Multimedia Systems

Frequency domain

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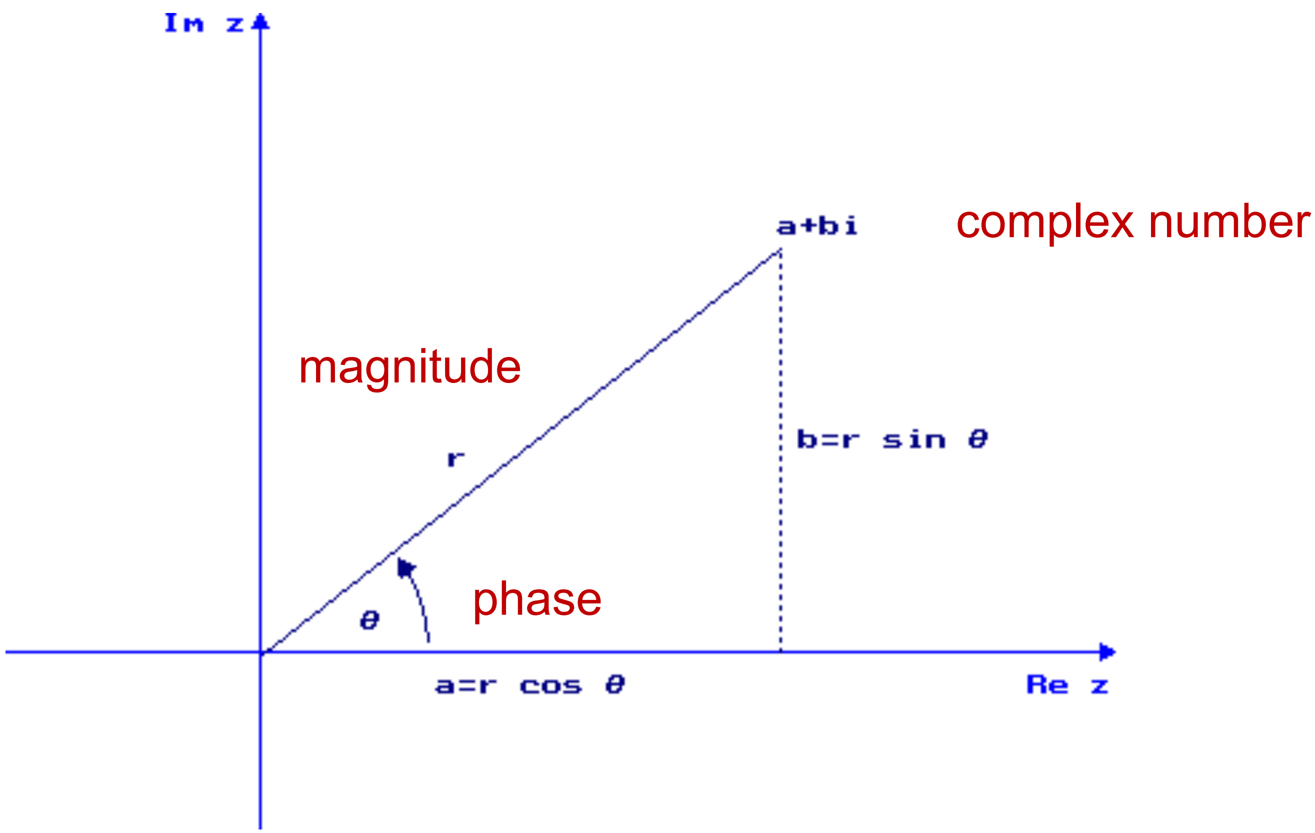
Signal processing

- Signal analysis
  - time domain
  - frequency domain
- Classical approaches
  - Discrete Time Fourier Transform (DTFT)
  - z Transform
  - Discrete Fourier Transform (DFT)
    - FFT
  - Discrete Cosine Tramsform (DCT)
  - Cepstrum

Polar coordinate system



Polar Coordinates for Complex Numbers

Exponential operator

- Euler formula

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- Magnitude

$$|z| = |a + ib| = \sqrt{a^2 + b^2} = r$$

- Sequence

$$x(n) = Ae^{j(2\pi fn + \varphi_0)}$$

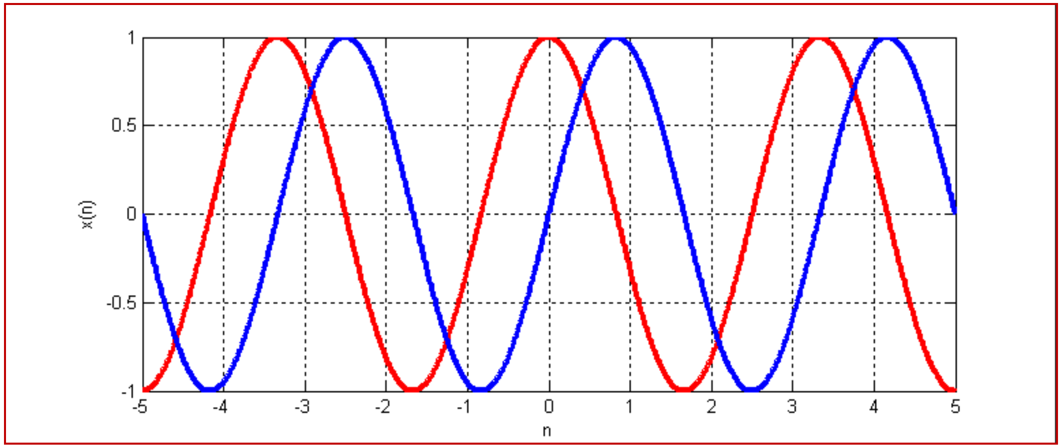
$$|x(n)| = |A| = \rho$$ magnitude

$$(2\pi fn + \varphi_0) = \vartheta$$ phase

$$x(n) = Ae^{j(2\pi fn + \varphi_0)} = A(\cos(2\pi fn + \varphi_0) + j\sin(2\pi fn + \varphi_0))$$

Sinusoidal content

$$x(n) = e^{j(2\pi 0.3n)}$$

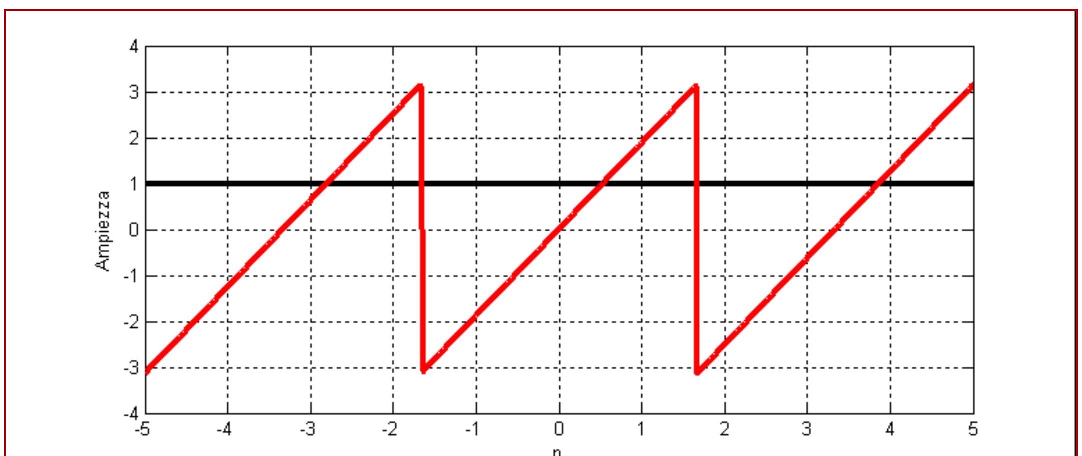


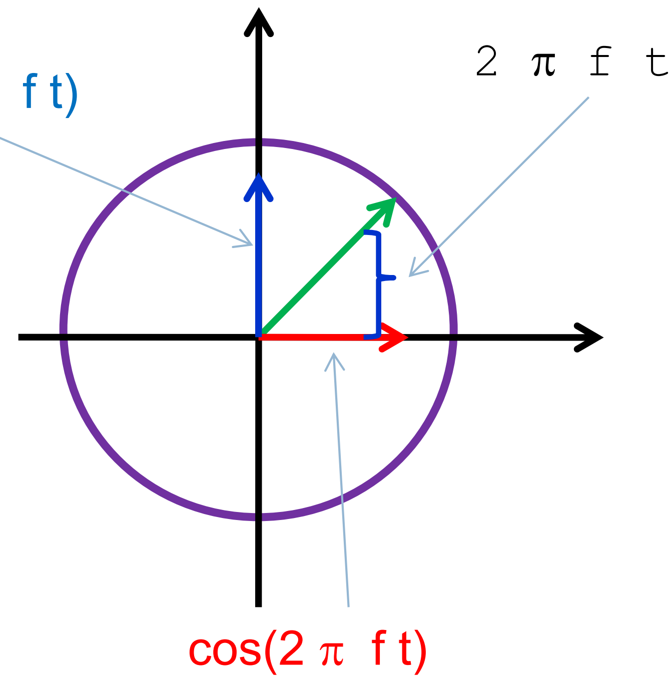
Real

Immaginary

Phase

Amplitude





Fourier theorem

- Theorem

any continuous periodic signal can be obtained by the superposition of simple sine waves, each with its amplitude and phase, and whose frequencies are harmonics of the fundamental frequency of the signal

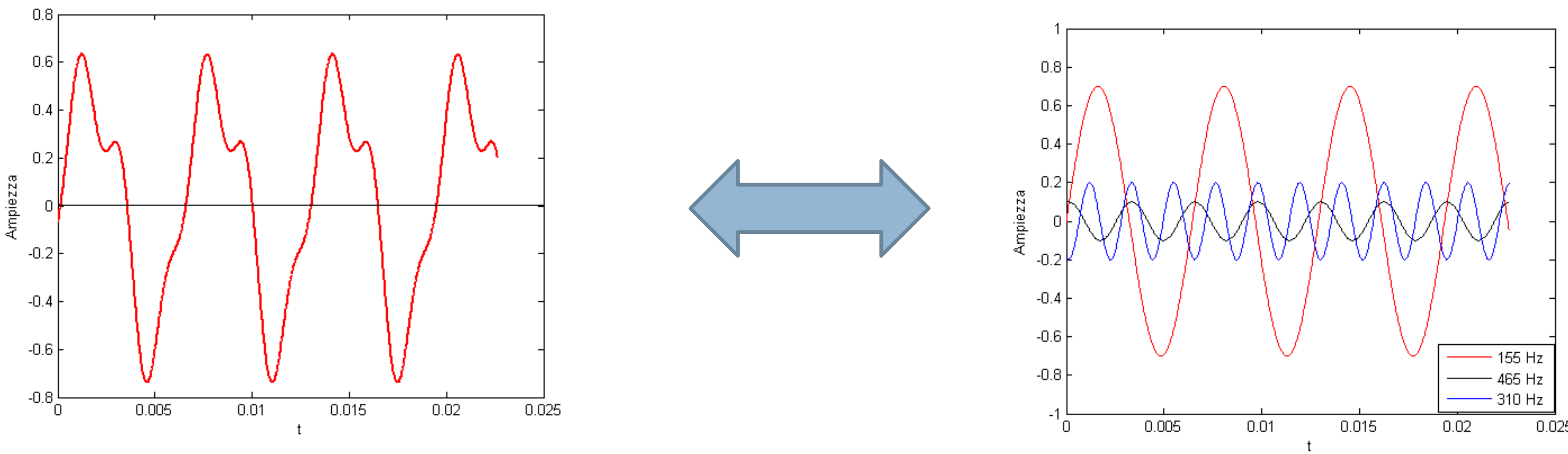
Continuous periodic signal with period T_0

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \theta_k) = A_0 + \sum_{k=1}^{\infty} A_k \cos(\omega_0 k t + \theta_k)$$

$$f_0 = 1/T_0 \qquad \omega_0 = 2\pi f_0$$

Fourier series

Fourier theorem



Fuorier example

Fourier transform

- Real environments
  - non-periodic signals

Inverse Fourier Transform
(Analysis)

$$x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df.$$

$$X(f)df = A_k$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt.$$

Direct Fourier Transform
(Syntesis)

Discrete Time Fourier Transform

- For a discrete sequence x(*n*)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

DTFT

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Trasformata di Fourier
a tempo discreto inversa

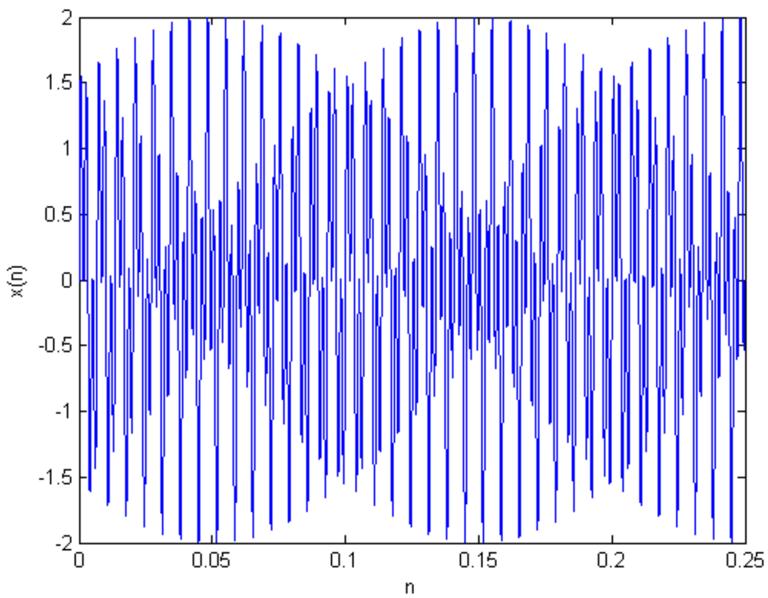
DTFT in C++


```
template <class Type, class Type_1>
class DTFT : public  sequenze<Type>{
public:

        DTFT(double wa, double wb, int n):sequenze<Type>(n){

                t = vector<double>(size);
                x = vector<Type>(size);

        double dw = (wb-wa)/size;

        for (int k=0; k<size; k++)
                {
                    t[k] = (double)wa + k*dw;
                        x[k].put(0.0,0.0);
            }
        }

void dtft( vector<Type> x, int L){ . . .}
. . .

private:
string Name;

}
```

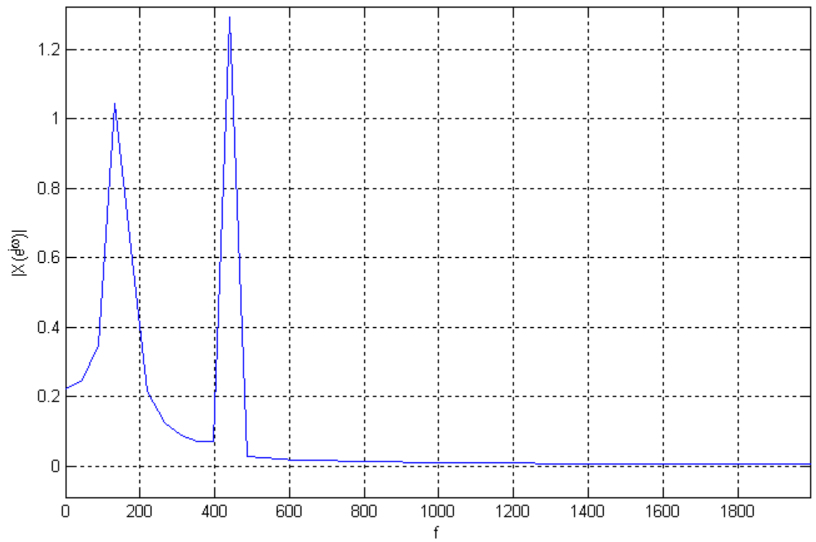
DTFT in C++

```
void dtft( vector<Type_1> x_i, int L){


        complex Z(0.0,0.0);

        for (int k=0; k<size; k++)
              {

              Z.c_exp(1.0,-t[k]);
      for (int i=0; i<L; i++)
          x[k].prod(Z,x[k]).add(complex(x_i[i],0.0));
        }
}
```

DTFT in C++



Sequence with two pure tones with 150 and 440 Hz, respectively. Sampling frquence of 44100 Hz.



Absolute value of the DTFT

$0 \leq \omega \leq \pi \rightarrow 0 \leq f \leq 22050$

Convolution theorem

$$x(n) \overset{\mathcal{F}}{\Longleftrightarrow} X\left(e^{j\omega}\right)$$

$$h(n) \overset{\mathcal{F}}{\Longleftrightarrow} H\left(e^{j\omega}\right)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

$$\Downarrow$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Highlighting
  - A convolution in the time domain corresponds to a product in the frequency domain

Modulation theorem (windowing)

$$x(n) \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

$$w(n) \stackrel{\mathcal{F}}{\longleftrightarrow} W\left(e^{j\omega}\right)$$

$$y(n) = x(n)w(n)$$

$$\Downarrow$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

- Highlighting
  - The DTFT of the product of sequence corresesopnds to a periodic convolution of the single DTFTs
  - e.g., FIR con windowing

Teorema di Parseval

$$x(n) \overset{\mathcal{F}}{\Longleftrightarrow} X\left(e^{j\omega}\right)$$

- Energy

$$E = \sum_{n=-\infty}^{\infty} \left| x(n) \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) \right|^2 d\omega$$

- $|X(e^{j\omega})|^2$ Energy Spectral Density