Course of “Automatic Control Systems”
2024/25

Asymptotic Bode diagrams

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

Università degli Studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code: **tz3jpwb**

Bode diagrams definition

- Let us consider a transfer function $W(s)$ an LTI system

$$W(s) = K\frac{s^{\nu}\prod_i (1+\sigma_i s)^{m_i}\prod_q \left(1+\frac{2\xi_q}{\omega_{nq}}s+\frac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_j (1+\tau_j s)^{n_j}\prod_p \left(1+\frac{2\zeta_p}{\omega_{np}}s+\frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}}$$

- The aim of the Bode diagrams is to represent the magnitude and the phase of transfer function $W(s)|_{s=j\omega}$ as function of ω.

- The function $W(s)|_{s=j\omega}$ corresponds to the harmonic response function only if the LTI system is asymptotically stable.

- In the following we will generally refer to Bode diagrams of a transfer function assuming implicitly that $W(s)$ is evaluated for $s = j\omega$

Bode diagrams definition

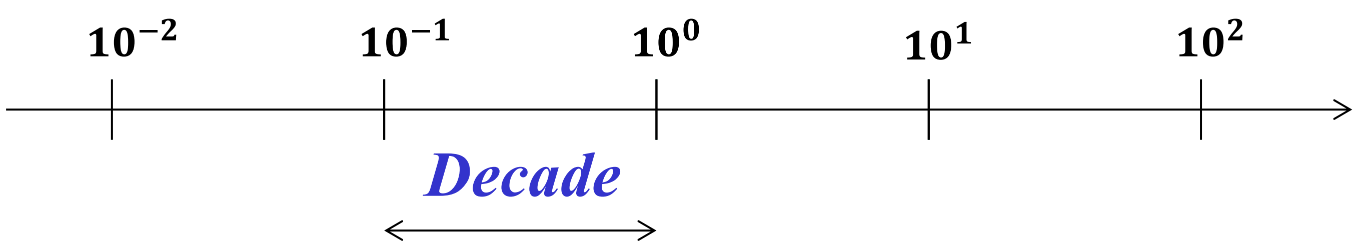
- In the Bode diagrams magnitude and phase of W(jω) are represented on two different Cartesian planes.

- ***The x-axis of both magnitude and phase Bode diagrams are in a logarithmic scale ($log_{10}\omega$)***

On a logarithmic scale, the distance between two frequencies ω_1 and ω_2 depends on the difference of the logarithms and hence on the ratio on the frequencies

$$\log(\omega_2) - \log(\omega_1) = \log\left(\frac{\omega_2}{\omega_1}\right)$$

A decade is defined as the distance between two frequencies whose ratio is 10.



Bode diagrams definition

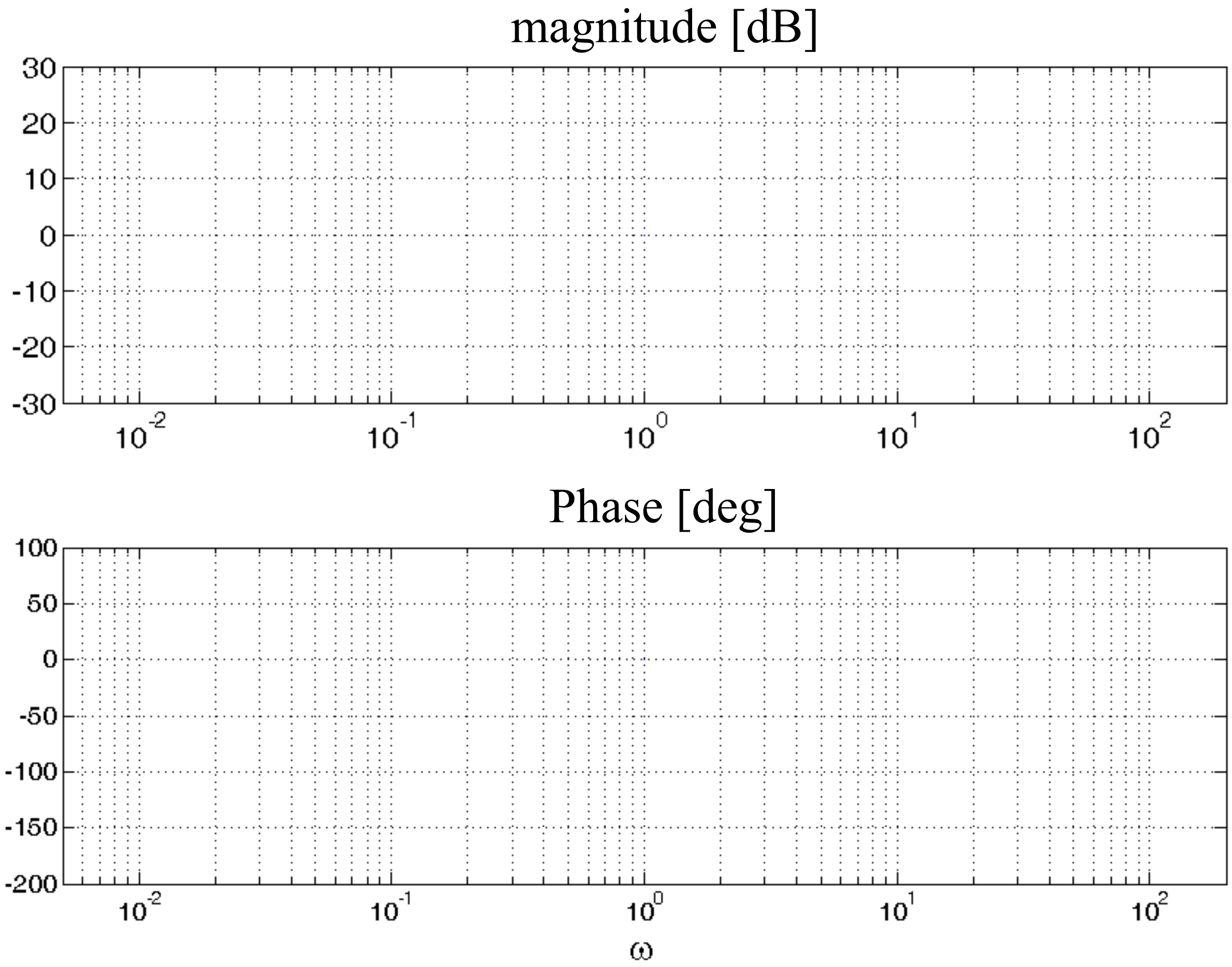
- The y-axis of the magnitude and phase Bode diagrams indicate respectively
  - ***the magnitude of the transfer function in dB (decibel)***

$$|W(j\omega)|_{db} = 20\log_{10}|W(j\omega)|$$

  - ***the phase of the transfer function in degrees or radians***

$$\angle W(j\omega)$$

Magnitude and phase diagrams



Bode diagrams of the magnitude

- The magnitude of the Bode diagrams is expressed in decibel firstly because the logarithmic scale allows to consider ***large magnitude intervals with limited space*** (ex: $|10|_{db} = 20$, $|100|_{db} = 40$, $|1000|_{db} = 60$)

- Moreover, the magnitude of $W(s)|_{s=j\omega}$ in decibel can be written has

$$\left|W(j\omega)\right|_{db} = 20\log_{10}\left(K \frac{s^{\nu}\prod_i (1+\sigma_i s)^{m_i} \prod_q \left(1+\frac{2\xi_q}{\omega_{nq}}s+\frac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_j (1+\tau_j s)^{n_j} \prod_p \left(1+\frac{2\zeta_p}{\omega_{np}}s+\frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}} \right|_{s=j\omega} \right) =$$

and using the main properties of the logarithm….

Bode diagrams of the magnitude

$$\left|W(j\omega)\right|_{db} = 20\log_{10} K +$$ ***Constant term***

$$+20\log_{10} s^{\nu} +$$ ***Monomial term***

$$+\sum_i 20\log_{10}(1+\sigma_i s)^{m_i} - \sum_j 20\log_{10}(1+\tau_j s)^{n_j}$$ ***Binomial terms***

Trinomial terms

$$+\sum_q 20\log_{10}\left(1+\frac{2\xi_q}{\omega_{nq}}s+\frac{s^2}{\omega_{nq}^2}\right)^{\eta_q} - \sum_p 20\log_{10}\left(1+\frac{2\zeta_p}{\omega_{np}}s+\frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}$$

- The magnitude of $W(s)|_{s=j\omega}$ in decibel is given by the sum of four terms: ***constant, monomial, binomial and trinomial terms***
- The phase function has the some product property of the logarithm. Hence in the following we will construct the magnitude and phase Bode diagrams considering these four terms separately.

Basic terms for the Bode diagrams

- ***Constant term:*** K

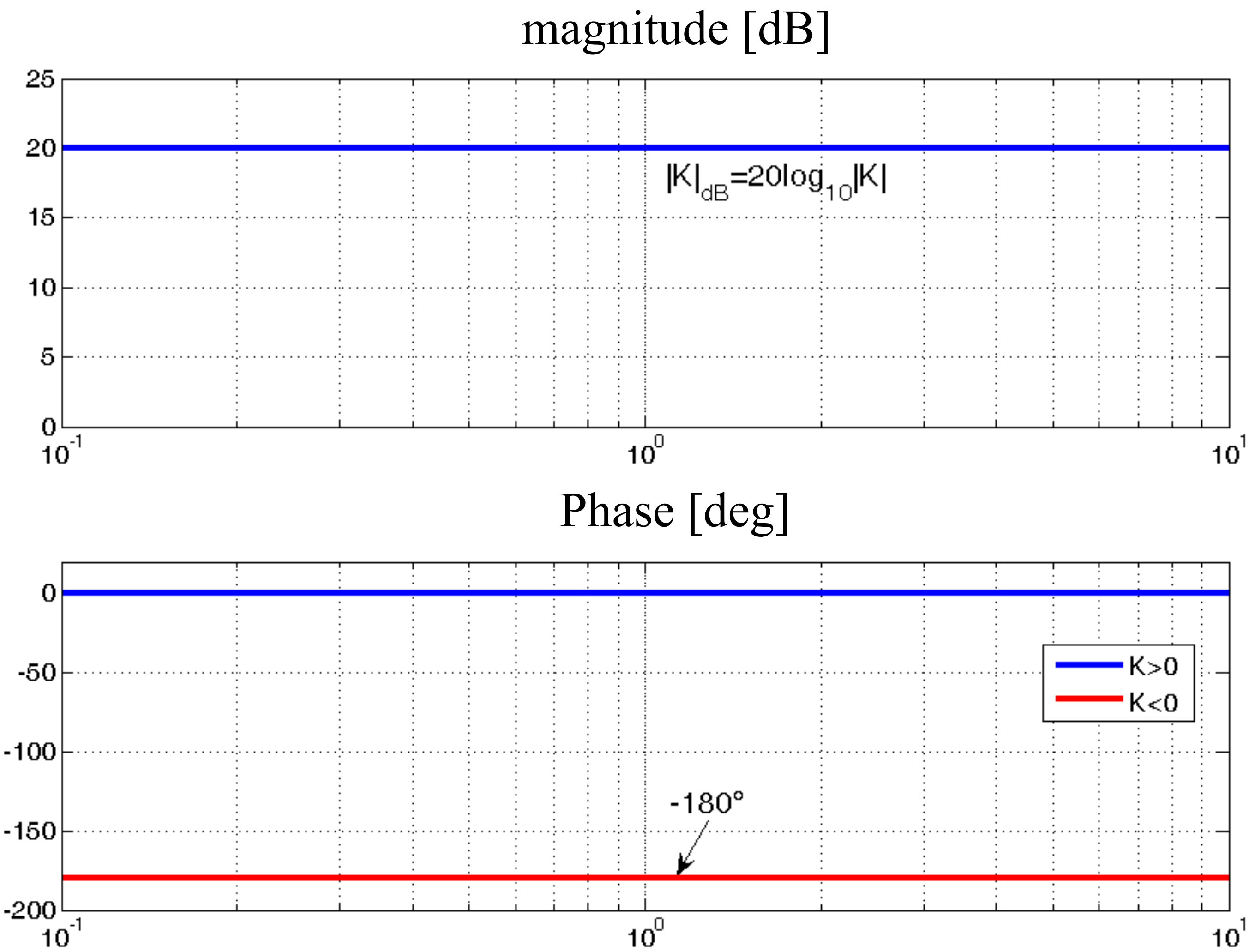
- ***Monomial term:*** Zero/Pole in the origin of multiplicity ν: $\mathbf{20\log_{10} s^{\nu}}$

- ***Binomial term*** **:** Real zero/pole of multiplicity ν: $\mathbf{20\log_{10}(1+\tau s)^{\pm\nu}}$

- ***Trinomial term*** **:** Complex conjugate zero/pole of multiplicity ν:

$$\mathbf{20\log_{10}\left(1+\frac{2\zeta s}{\omega_n}+\frac{s^2}{\omega_n^2}\right)^{\pm\nu}}$$

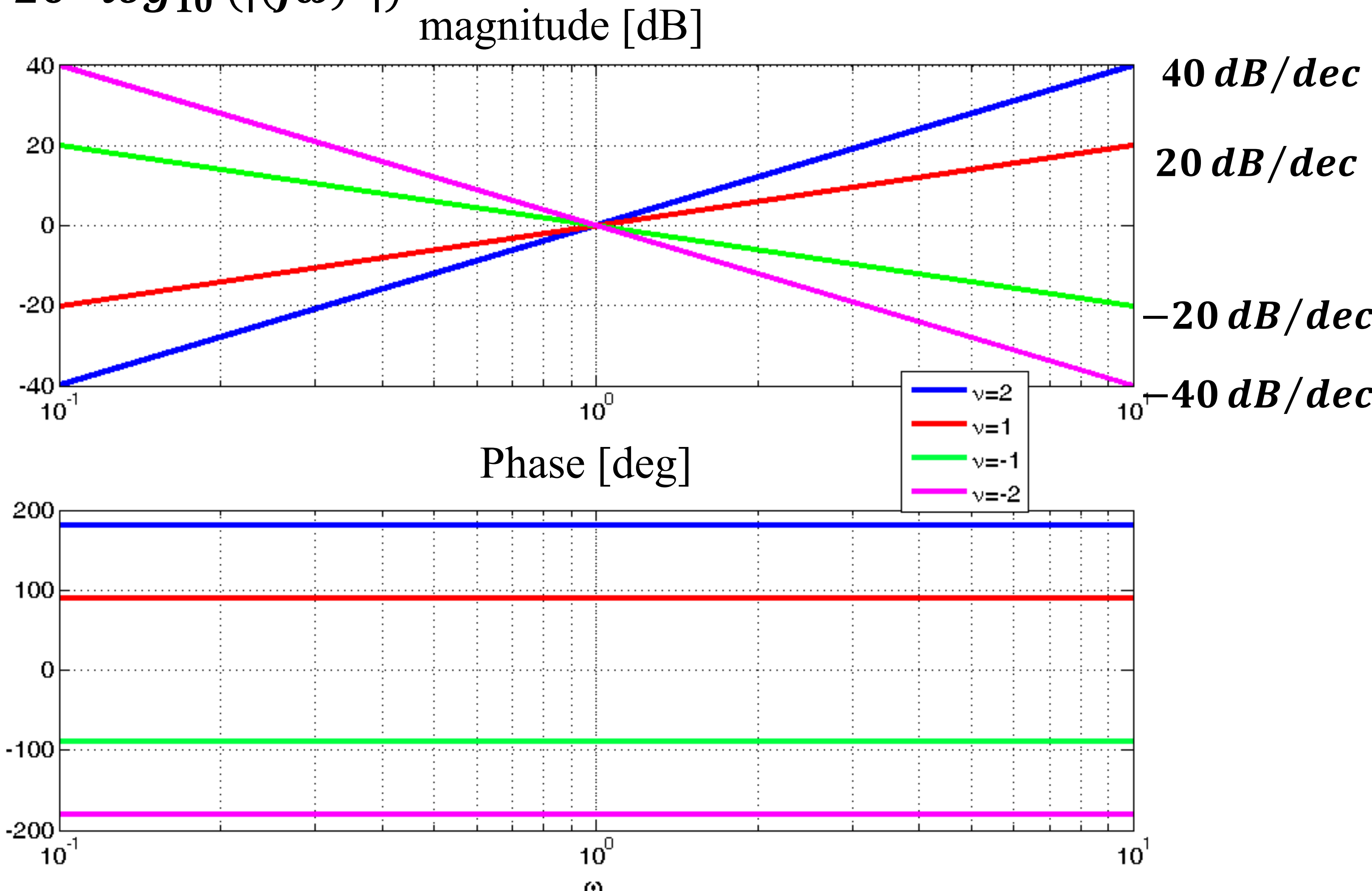
Constant term: K



$\|K\|$	$\|K\|_{dB}$
0.01	-40
0.1	-20
1	0
2	6
3	10
5	14
10	20
100	40
1000	60

Monomial terms: $(j\omega)^\nu$

$$|W(j\omega)|_{dB} = 20 \cdot log_{10}\,(|(j\omega)^\nu|)$$



$$\angle W(j\omega) = \angle\,(j\omega)^\nu$$

Asymptotic Bode diagrams

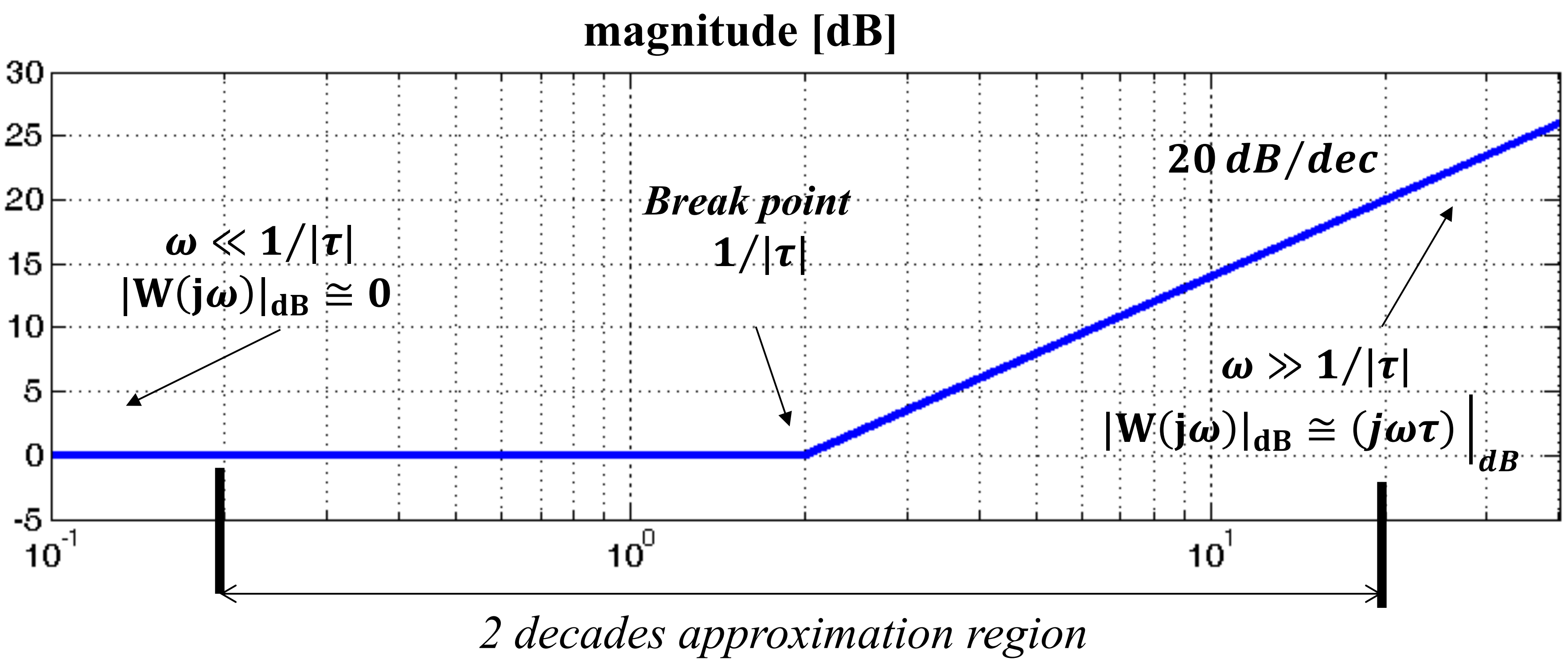
- ***The Bode diagrams definition of binomial and trinomial terms is more demanding***

- For these cases we will first consider the ***asymptotic Bode diagrams*** that give a correct information of magnitude and phase of the considered terms for $\omega \to 0$ and $\omega \to +\infty$ (***or at least 1 decade before and after the break point of the binomial term***)

- In the interval ***from a decade before to a decade after the break point*** the magnitude and phase asymptotic Bode diagrams are linked with linear connections

Case 1: real zero of multiplicity one $(1 + s\tau)$

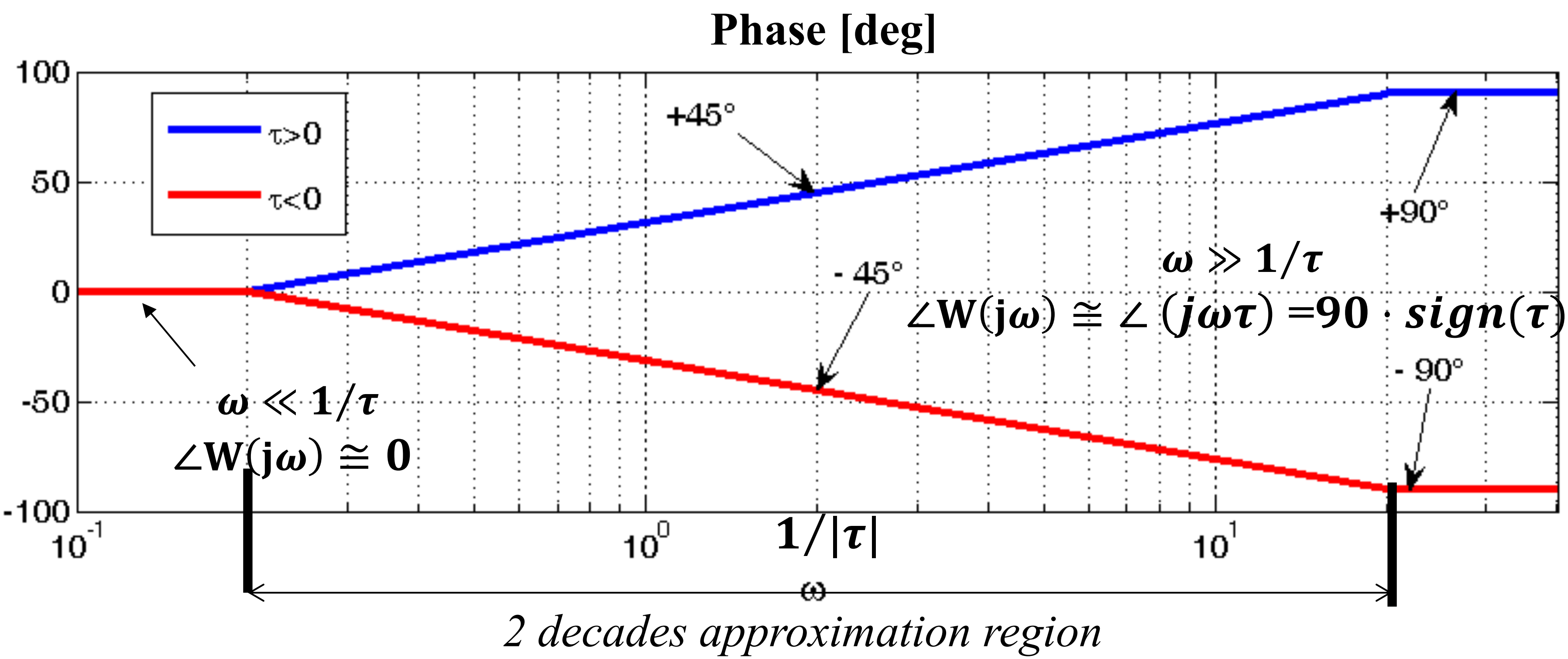
magnitude in decibel $|W(j\omega)|_{dB} = 20 \cdot log_{10}(|1 + j\omega\tau|) = 20 \cdot log_{10}\sqrt{1 + (\omega\tau)^2}$



The magnitude diagram is independent of the sign of τ

Case 1: real zero of multiplicity one $(1 + s\tau)$

Phase in degree $\angle W(j\omega) = \angle(1 + j\omega\tau) = \tan^{-1}(\omega\tau)$



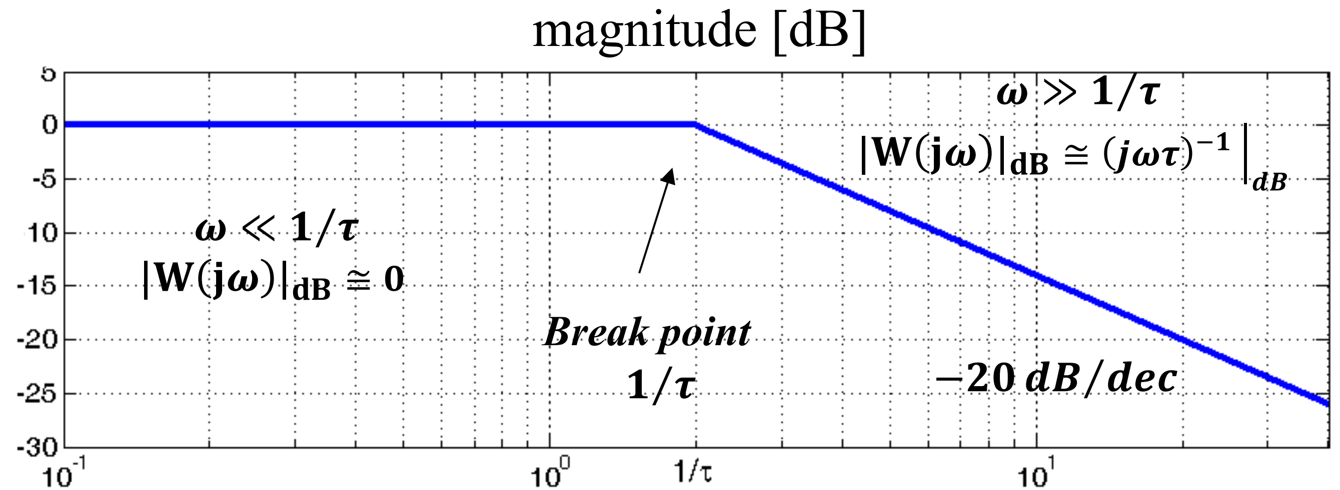
The phase diagram depends on the sign of τ

Case 2: real pole of multiplicity one $(1+s\tau)^{-1}$

magnitude in decibel

$-20 \cdot log_{10}(1+j\omega\tau)$

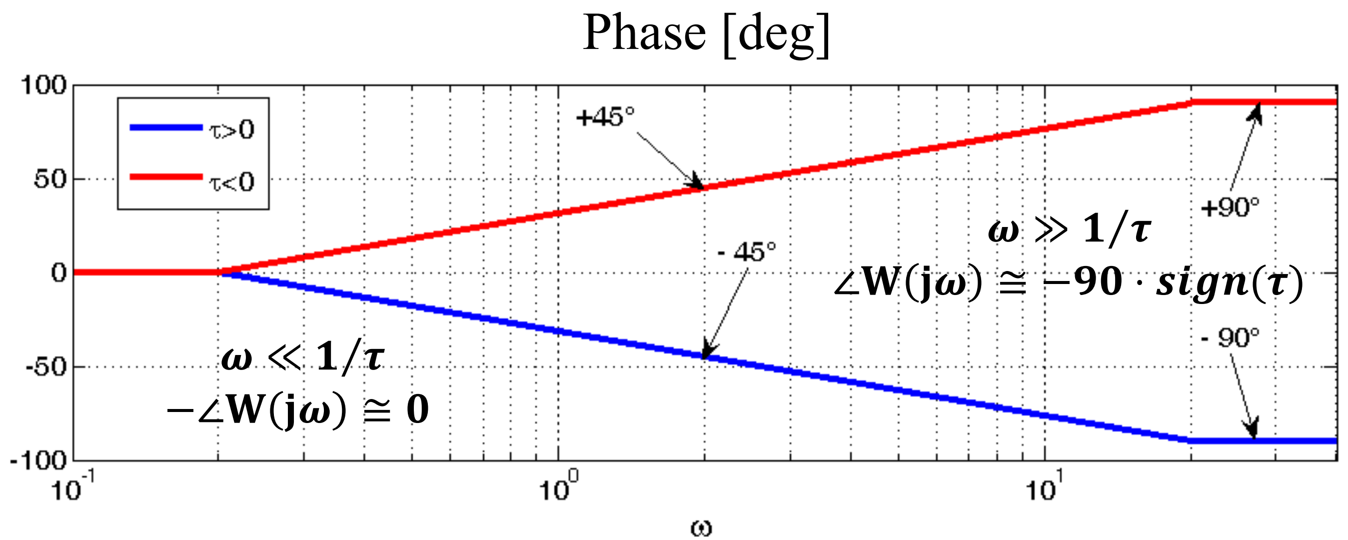
The magnitude diagram is independent of the sign of $\boldsymbol{\tau}$



Phase in degree

$-\angle(1+j\omega\tau)$

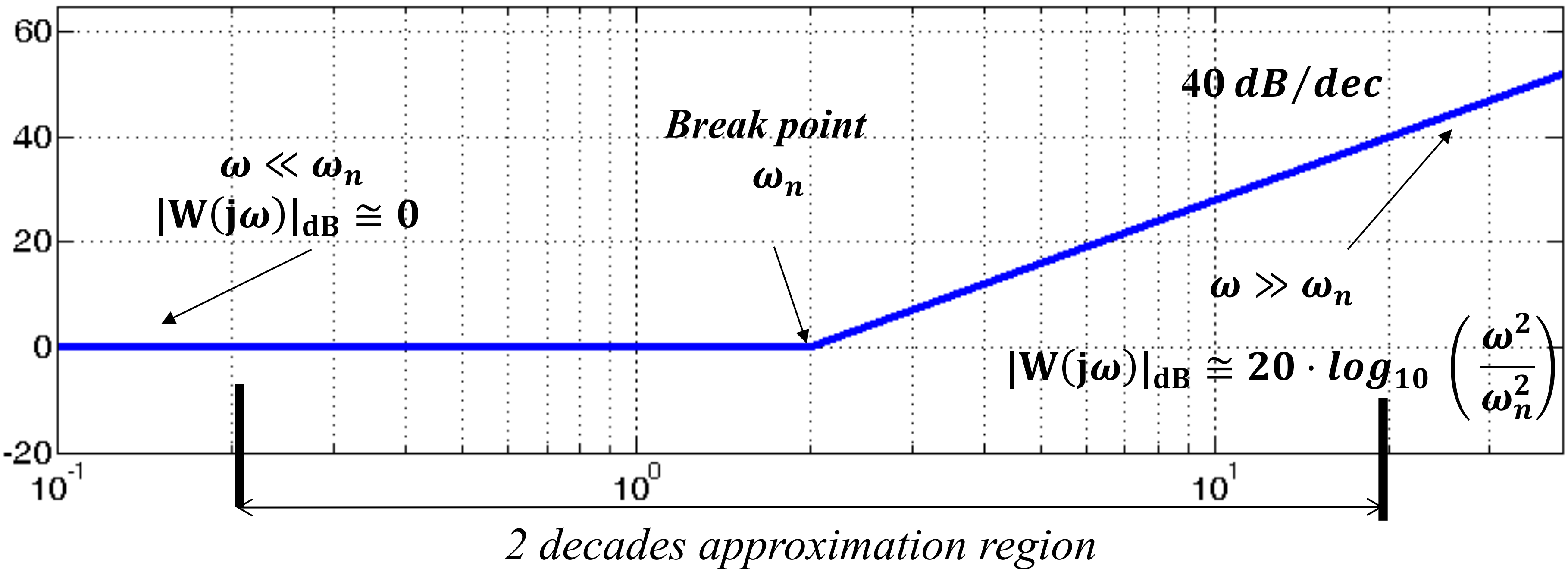
The phase diagram depends on the sign of τ



Case 1: complex conjugate zeros of multiplicity one $\left(1+\frac{2\zeta s}{\omega_n}+\frac{s^2}{\omega_n^2}\right)$

magnitude in decibel $|W(j\omega)|_{dB} = 20\cdot log_{10}\left(1+j\frac{2\zeta\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right) = 20\cdot log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta}{\omega_n}\right)^2}$

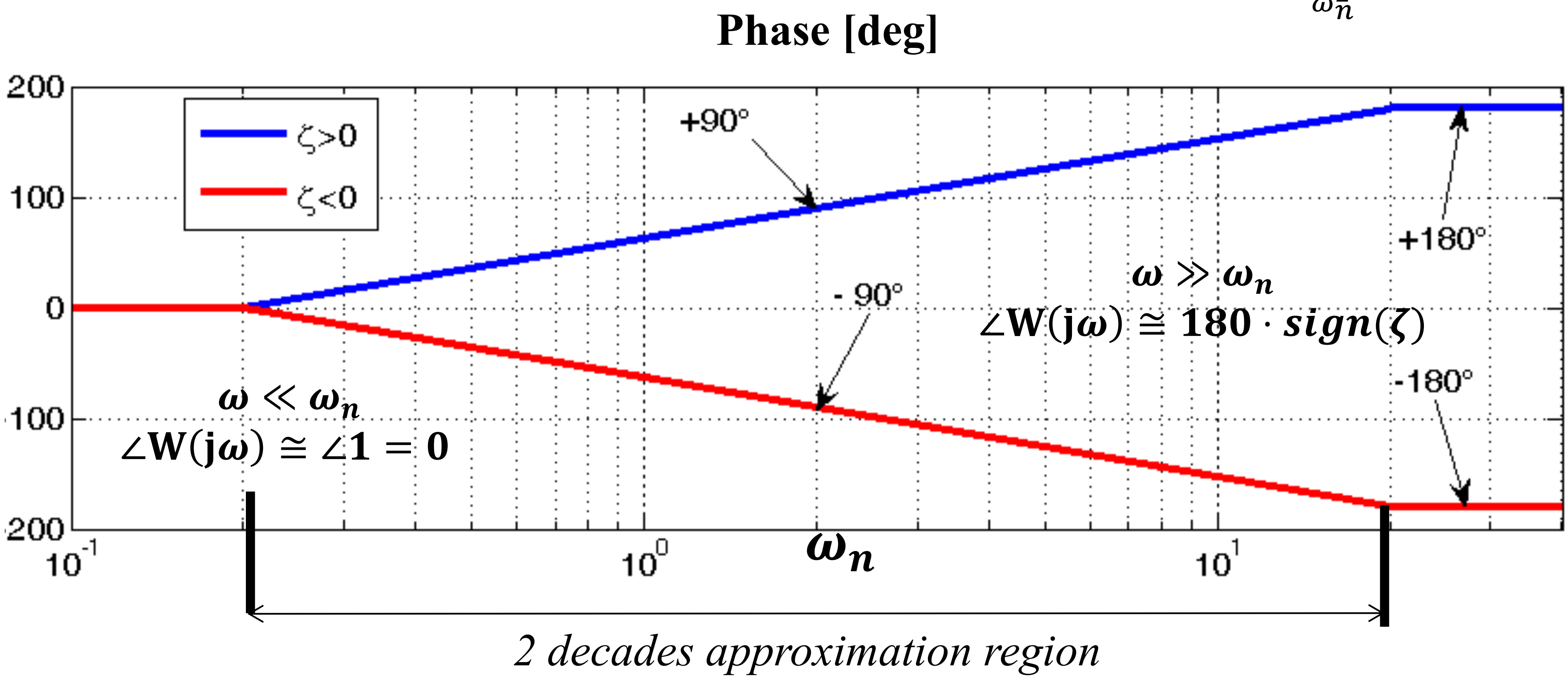
magnitude [dB]



The magnitude diagram is independent of the sign of ζ.

Case 1: complex conjugate zeros of multiplicity one $\left(1+\frac{2\zeta s}{\omega_n}+\frac{s^2}{\omega_n^2}\right)$

Phase in degree $\angle W(j\omega) = \angle\left(1 + j\frac{2\zeta\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right) = \tan^{-1}\frac{\frac{2\zeta}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$



The phase diagram depends on the sign of ζ.

Asymptotic Bode diagrams: trinomial term (3/3)

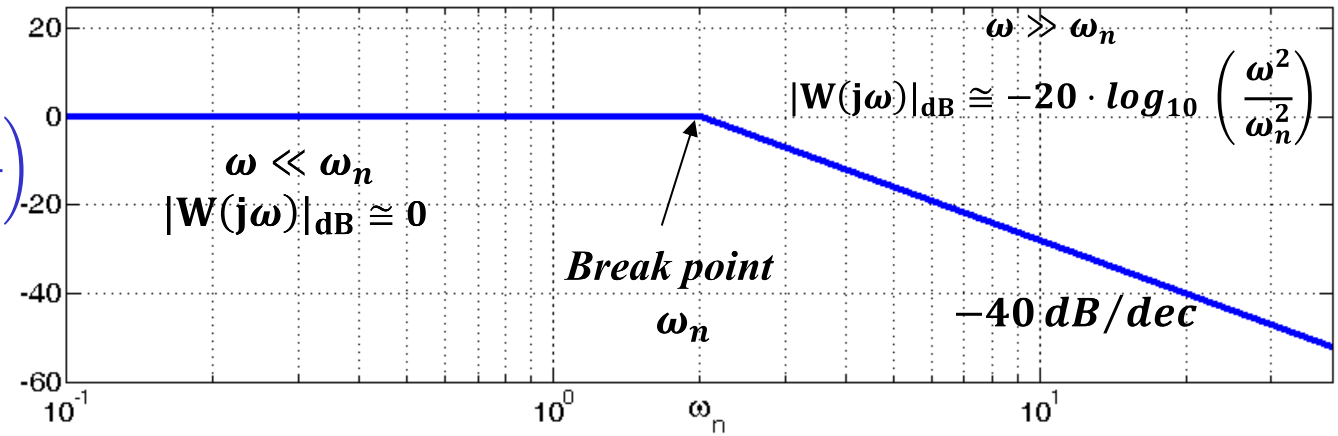
Case 2: complex conjugate poles of multiplicity one $\left(1+\frac{2\zeta s}{\omega_n}+\frac{s^2}{\omega_n^2}\right)^{-1}$

magnitude in decibel

$$-20\cdot log_{10}\left(1+j\frac{2\zeta\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right)$$

The magnitude diagram is independent of the sign of ζ

magnitude [dB]

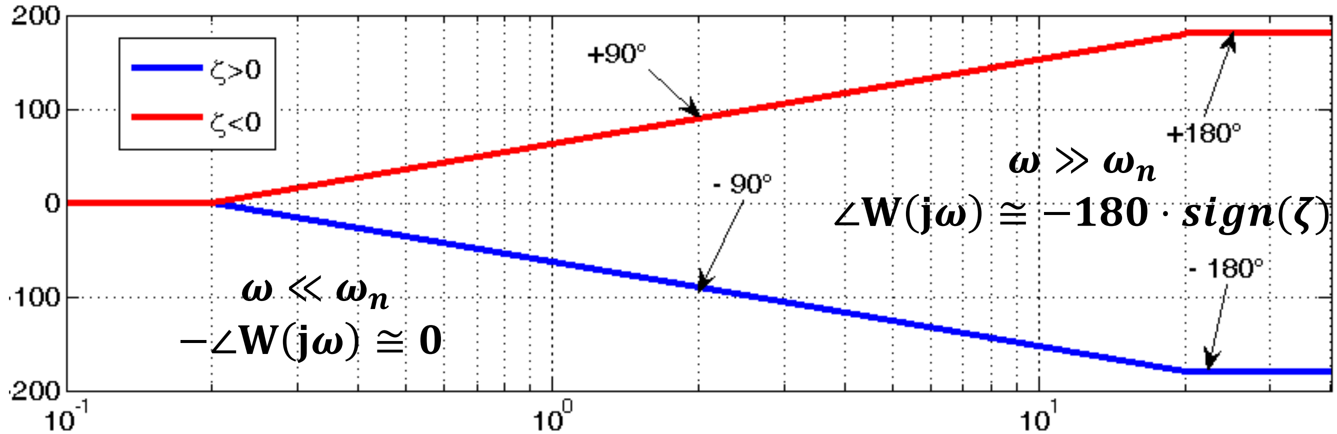


Phase in degree

$$-\angle\left(1+j\frac{2\zeta\omega}{\omega_n}-\frac{\omega^2}{\omega_n^2}\right)$$

The phase diagram depends on the sign of ζ

Phase [deg]



Example

- Let us consider the transfer function of an LTI system

$$W(s) = \frac{10(s+5)}{(s+1)(s+10)} = \frac{10 \cdot 5(\frac{s}{5}+1)}{(s+1)10\left(\frac{s}{10}+1\right)}$$

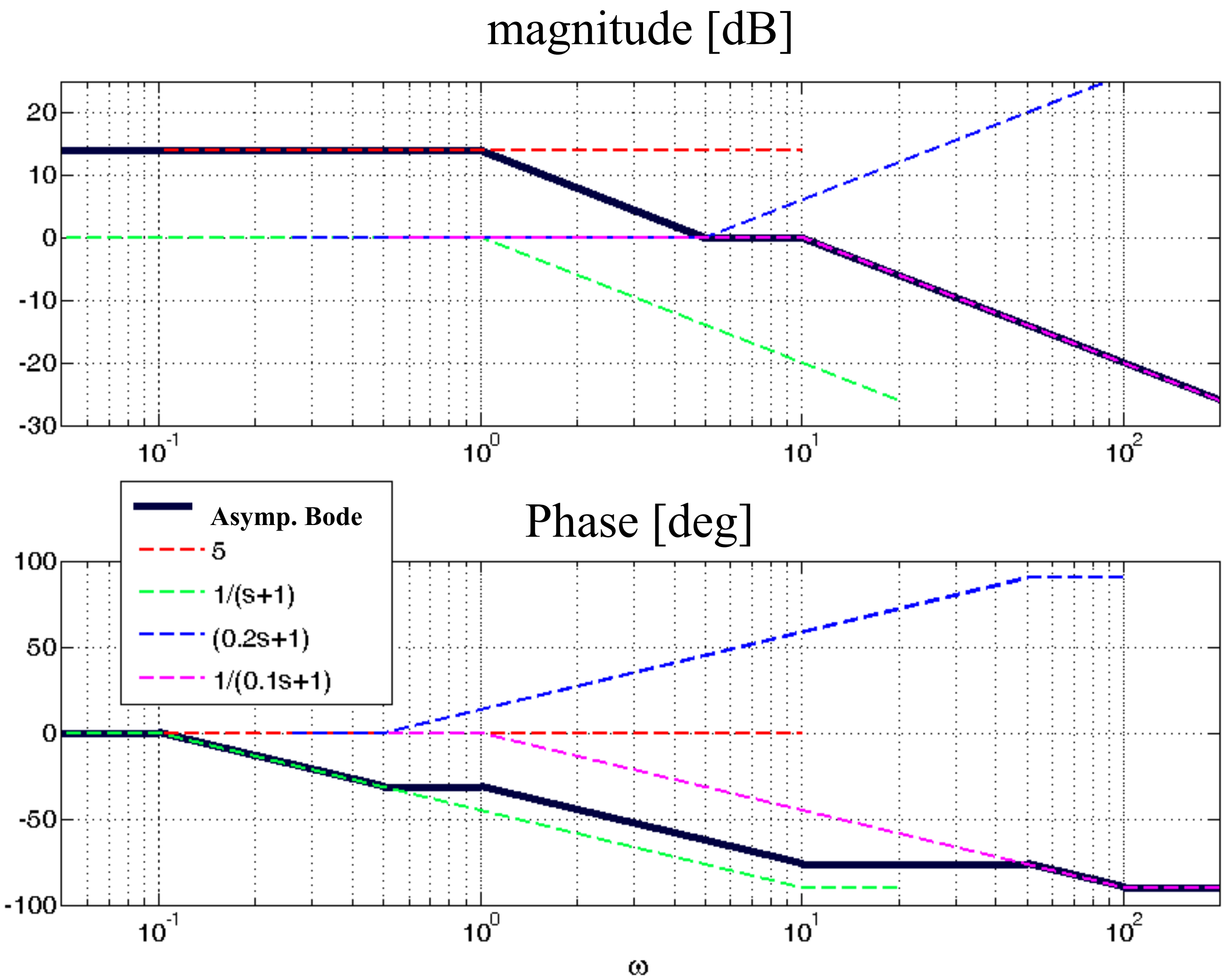
- The LTI system is asymptotically stable and the harmonic response is

$$W(j\omega) = \frac{5(\frac{j\omega}{5}+1)}{(j\omega+1)10\left(\frac{j\omega}{10}+1\right)} = 5\frac{(j0.2\omega+1)}{(j\omega+1)(j0.1\omega+1)}$$

- The harmonic response is composed by a constant term $k = 5$, and three binomial terms $(1+j0.2\omega), (1+j\omega)^{-1}, (1+j0.1\omega)^{-1}$

- Summing magnitude and phase of the four contributions we can evaluate the Bode diagrams of the system

Example


magnitude [dB]
20
10
0
-10
-20
-30
10^-1
10^0
10^1
10^2
Asymp. Bode
5
1/(s+1)
(0.2s+1)
1/(0.1s+1)
Phase [deg]
100
50
0
-50
-100
10^-1
10^0
10^1
10^2
ω