Course of
"Automatic Control Systems"
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Real Bode diagrams – Fourier analysis

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

Università degli Studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code**: tz3jpwb**

Real Bode diagrams

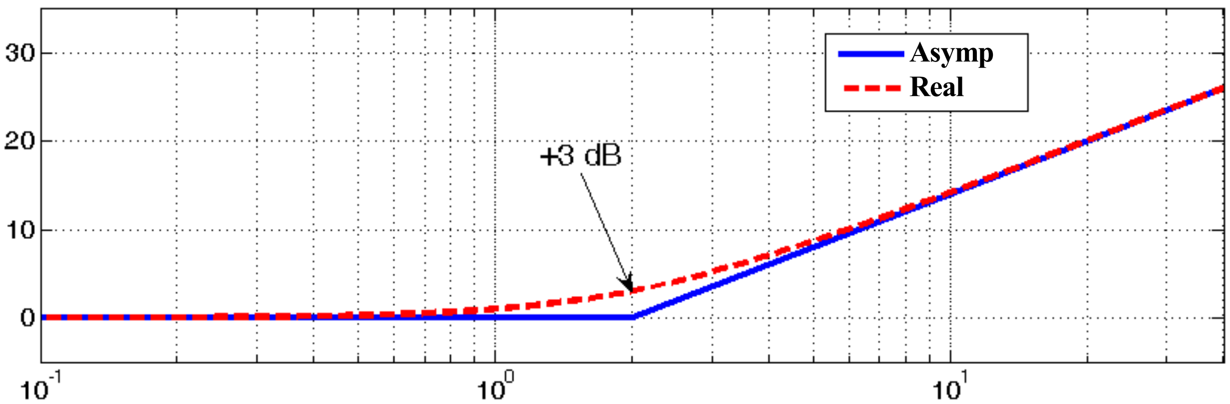
- ***In the real Bode diagrams the magnitude and phase of a transfer function $W(s)$ with $s = j\omega$ are drawn accurately*** also the in two decades around the break points of the binomial and trinomial terms.

- The real Bode diagrams are usually traced applying some ***corrections to the asymptotic Bode diagrams***

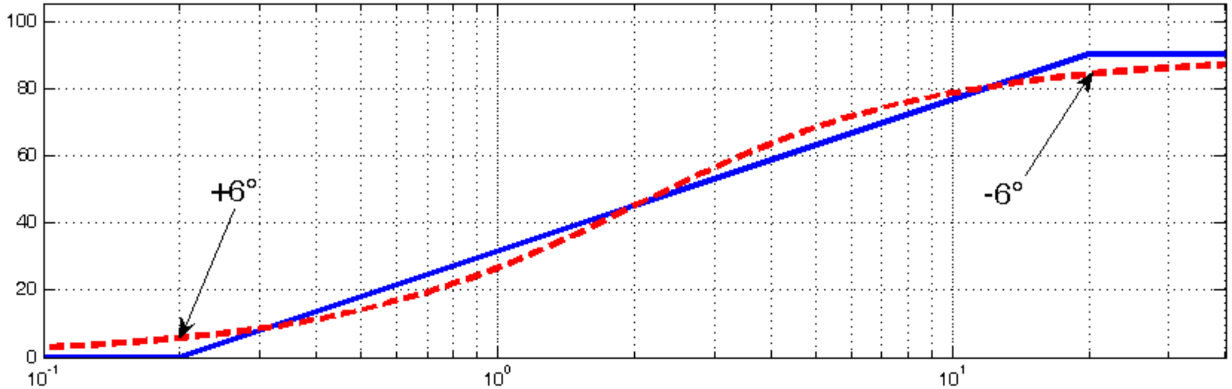
- The real Bode diagrams ***can be drawn in MATLAB using the command 'bode'***

Zero of multiplicity one $W(s) = (1 + s\tau)$

Magnitude [dB]



Phase [deg]



This result can be easily generalized to a generic binomial term

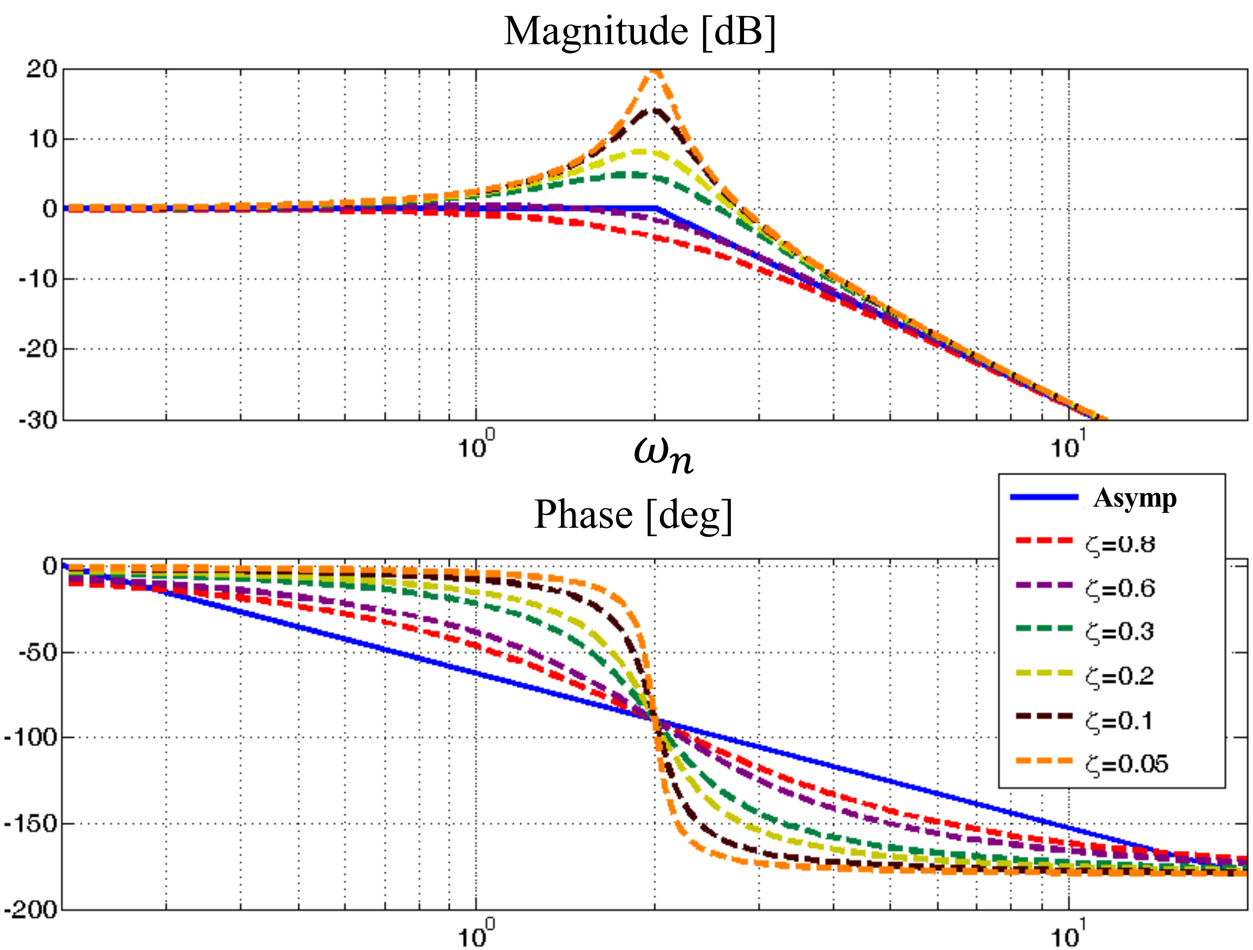
Complex conjugate poles of multiplicity one $W(s) = \left(1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}\right)^{-1}$

Peak module

$$M_p = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$$

Peak frequency

$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$

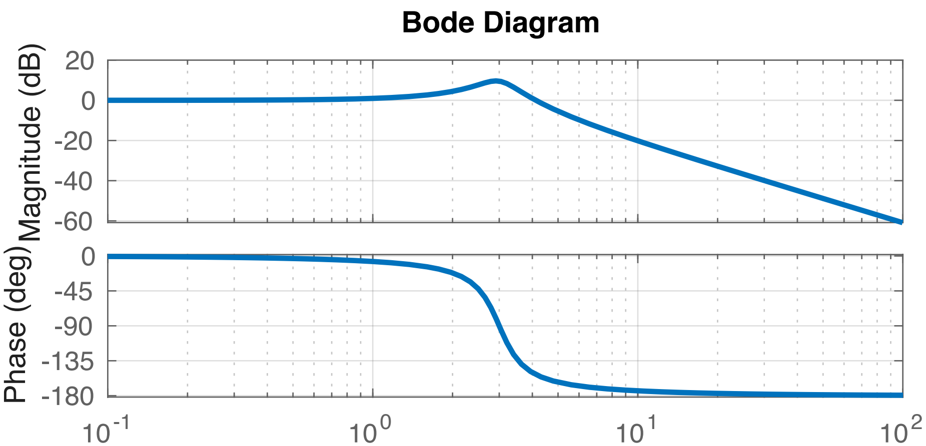


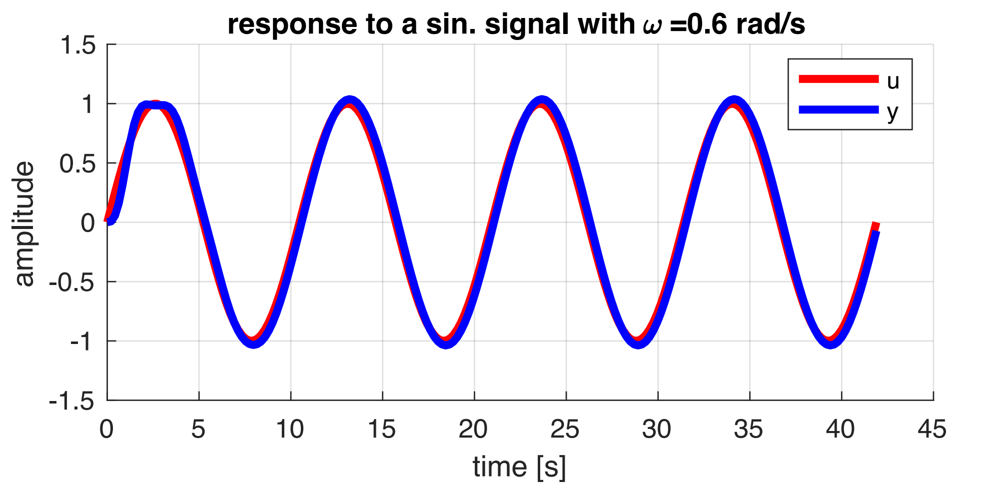
This result can be easily generalized to a generic trinomial term

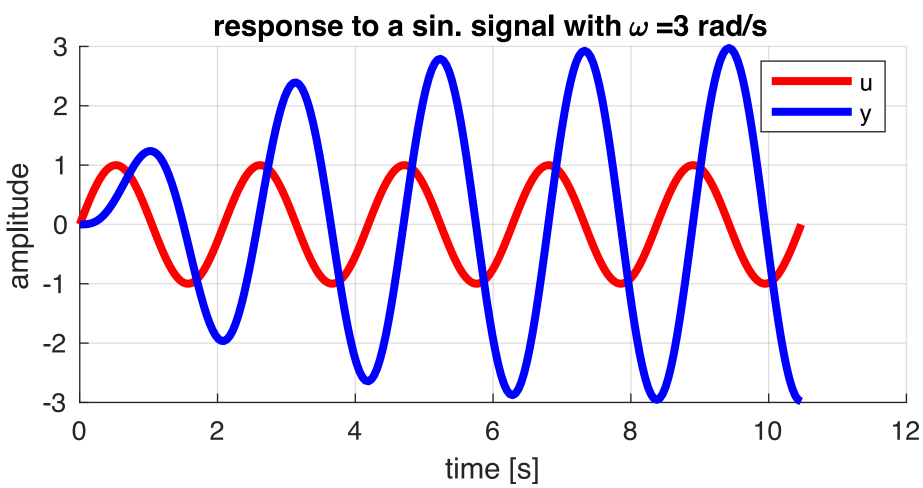
Response to sinusoidal inputs of a second order LTI system

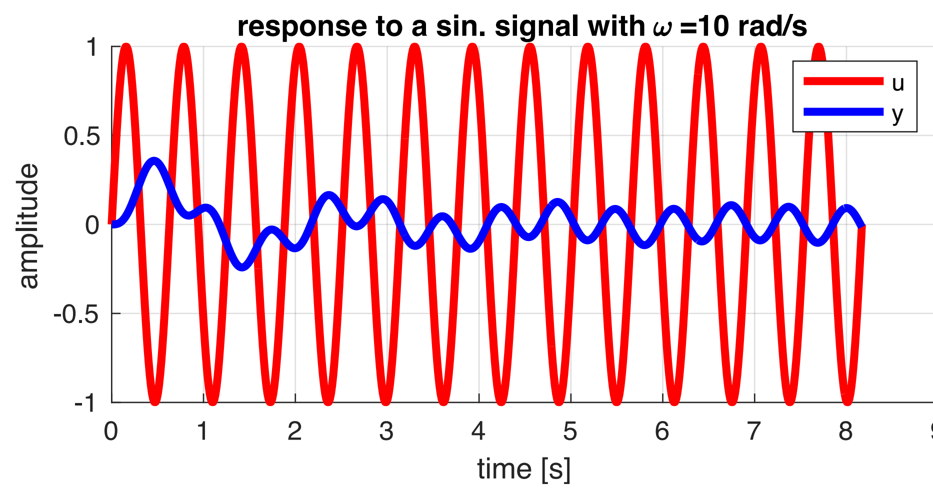
Let assume a second order LTI system described by the following t.f.:

$$G(s) = \frac{9}{s^2+s+9}$$









Bode magnitude table

- Monomial terms of multiplicity 1. The slope is constant in $\omega \in [0 \ \ \infty[$

Zero in the origin	**+20 dB/decade**
Pole in the origin	**-20 dB/decade**

- Binomial and trinomial terms of multiplicity 1. The slope changes on the break point

	Indipendent from the sign of the real part
Real Zero	**+20 dB/decade**
Real Pole	**-20 dB/decade**
Comp. Conjug. zeros	**+40 dB/decade**
Comp. Conjug. poles	**-40 dB/decade**

- When the term has a multiplicity greater than one, the slopes should be multiplied by the multiplicity.

Bode phase table

- Constant and monomial terms of multiplicity 1. The slope is constant in $\omega \in [0 \;\; \infty[$

K < 0	**-180° per $\omega \in [0,\infty)$**
Zero in the origin	**+90° per $\omega \in [0,\infty)$**
Pole in the origin	**-90° per $\omega \in [0,\infty)$**

- Binomial and trinomial terms of multiplicity 1. The slope changes one decade before and after the breaking point.

	Negative real part	**Positive real part**
Real Zero	**+90° +45→ -45 °/decade**	**-90° -45→ +45 °/decade**
Real Pole	**-90° -45→ +45 °/decade**	**+90° +45→ -45 °/decade**
Comp. Conjug. zeros	**+180° +90→ -90 °/decade**	**-180° -90→ +90 °/decade**
Comp. Conjug. poles	**-180° -90→ +90 °/decade**	**+180° +90→ -90 °/decade**

- When the term has a multiplicity greater than one, the phase variation should be multiplied by the multiplicity.

Examples

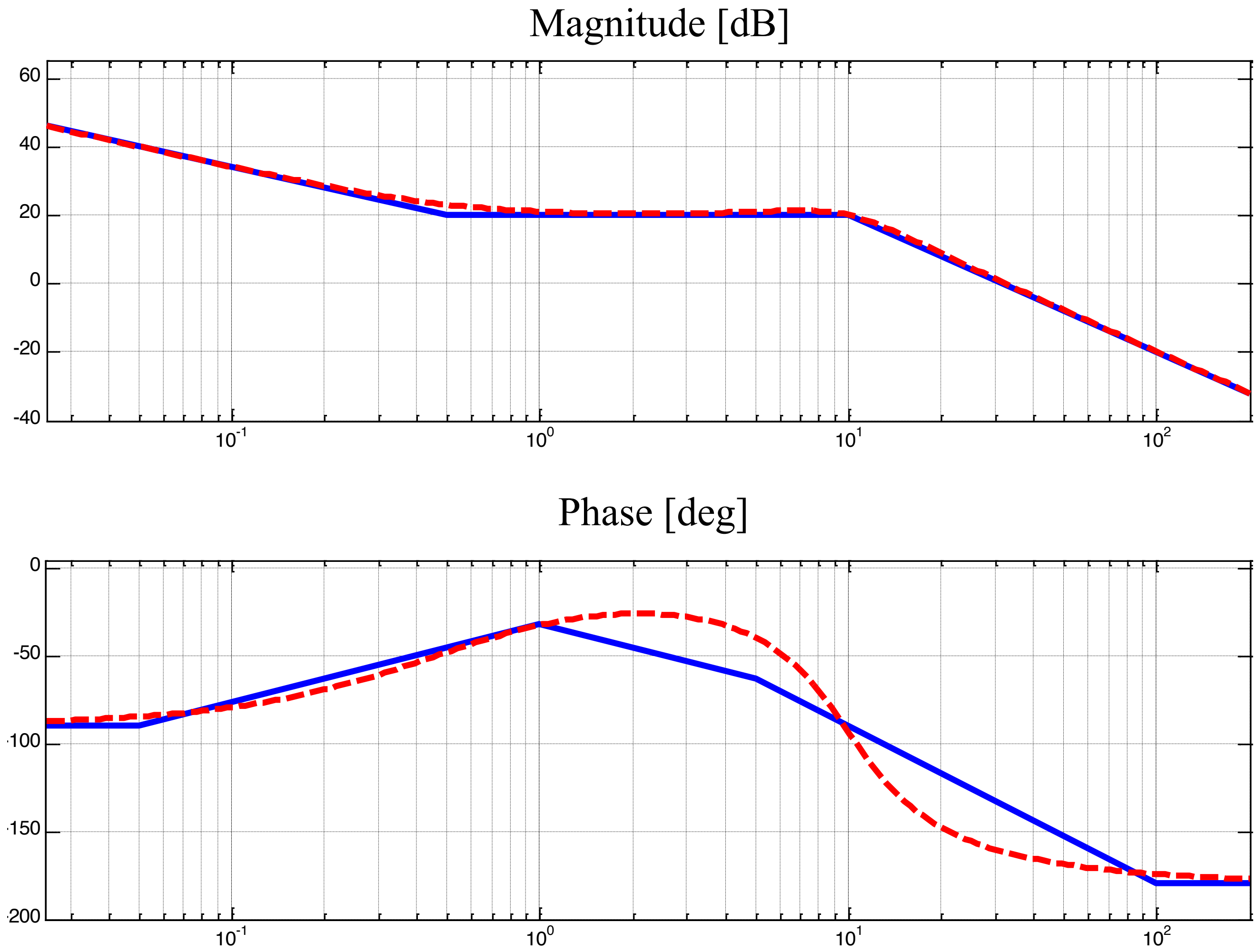
- Trace the real Bode diagrams of the functions

$$W(s) = \frac{1000(s+0.5)}{s(s^2+10s+100)}$$

$$W(s) = \frac{s(s-2)}{(s^2+5s+25)}$$

Example 1

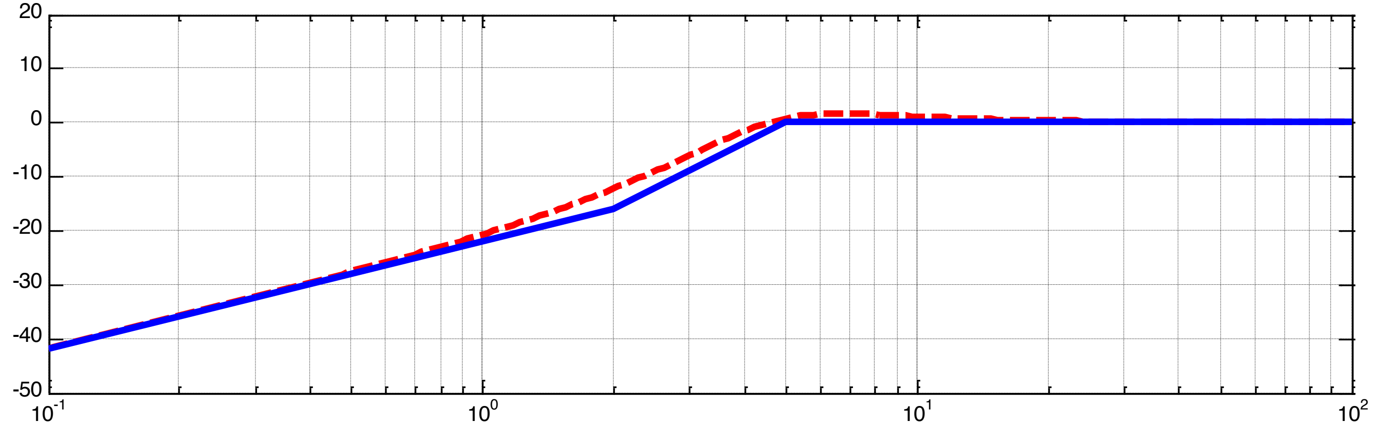
$$W(s) = \frac{1000(s+0.5)}{s(s^2+10s+100)}$$



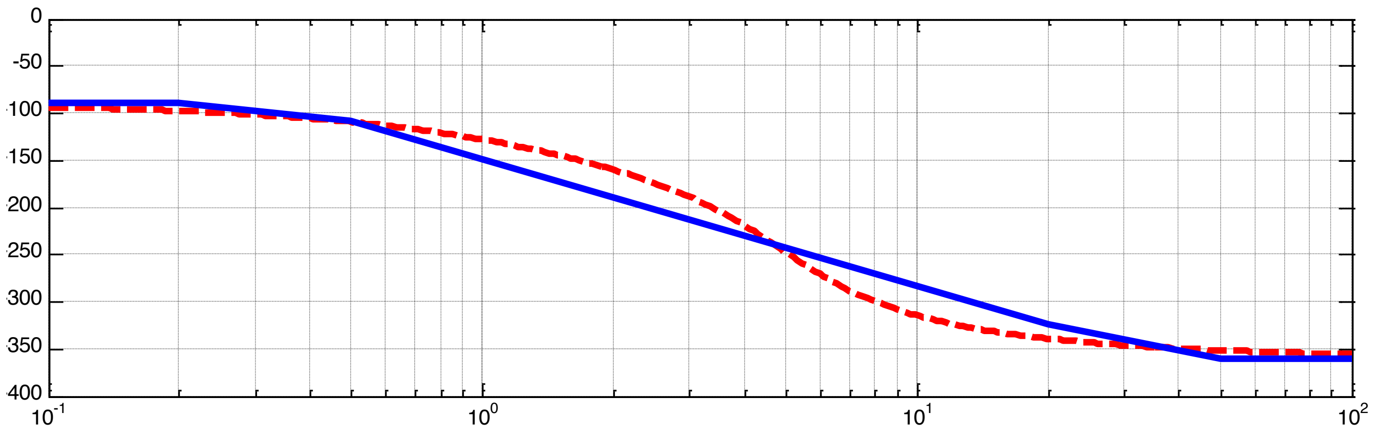
Exmple 2

$$W(s)=\frac{s(s-2)}{(s^2+5s+25)}$$

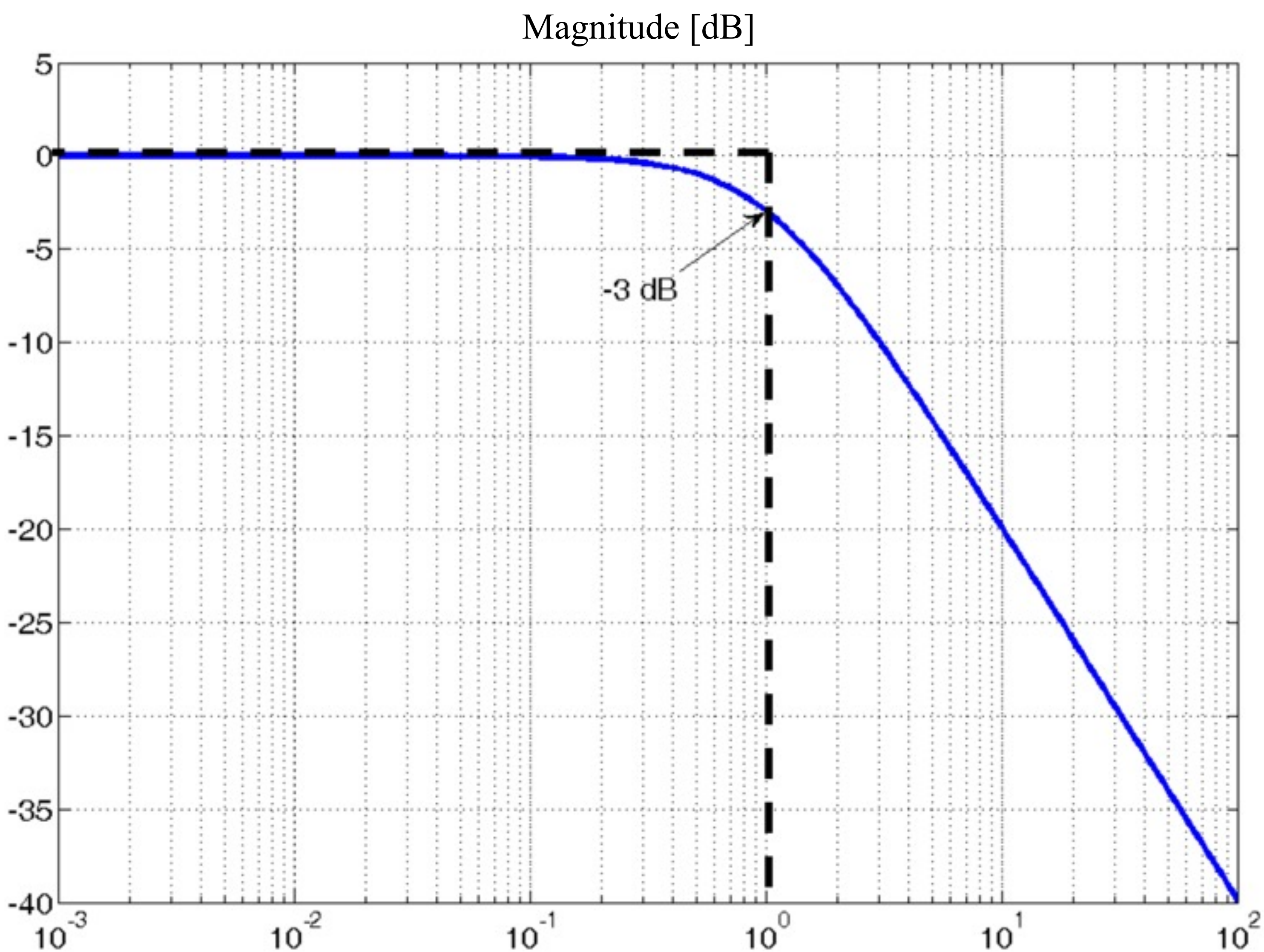
Magnitude [dB]



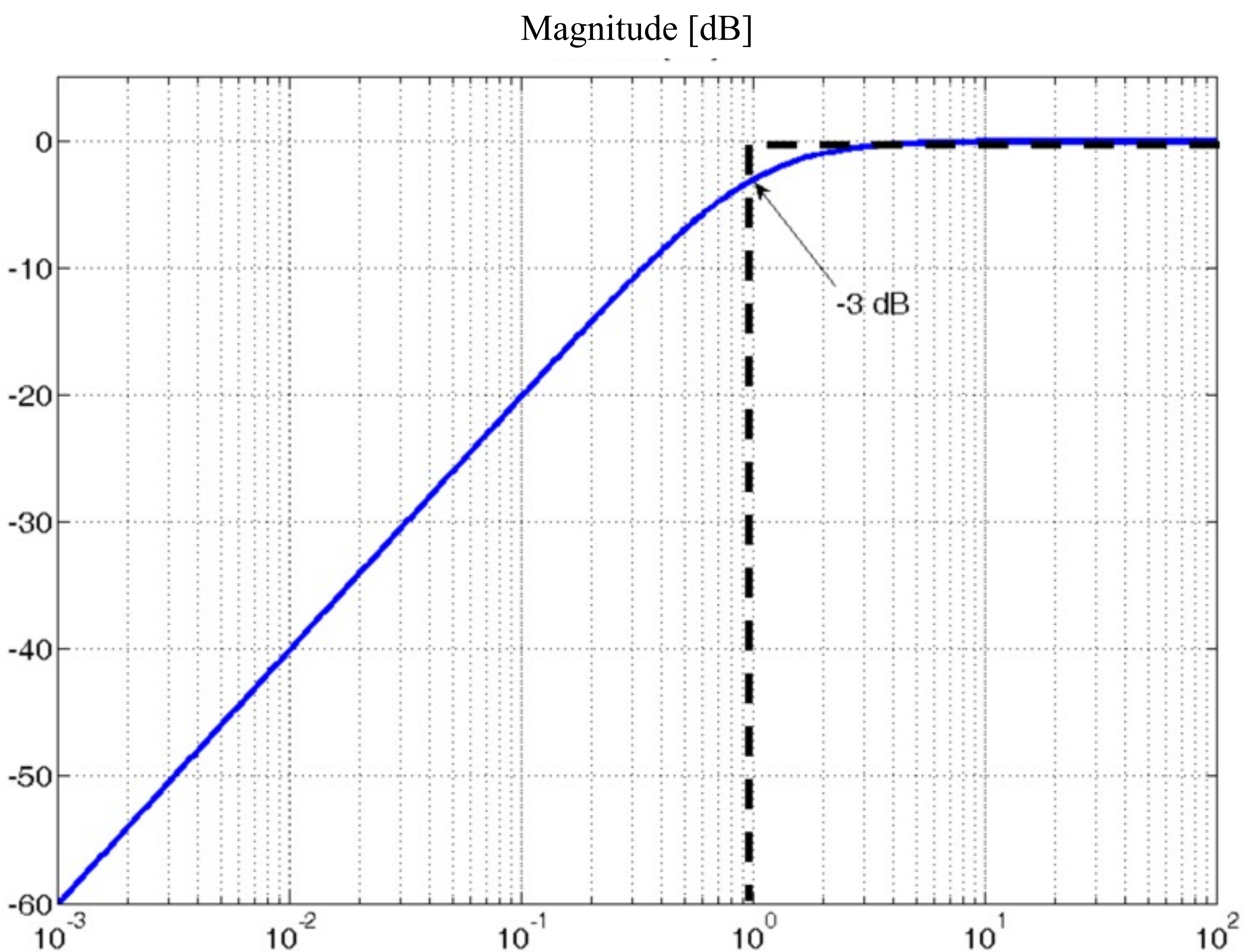
Phase [deg]



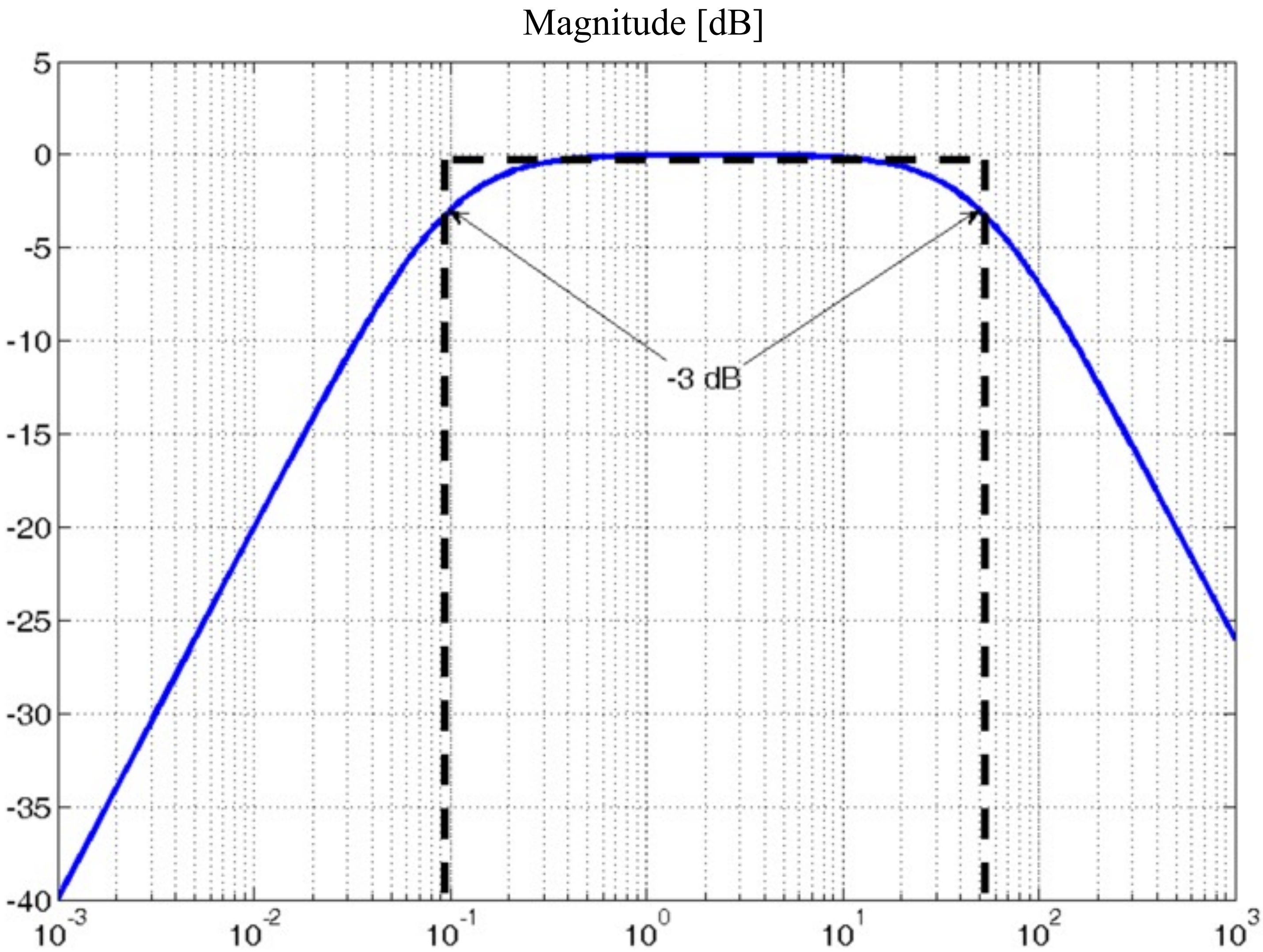
Low-pass filter



High-pass filter



Band-pass filter



- **Any periodic function $f(t)$ with period T,**

$$f(t) = f(t+T),$$

can be written as

$$f(t) = F_0 + \sum_{n=1}^{\infty}\left[F_{cn}\cos(n\omega_0 t) + F_{sn}\sin(n\omega_0 t)\right]$$

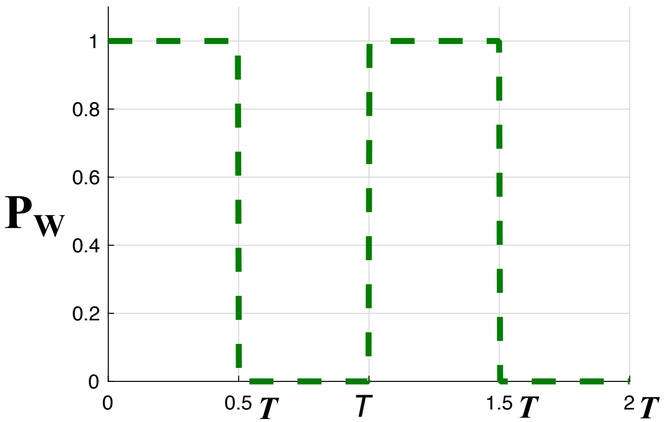
where $\omega_0 = \frac{2\pi}{T}$ **,**

$$F_0 = \frac{1}{T}\int_T f(t)dt \qquad F_{cn} = \frac{2}{T}\int_T f(t)\cos(n\omega_0 t)dt \qquad F_{sn} = \frac{2}{T}\int_T f(t)\sin(n\omega_0 t)dt \cdot$$

F_0 **is the average value of** f **over a single period.**

The component with ω_0 **is the fundemental harmonic or 1st harmonic, that with** $n\omega_0$ **is n-th harmonic.**

Example: square wave



$$P_{\mathrm{w}}(t) = \begin{cases} 1 & \text{if} \quad 0 < t \leq T/2 \\ 0 & \text{if} \quad T/2 < t \leq T \end{cases}$$

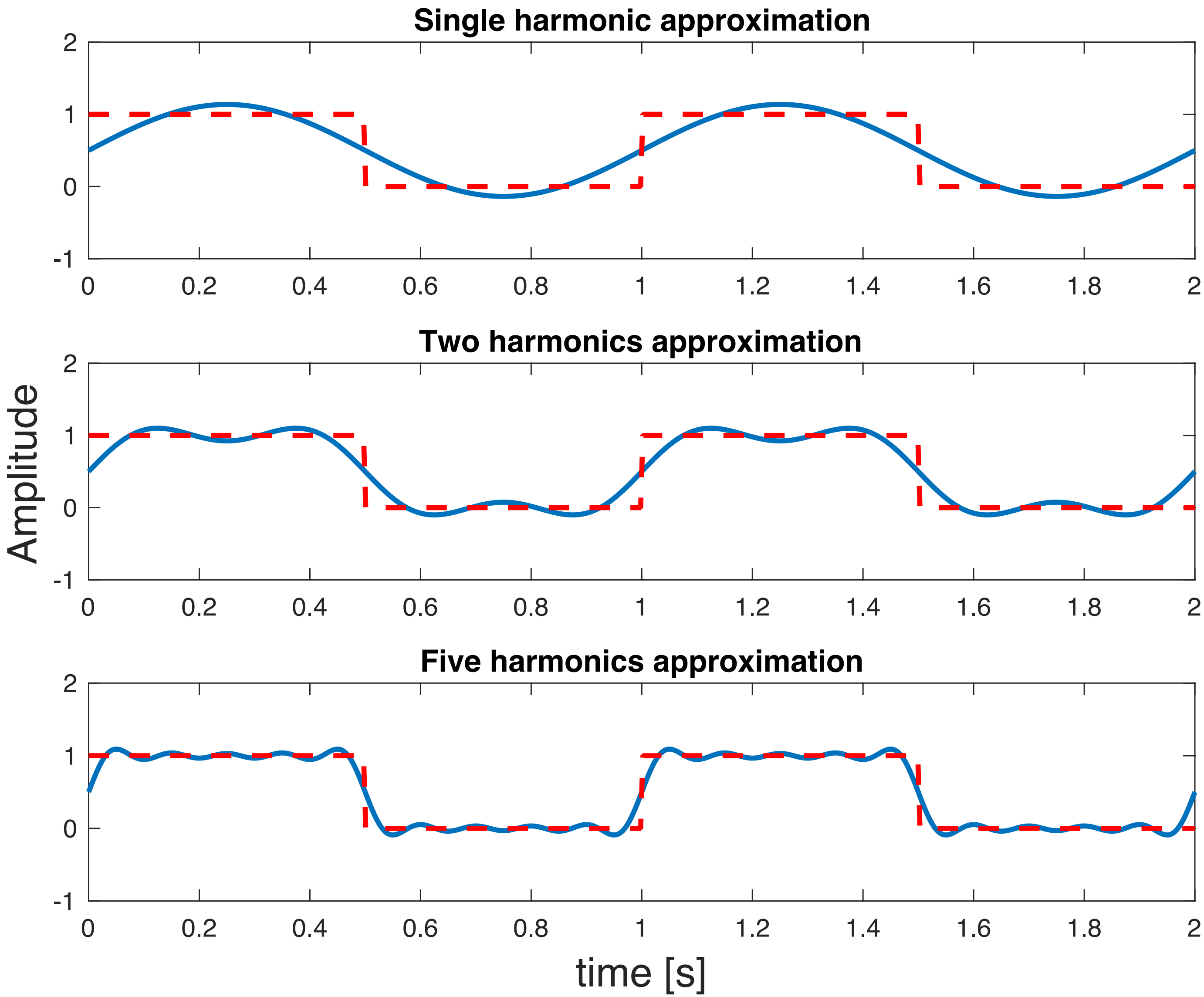
Using Fourier analysis:

$$F_0 = \frac{1}{2}\,, \quad F_{\mathrm{cn}} = 0 \quad \forall n \in \mathbb{N}\,, \quad F_{\mathrm{sn}} = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

Therefore, the square wave can be written

$$P_{\mathrm{w}}(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \frac{2}{5\pi}\sin(5\omega_0 t) + \cdots$$

Example: approximation of a square wave



$\omega_0 = 2\pi$ rad/s

$\omega_0, 3\omega_0$

$\omega_0, 3\omega_0,$
$5\omega_0, 7\omega_0,$
$9\omega_0,$

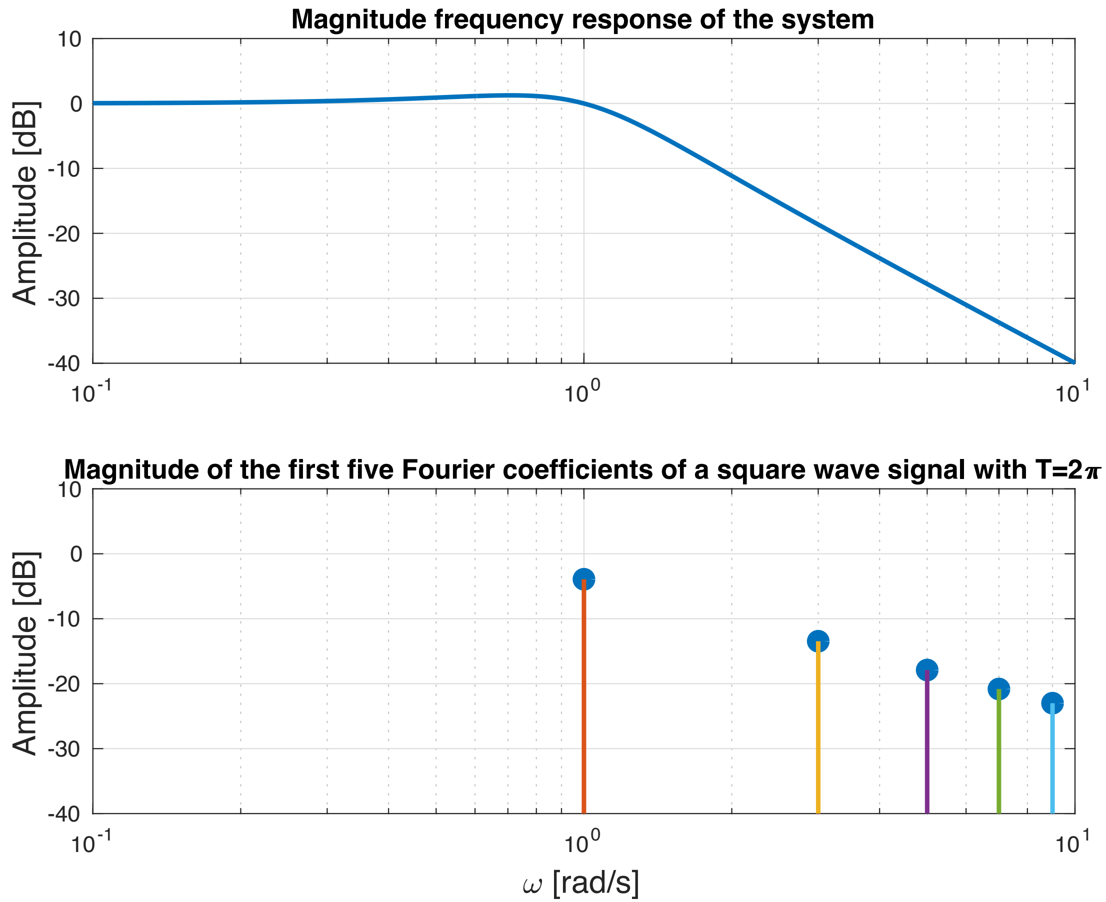
Example: steady state response to a square wave - 1

Let us consider the system with transfer function:

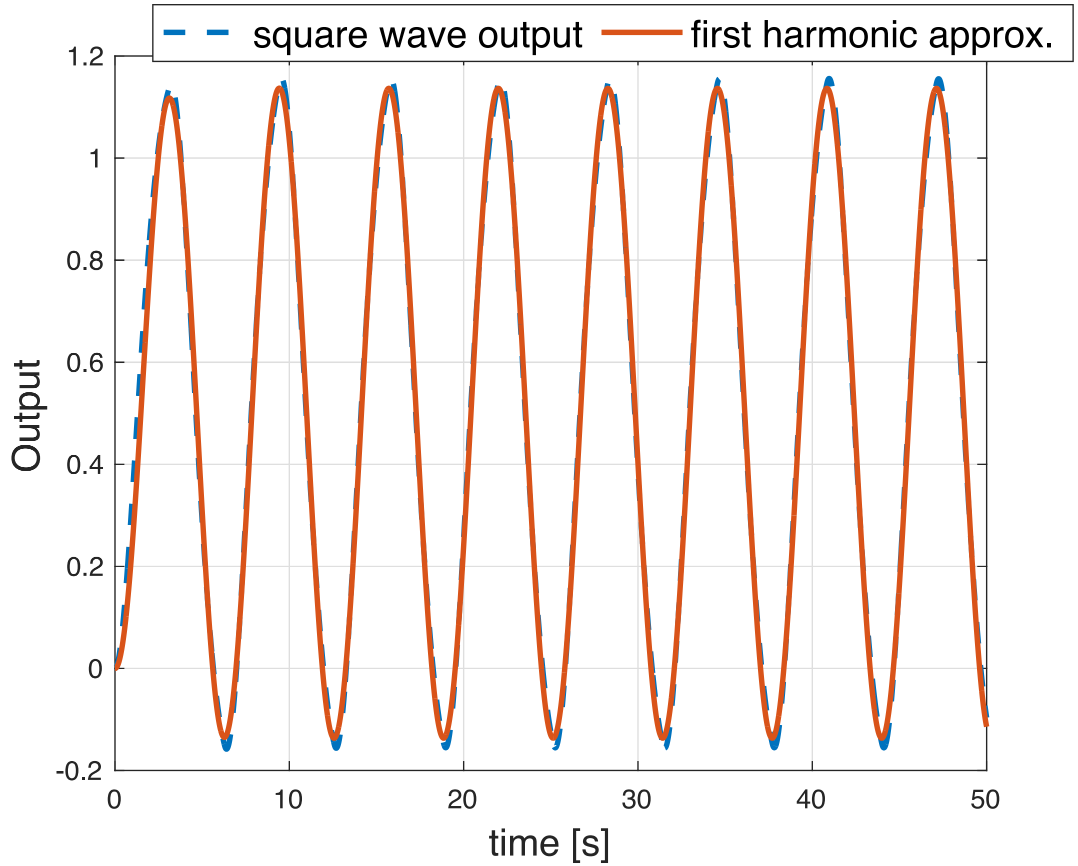
$$G(s) = \frac{1}{s^2 + s + 1}$$

and assume we want to compute the steady-state response to the square wave with period $T=2\pi$.

- $u(t) = \frac{1}{2} + \frac{2}{\pi}\sin t + \frac{2}{3\pi}\sin(3t) + \frac{2}{5\pi}\sin(5t) + \cdots$



Example: steady state response to a square wave - 2



The stead state response of the system with transfer function

$$G(s) = \frac{1}{s^2 + s + 1}$$

is practically identical to the response assuming just the first two terms of the Fourier expansion (the average value plus the first harmonic)

Fourier transform

- So far we have assumed that the signal are periodic. In this case, the frequency spectrum (i.e., the coefficients of the Fourier series) of the signal is discrete (i.e., it is defined only a certain frequencies)

- When the signal is aperiodic, we can assume it as a signal with period $T = \infty$. Thus, the interval between two consecutive harmonics $n\omega_0 = n\frac{2\pi}{T}$ tends to zero and the frequency spectrum becomes a continuous function of w (i.e. defined for all the frequency values)

- Formally, given a aperiodic signal $f(t)$, it can be analysed in the frequency domain by applying the Fourier transform, defined as

$$\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\, dt$$

➢ ***Draw the asymptotic Bode diagrams of the following LTI systems:***

- $G_1(s) = \frac{10s}{s^2+6s+5}$; $G_2(s) = \frac{s}{s^2+s+1}$;

- $G_3(s) = \frac{10(s+0.1)}{s(s^2+2s+1)}$; $G_4(s) = \frac{16s}{s^2+s+16}$;

➢ ***Compute the steady state responses of the above LTI systems to the following inputs:***

- $u_1(t) = 2\sin(t)1(t)$; $u_2(t) = \frac{1}{2}\sin(10t)1(t)$;

➢ ***For $G_1(s)$ and $G_4(s)$, compute the steady state response to the following square wave signal, u:***

- $u(t) = \begin{cases} 1 \text{ if } 0 < t \leq T/2 \\ 0 \text{ if } T/2 < t \leq T \end{cases}$, *where* T=2 s *is the period of the signal u.*