Course of
"Automatic Control Systems"
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Harmonic response function

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Transient and Steady-state

- Let us consider an ***asymptotically stable LTI system.***

- Given an input signal $u(t)$ and an initial condition $x(0)$, we define

  - ***steady-state response*** $\boldsymbol{y_{ss}(t)}$, the regular behavior of the total response $y(t)$ *(if exist)* after an infinite time from the application of the input.

  - ***transient response*** $\boldsymbol{y_t(t)}$, the difference between the total response of the system and the steady-state response $\boldsymbol{y_t(t) = y(t) - y_{ss}(t)}$.

Transient and Steady-state

- ***The steady-state response*** of asymptotically stable system is independent from the initial condition.

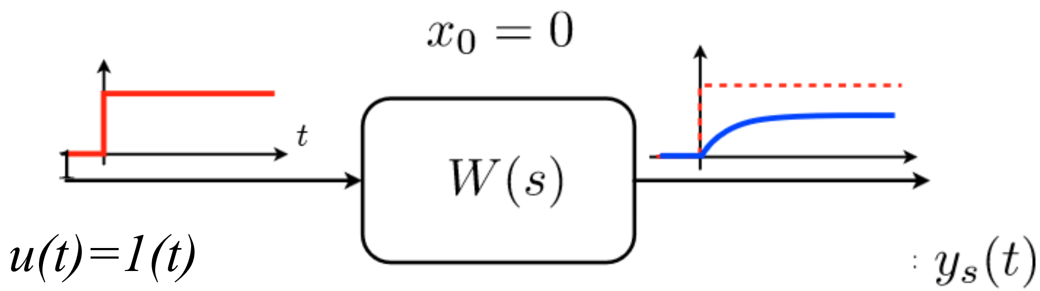
- It depends on the particular input applied to the system

polynomial inputs → polynomial steady state

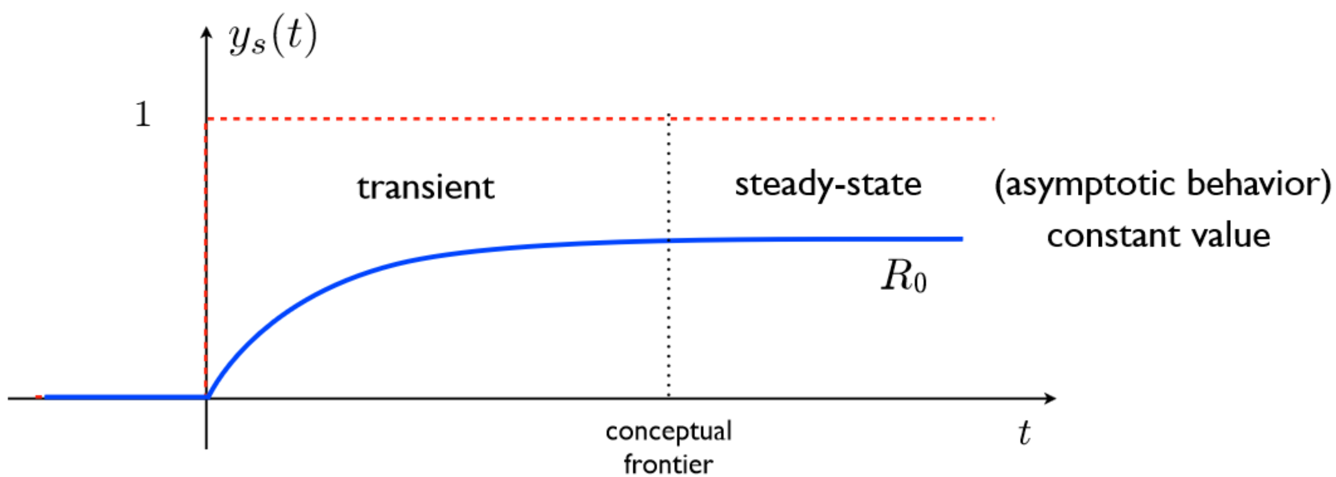
sinusoidal inputs → sinusoidal steady state

Step response: Transient and Steady-state

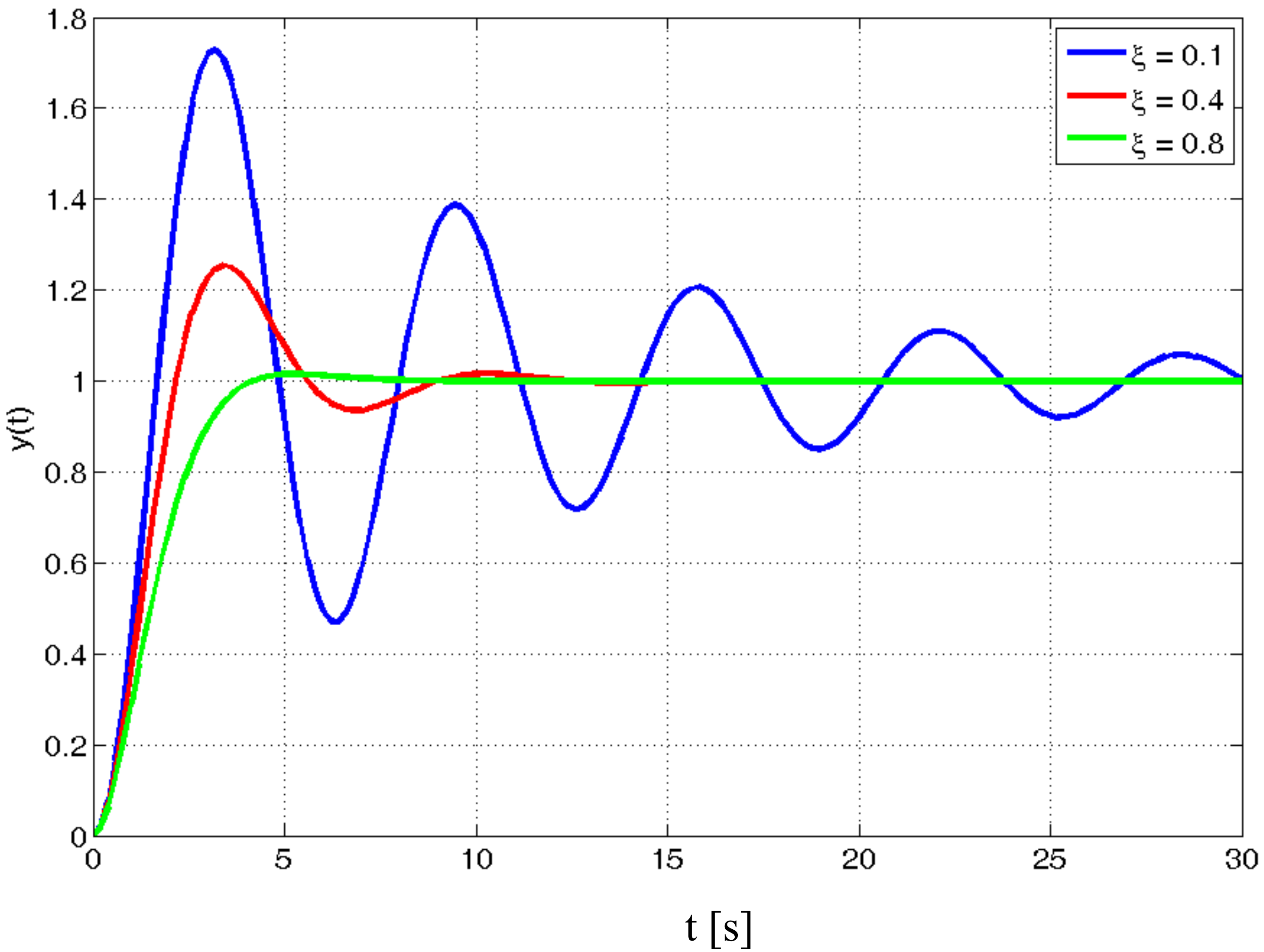
- The step response is characterized by "decaying" exponential functions related to the system evolution modes and a constant value



- The "decaying" exponential functions determine ***the transient*** part of the response while the constant term is the ***steady-state*** value.



- Different evolution modes determine different transient responses.



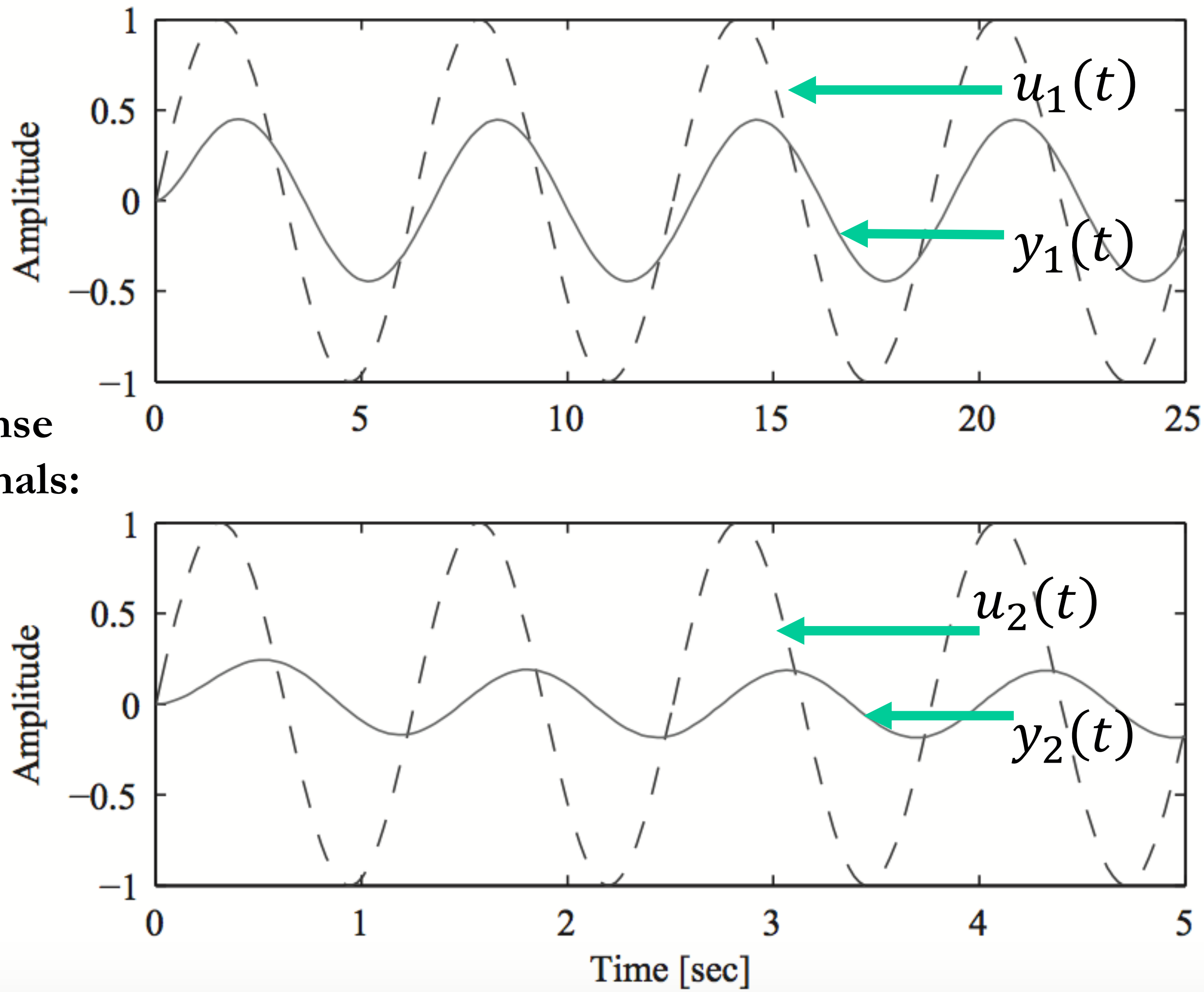
Response at sinusoidal inputs of LTI systems

Let assume a first order LTI system:

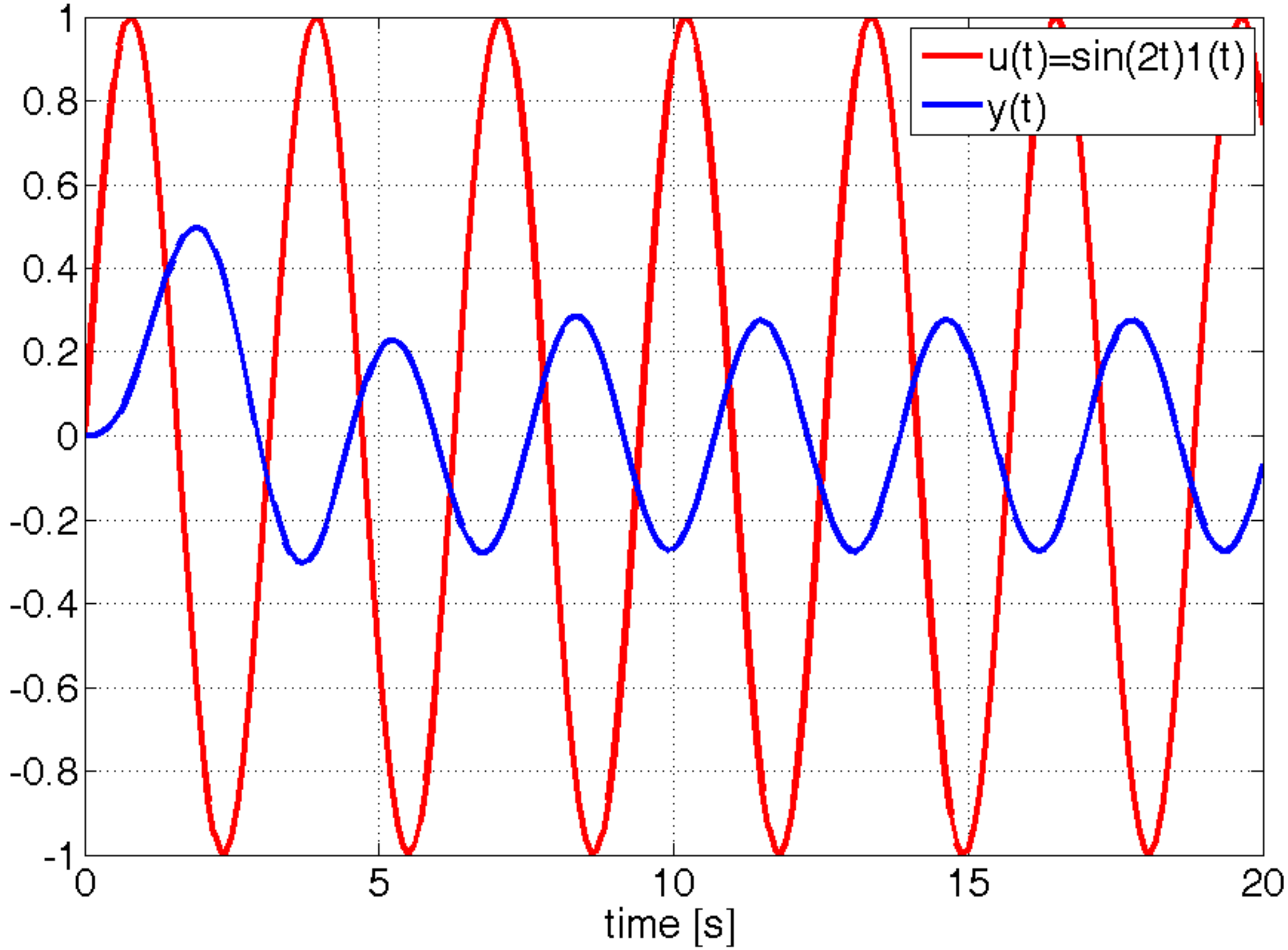
$$\dot{y}(t) + 2\,y(t) = u(t)$$

Compute the response to the following signals:

- $u_1(t) = \sin t$
- $u_2(t) = \sin(5t)$



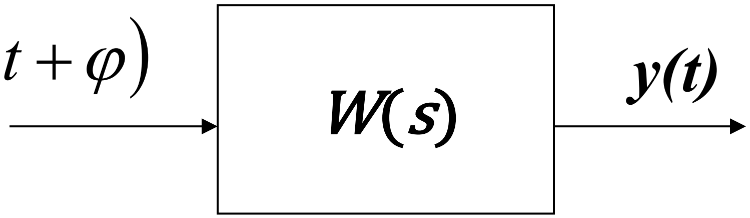
Total response of system $W(s) = 1/(s^2+s+1)$ to the input $u(t) = \sin(2t) \cdot 1(t)$.



Steady state response at sinusoidal inputs

- Let us consider an asymptotically stable LTI system with a transfer function $W(s)$ subject to a sinusoidal input signal

$$u(t) = U_0 \sin(\omega_0 t + \varphi)$$



- The evaluation of the steady state response of LTI system to sinusoidal inputs is very interest taking into account that ***any periodic signal,*** $f(t) = f(t+T)$***, with period T*** ($\omega_0 = \frac{2\pi}{T}$)***, can be decomposed in the sum of a finite or infinite sinusoids by means of the Fourier series, as***

$$f(t) = F_0 + \sum_{n=1}^{\infty} \left[F_{cn} \cos(n\omega_0 t) + F_{sn} \sin(n\omega_0 t) \right]$$

- In this case, the frequency spectrum (i.e., the coefficients of the Fourier series) of the signal is discrete (i.e., it is defined only a certain frequencies)

- ***An aperiodic signal*** can be analysed in the frequency domain by applying the Fourier transform, defined as $\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt$. The spectrum becomes a continuous function of ω (i.e. defined for all the frequency values).

- It is possible to prove that the steady state response of an LTI system with transfer function $W(s)$ to a sinusoidal inputs $u(t) = \mathrm{U}_0 \sin(\omega_0 t + \phi)$ can be written in the time domain as

$$y_{ss}(t) = U_0 \left| W(s) \right|_{s=j\omega_0} \sin\left(\omega_0 t + \varphi + \angle W(s)_{s=j\omega_0}\right)$$

where

- $|W(s)|_{s=j\omega_0}$ is the magnitude of the Laplace transform of $W(s)$ evaluated in $s = j\omega_0$.
- $\angle W(s)|_{s=j\omega_0}$ is the phase of the Laplace transform of $W(s)$ evaluated in $s = j\omega_0$.

The function $W(j\omega)$ is called *harmonic response function* of the system.

Filters

- The proposed result can be summarized as follows:

  - The magnitude of a sinusoidal input signal $u(t) = \sin(\omega_0 t + \phi)$ is ***amplified or reduced*** by a linear system depending on the value of $|W(s)|_{s=j\omega_0}$.

  - An input signal $u(t) = \sin(\omega_0 t + \phi)$ is ***phase shifted*** by a linear system depending on the value of $\angle W(s)|_{s=j\omega_0}$.

- In other terms, ***a linear system can be designed as a filter*** able to amplify without distortion a certain set of input signals Ω_1 and reduce or eliminate another signals.

- Possible structures of filters will be discussed in the following lessons.

LTI system response to exponential inputs

Let us consider input signal belonging to the class of complex exponential functions:

- $u(t) = e^{st}, s = \alpha + j\omega$

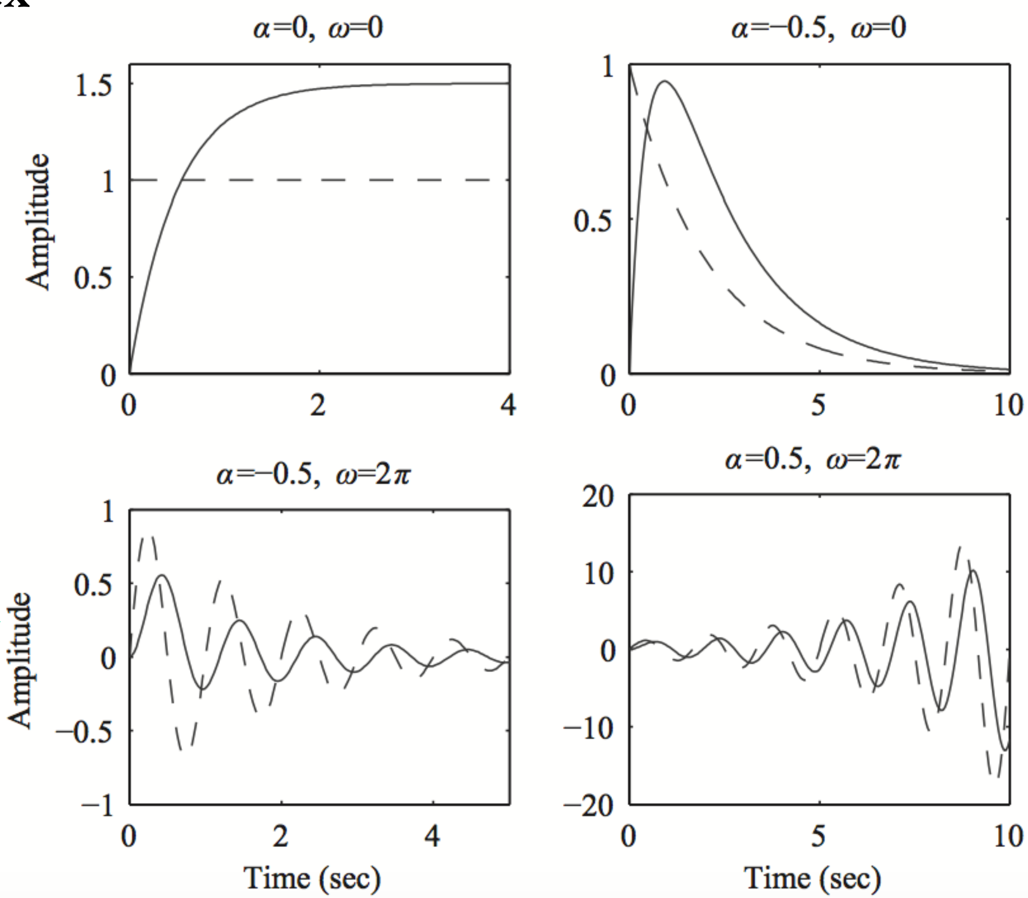
Recall that

- $e^{st} = e^{(\alpha + j\omega)\ t}$
  $= e^{\alpha t}(\cos(\omega t) + j\sin(\omega t))$

Many signals may be written as a linear combination of complex exponential functions.

After an initial transient, the LTI response is proportional to the input (i.e. exhibits the same form of the input).

LTI: $\dot{y}(t) + 2\,y(t) = 3u(t)$



LTI system response to exponential input

- **A SISO system of *n-th* order, t_0=0, $x(0)$=x_0:**

$$\dot{x}(t) = A\,x(t) + Bu(t) \qquad \xrightarrow{\mathcal{L}} \qquad sX(s) - x_0 = A\,X(s) + BU(s)$$

$$y(t) = C\,x(t) + D\,u(t) \qquad\qquad Y(s) = C\,X(s) + DU(s)$$

$$X(s) = (sI - A)^{-1}\,x_0 + (sI - A)^{-1}B\,U(s)$$

$$W(s) = C(sI - A)^{-1}B + D$$

$$Y(s) = \underbrace{C(sI - A)^{-1}x_0}_{Y_l} + \underbrace{\left(C(sI - A)^{-1}B + D\right)U(s)}_{Y_f}$$

- **For exponential input:**

$$u(t) = e^{\lambda t}, t \geq 0 \quad \xrightarrow{\mathcal{L}} \quad Y_f(s) = \boldsymbol{W}(s)U(s) = W(s)\frac{1}{s - \lambda} = \cdots + \frac{k}{s - \lambda}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad \cdots + ke^{\lambda t}$$

LTI system response to exponential input

- ***Initial condition x*(0) that nullifies the evolution modes**

$\Rightarrow$ $x(t) = x(0)e^{\lambda t}$

By using the state equation

$\dot{x}(t) = A\,x(t) + Bu(t)$ $\Rightarrow$ $\lambda x(0)e^{\lambda t} = Ax(0)e^{\lambda t} + Be^{\lambda t}$

$\Rightarrow$ $(\lambda I - A)x(0) = B$ $\Rightarrow$ $x(0) = (\lambda I - A)^{-1}B$**, if** λ **is not an eigenvalue of A**

Then,

$x(t) = x(0)e^{\lambda t} = (\lambda I - A)^{-1}Be^{\lambda t}, t > 0$

$y(t) = C\,x(t) + D\,u(t) = C(\lambda I - A)^{-1}Be^{\lambda t} + De^{\lambda t} = (C(\lambda I - A)^{-1}B + D)e^{\lambda t} = \boldsymbol{W}(\boldsymbol{\lambda})e^{\lambda t}.$

If the system is asymptotically stable, $x(t) = (\lambda I - A)^{-1}Be^{\lambda t}$, $y(t) = \boldsymbol{W}(\boldsymbol{\lambda})e^{\lambda t}$,

these functions represent the asymptotic movements of the state and the output of the system, for any initial condition *x*(0).

LTI system response to sinusoidal input

For a sinusoidal input,

$$u(t) = \sin(\omega t)\,, t \geq 0, \omega = \frac{2\pi}{T},$$

we exploit the results achieved for an exponential input.

Indeed, $\sin(\omega t) = \mathrm{Im}\left(e^{j\omega t}\right)$. **Recall that** $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$.

For $\tilde{u}(t) = e^{j\omega t}$, **if A without eigenvalues in** $\pm j\omega$, **then there is an initial state** $x(0) = (j\omega I - A)^{-1}B$, **such that the movements of the state and the output:**

- $\tilde{x}(t) = x(0)e^{j\omega t} = (j\omega I - A)^{-1}Be^{j\omega t}, t > 0$
- $\tilde{y}(t) = C\,x(t) + D\,u(t) = C(j\omega I - A)^{-1}Be^{j\omega t} + De^{j\omega t}$
  $= (C(j\omega I - A)^{-1}B + D)e^{j\omega t} = W(j\omega)e^{j\omega t}$
  $= |W(j\omega)|e^{j\arg\left(\boldsymbol{W}(j\omega)\right)}e^{j\omega t} = |\boldsymbol{W}(j\omega)|e^{j\left(\omega t + \arg\left(\boldsymbol{W}(j\omega)\right)\right)}$

Recall that z= $a + ib = r(\cos\theta + i\sin\theta) = re^{j\theta}$ **with** $\theta = \arg(z) = \tan^{-1}\frac{b}{a} + 2k\pi$

LTI system response to sinusoidal input

$\tilde{x}(t) = (j\omega I - A)^{-1} B e^{j\omega t}, \quad \tilde{y}(t) = |\boldsymbol{W}(j\omega)| e^{j\left(\omega t + \arg(\boldsymbol{W}(j\omega))\right)}, t > 0$

These functions represent the asymptotic movements, for LTI asymptotically stable with $\tilde{\boldsymbol{u}}(\boldsymbol{t}) = \boldsymbol{e}^{\boldsymbol{j\omega t}}$.

For $u(t) = \sin(\omega t) = \text{Im}\left(e^{j\omega t}\right)$, **then**

$x(t) = \text{Im}\left(\tilde{x}(t)\right), \; y(t) = \text{Im}\left(\tilde{y}(t)\right) = |\boldsymbol{W}(j\omega)| \sin\left(\omega t + \arg\left(\boldsymbol{W}(j\omega)\right)\right), t > 0$

In general for $u(t) = U \sin(\omega_0 t + \varphi), t > 0,$

there is an initial state such that the output is a sinusoidal signal:

$y(t) = Y \sin(\omega_0 t + \psi), t > 0$

with $\quad Y = |W(j\omega_0)| U,$

where $\boldsymbol{W}(j\omega_0) = C(j\omega_0 I - A)^{-1} B + D,$ **and** $\quad \psi = \varphi + \arg\left(\boldsymbol{W}(j\omega_0)\right).$

If the system is a.s. $\boldsymbol{y(t)}$ **(and** $\boldsymbol{x(t)}$**) represents the asymptotic movement of the output (state).**

Harmonic response function

- This result underlines the importance of the ***harmonic response function*** $W(j\omega)$ for the analysis of the forced response of LTI systems.

- In the following we present a method able to rapidly evaluate the magnitude and the phase $W(j\omega)$ as a function of ω: ***Bode diagrams***

$W(j\omega)$ general form

- ***Bode diagrams*** allows to extract the magnitude and the phase of $W(j\omega)$ as a function of ω
- Bode diagrams are a main tool for the closed loop control design
- For the closed loop control problems, we will be interested to analyze magnitude and the phase of transfer functions $W(s)$, also in case of stable and unstable systems
- In that cases, $W(s)|_{s=j\omega}$ will be not the harmonic function.

$W(j\omega)$ general form

- Given an asymptotically stable LTI system, the ***harmonic response function*** $W(j\omega)$ is given by the ratio of polynomial with real and complex conjugate roots

$$W(j\omega) = W(s)\big|_{s=j\omega} = K \left. \frac{s^{\nu} \prod_i (1+\sigma_i s)^{m_i} \prod_q \left(1 + \frac{2\xi_q}{\omega_{nq}} s + \frac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_j (1+\tau_j s)^{n_j} \prod_p \left(1 + \frac{2\zeta_p}{\omega_{np}} s + \frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}} \right|_{s=j\omega}$$