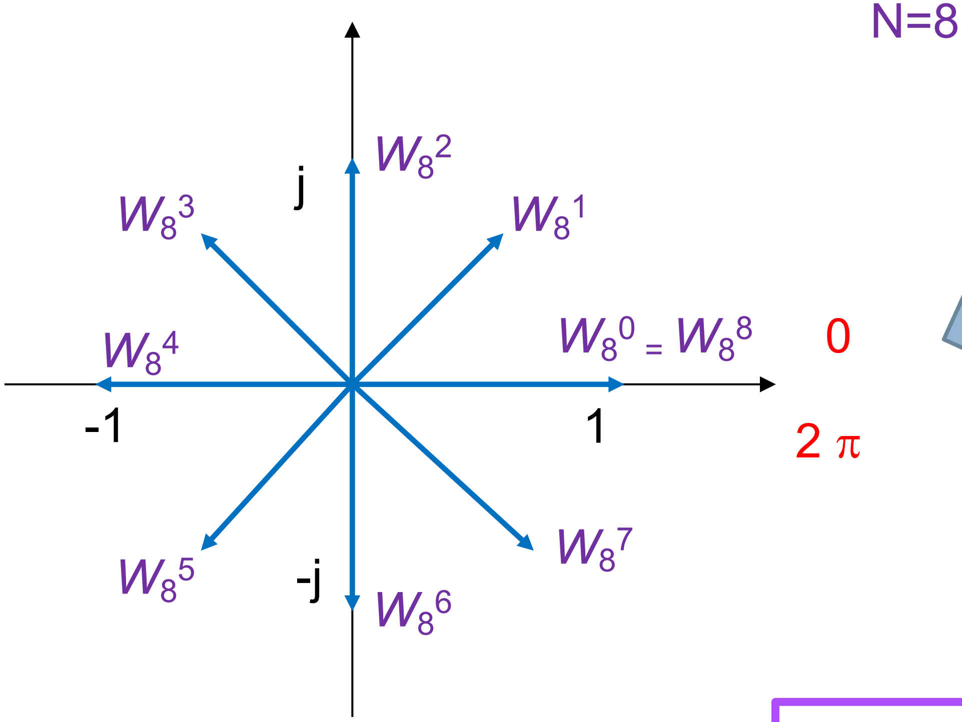
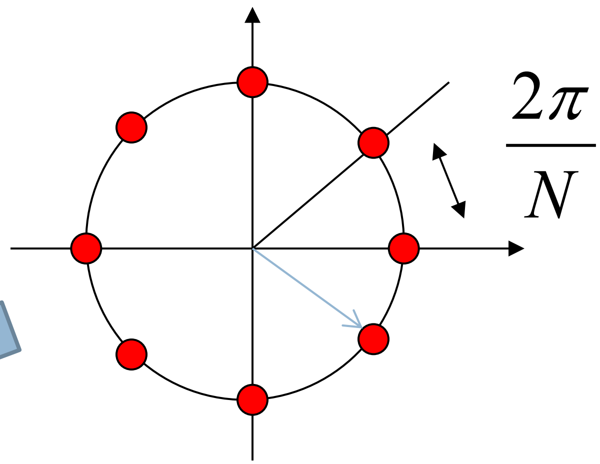
Discrete Fourier Transform

- Continuous time transforms
  - Fourier Theorem
  - Continuous Fourier Transform
  - Discrete Time Fourier Transform (DTFT)
  - z-transform

- Transformation for finite duration sequences
  - Discrete Fourier Transform (DFT)
  - Discrete Cosine Transform (DCT)

Roots of the unit circle





$$W_N = e^{-j\frac{2\pi}{N}}$$

The N-th root of the unity

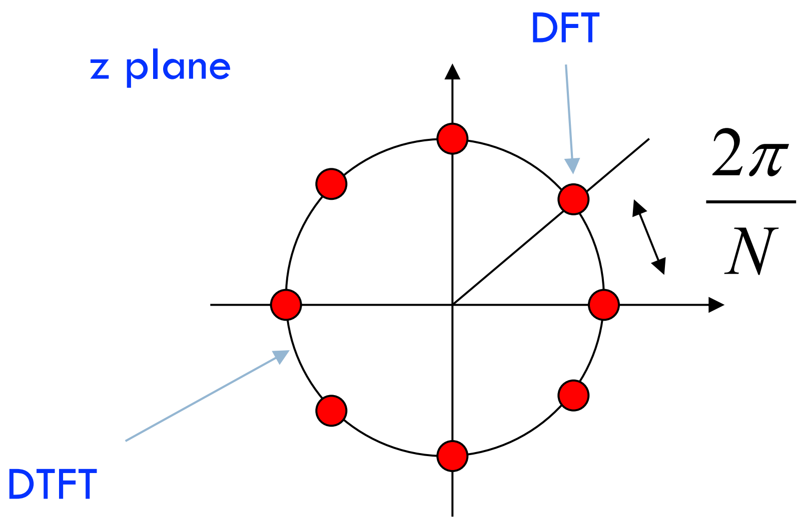
$$W_N^k = e^{-j\frac{2\pi}{N}k}$$

N roots of the unity

$$0 \le k \le N-1$$

DFT and z-Transform

- It corresponds to sample the z-transform, X (z), in N points equally spaced on the unit circle



DFT

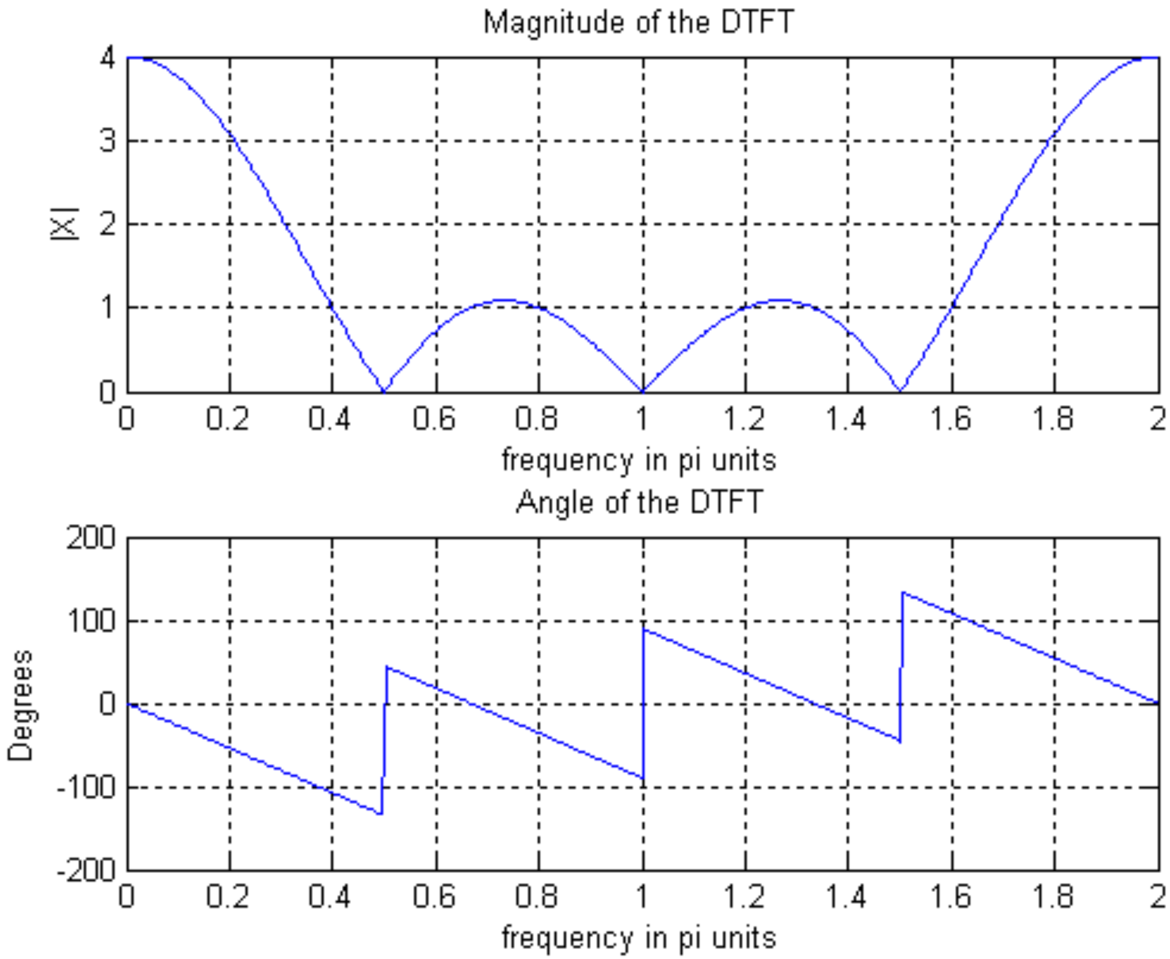
$$X(k)=\begin{cases}\sum_{n=0}^{N-1} x(n)W_N^{kn} & 0\le k\le N-1\\ 0 & \text{altrove}\end{cases}$$

Analysis

$$x(n)=\begin{cases}\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} & 0\le n\le N-1\\ 0 & \text{altrove}\end{cases}$$

Synthesis

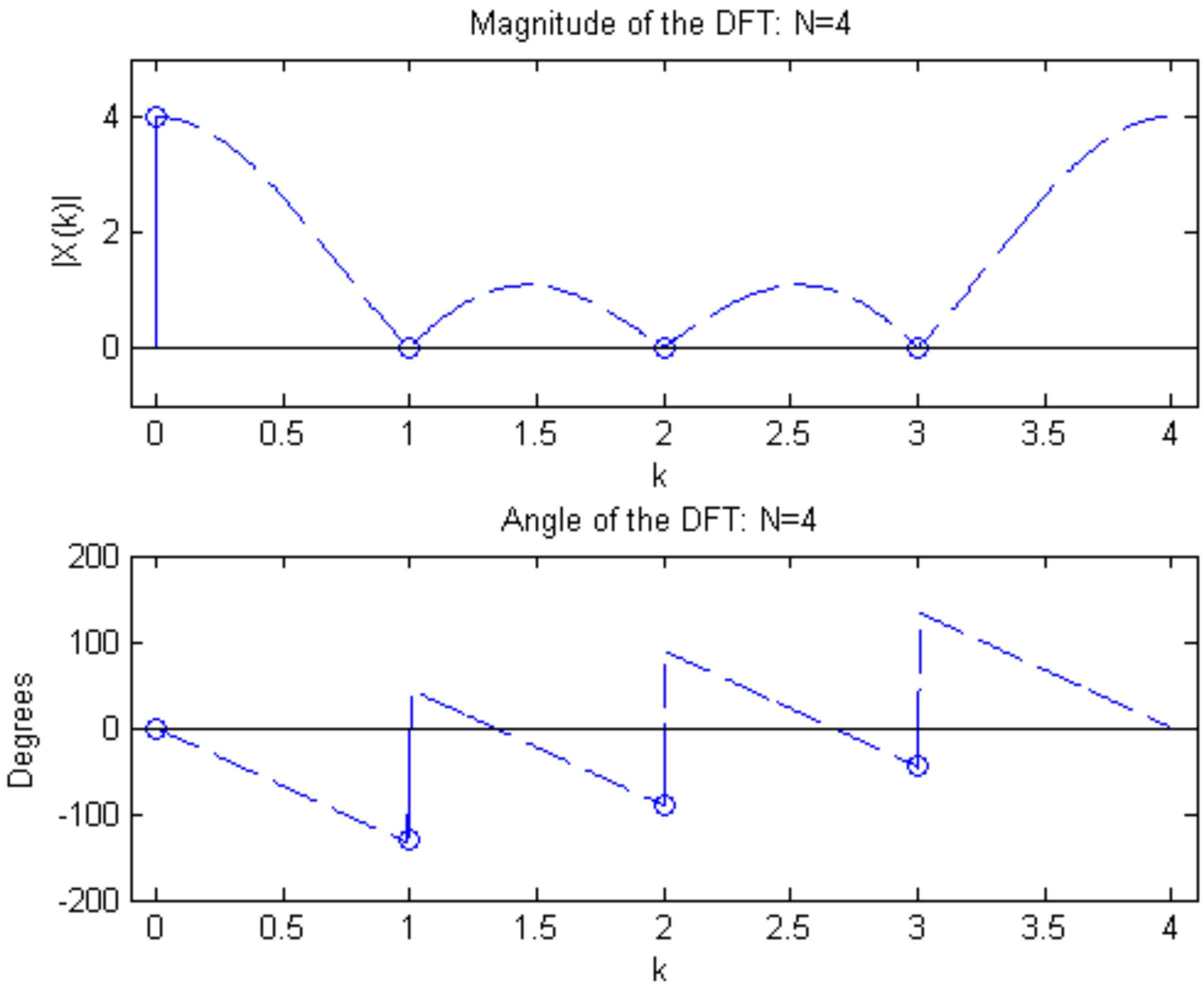
DFT



DTFT of a 4 points sequence

$$x(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & \text{altrove} \end{cases}$$

DFT



DFT of a 4 points sequence. The DFT is a sampling of the DTFT

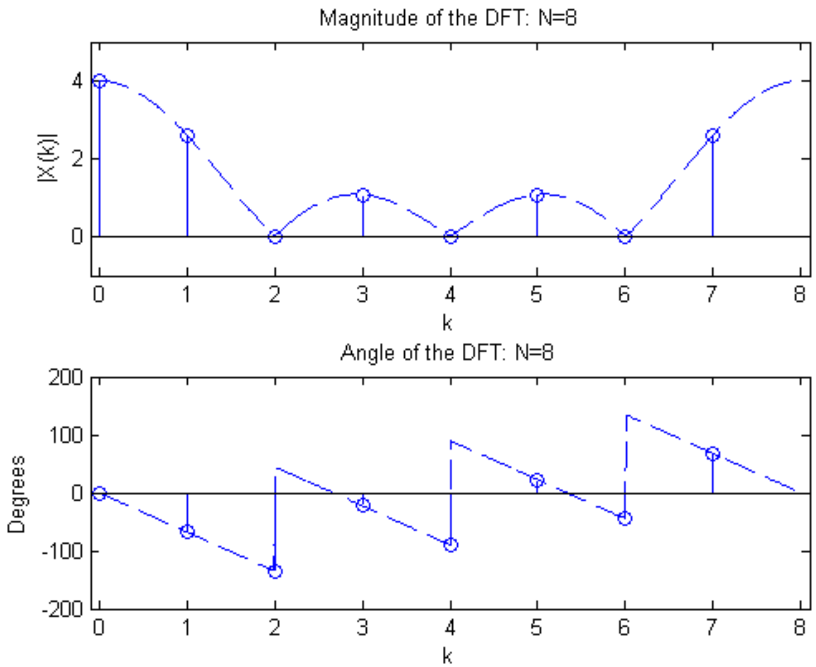
Padding

- We add some zeros to the previous sequence

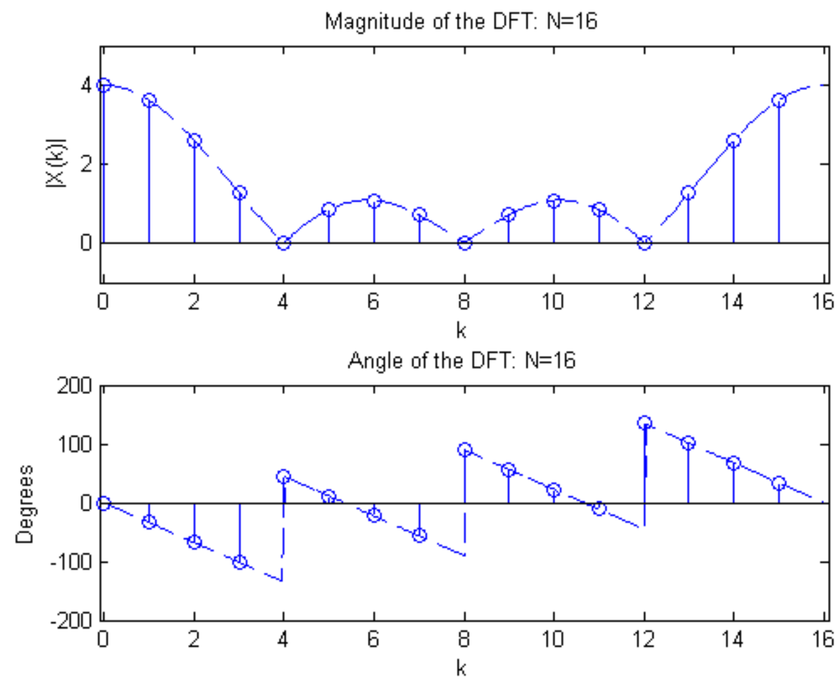
$$x(n)=[1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]$$

- This operation is named zero-padding
- It is needed to obtain a dense spectrum

DFT



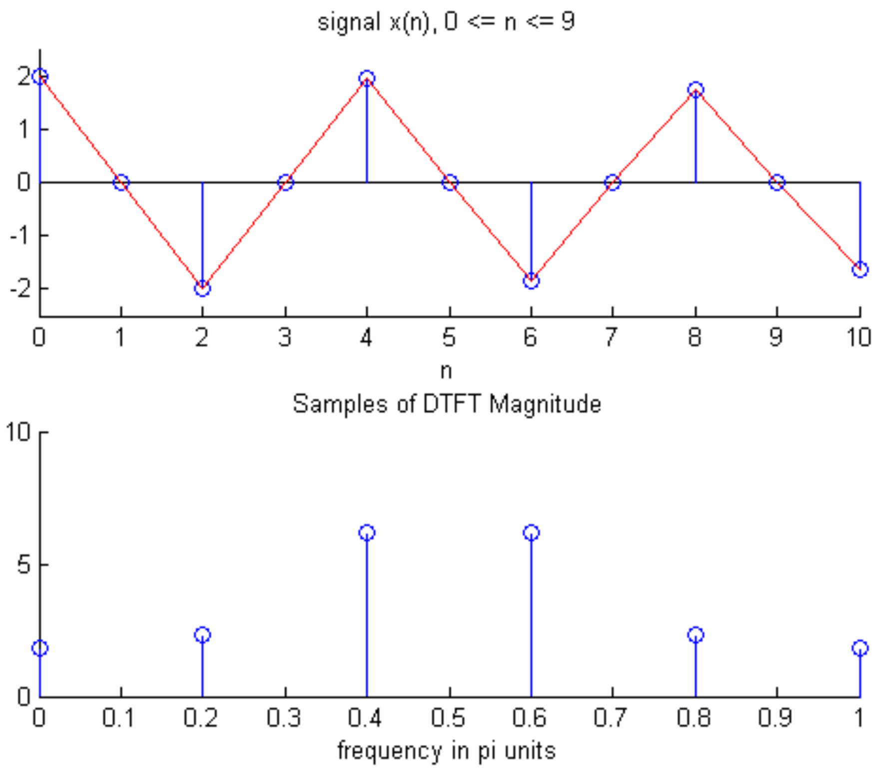
DFT after zero-padding



High density spectrum

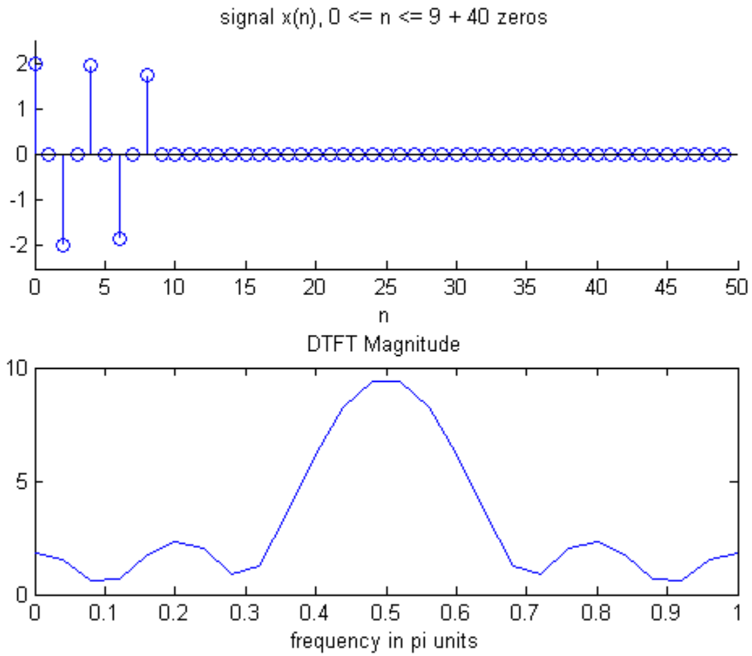
- We consider the following signal

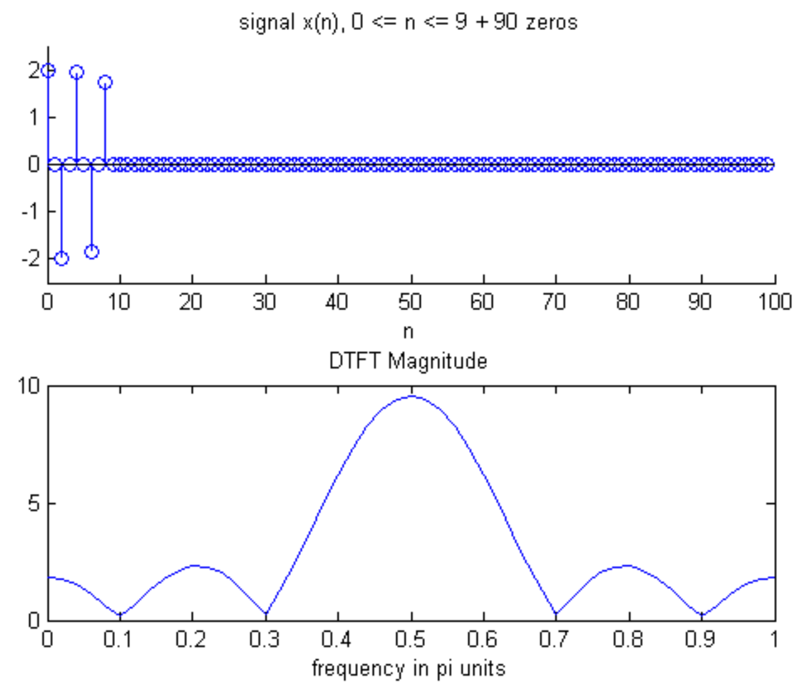
$$x(n) = \cos(0.48\pi n) + \cos(0.52\pi n)$$



DFT on 10 points

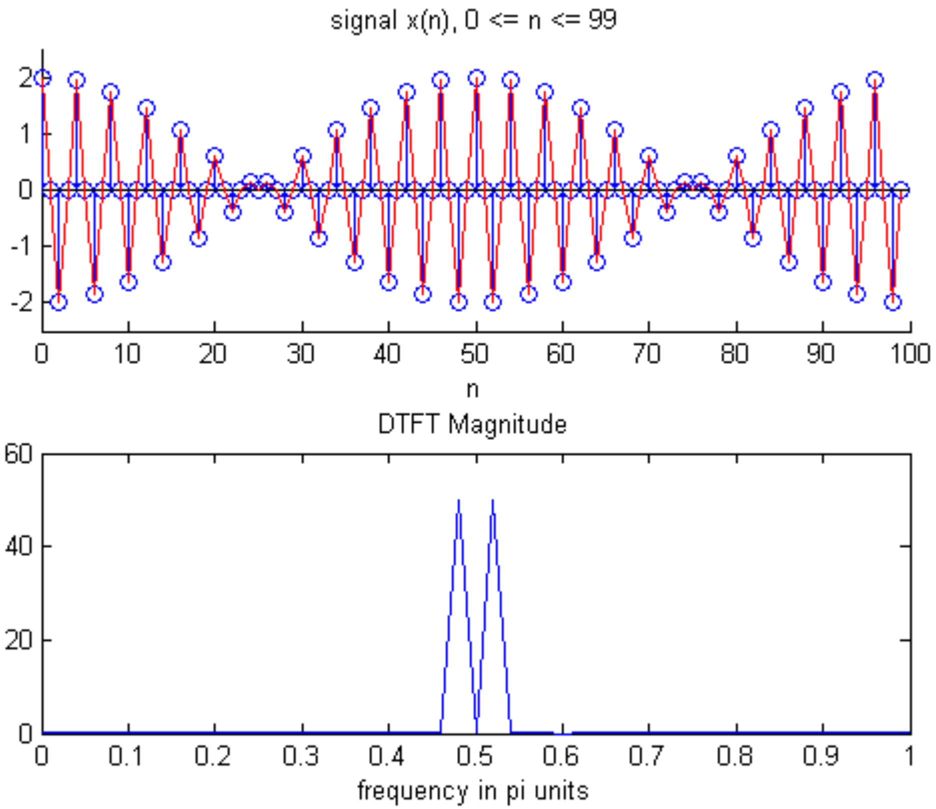
High density spectrum





Increasing the points …

High resolution spectrum



To obtain an High Resolution Spectrum we increse the sampling points of the source sequence. The estimated frequencies corrispond to the frequencies of the analyzed signal.

Discrete Cosine Transform

- The Discrete Cosine Transform (DCT) is very similar to DFT but in real domain

- DCT is used for feature extraction
  - data decorrelation
  - low loss compression
  - the basis functions are orthogonal
  - it is symmetric

1D DCT

$0 \le k \le$ N-1 — monodimensional signal

$$X(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{\pi(2n+1)k}{2N}\right)$$

Analysis

$$x(n) = \sum_{k=0}^{N-1} \alpha(k) X(k) \cos\left(\frac{\pi(2n+1)k}{2N}\right)$$

Synthesis

$$\alpha(k) = \begin{cases} \sqrt{\frac{1}{N}} & \text{se} \quad k = 0 \\ \sqrt{\frac{2}{N}} & \text{se} \quad k \neq 0 \end{cases}$$

1D DCT

$0 \le n \le$ N-1
$0 \le m \le$ N-1

bidimensional signal

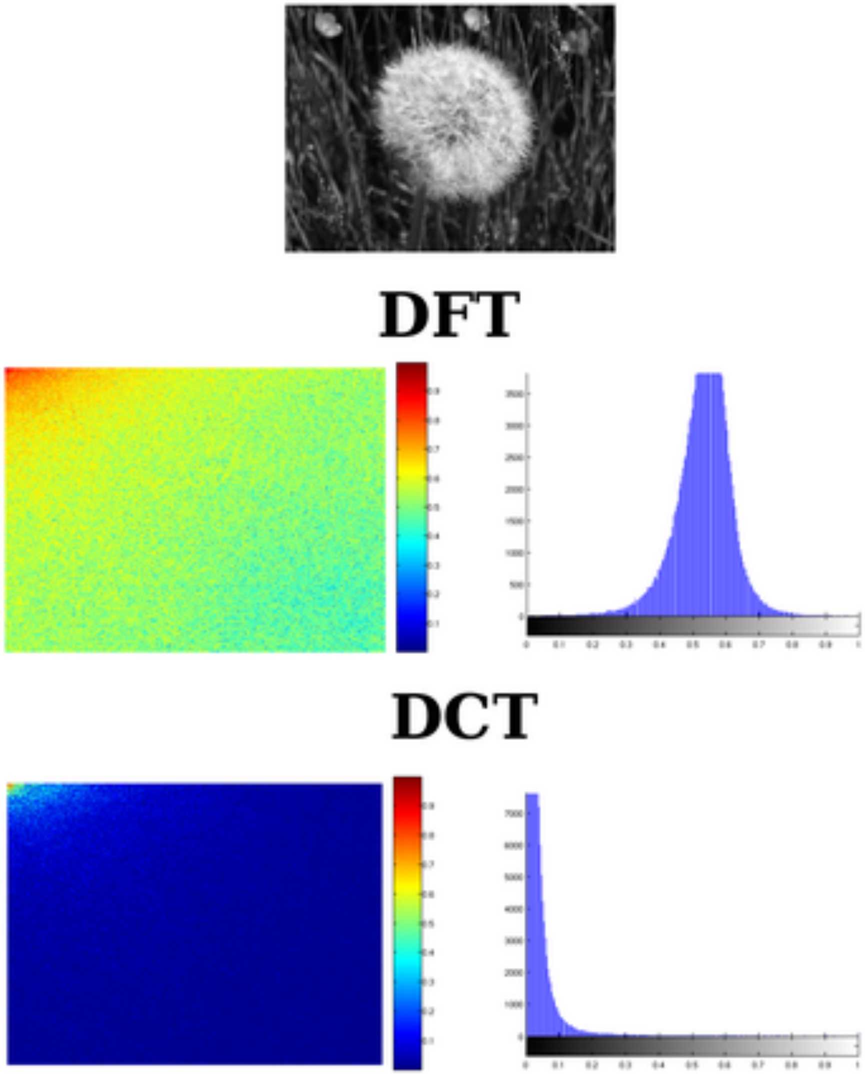
Analysis

$$X(n,m) = \alpha(n)\alpha(m)\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} x(i,j)\cos\left(\frac{\pi(2i+1)n}{2N}\right)\cos\left(\frac{\pi(2j+1)m}{2N}\right)$$

Synthesis

$$x(i,j) = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \alpha(n)\alpha(m)X(n,m)\cos\left(\frac{\pi(2i+1)n}{2N}\right)\cos\left(\frac{\pi(2j+1)m}{2N}\right)$$

DFT vs DCT



DCT provides the spatial compression, able to detect changes of information between contiguos area avoiding the repetitions