Course of
"Automatic Control Systems"
2024/25

Step response: quantitative and qualitative analysis

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Step response

- With the term ***step response*** or ***indicial response*** we indicate the forced response of an LTI system to a step input of unitary amplitude.

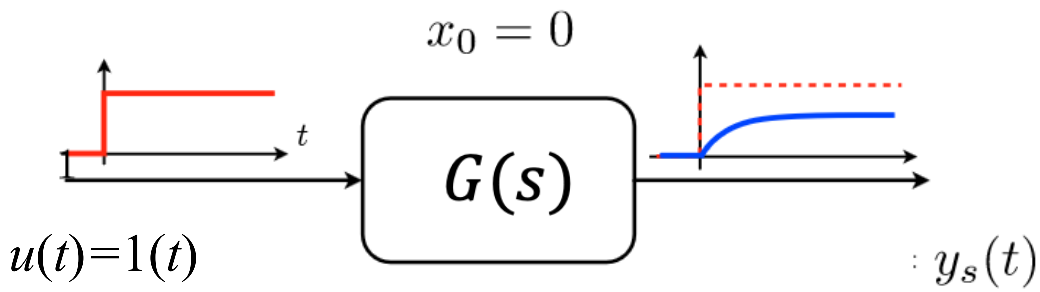
- The study of the step response is of interest for two important reasons

  - in many practical control problem, the input signal is constant or slowly time variant

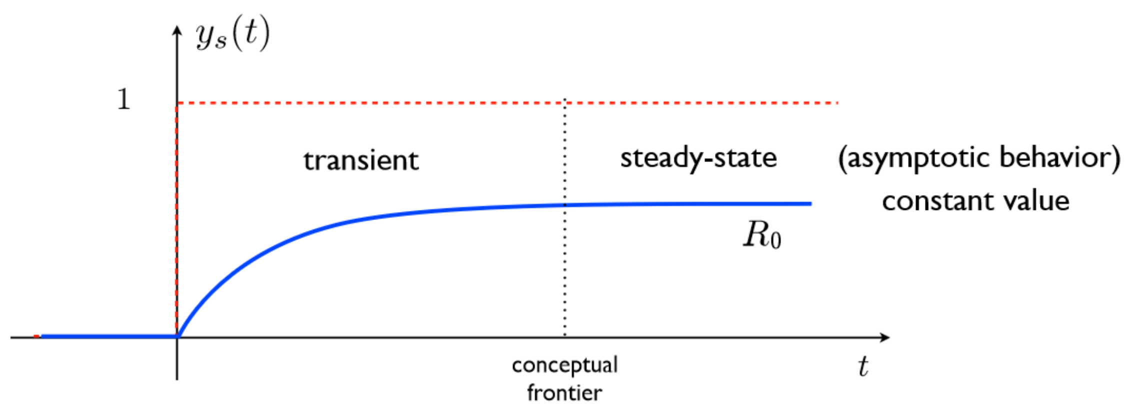
  - for system whose mathematical model is unknown, the experimental step response can be used to identify a linear approximation of the model

- In this course we will focus on the qualitative step response of a linear system assuming that the transfer function $G(s)$ is known.

- When the system is asymptotically stable, the step response is characterized by "decaying" exponential functions and a constant value



- The "decaying" exponential functions determine ***the transient*** part of the response while the constant term is the ***steady-state*** value.



- The concept of transient and steady-state can be generalized to different classes of inputs and initial conditions.

Step response: qualitative parameters

- When the system is asymptotically stable, the qualitative behavior of the step response can be described by a set of qualitative parameters:

- ***Initial value***

- ***Final value*** (steady-state value)

- Parameters indicating how rapidly the transient evolves and decays: ***rise-time, peak time, settling time***

- Parameters indicating the behavior of the response during the transient: ***overshoot, number of oscillations***

Initial value of the step response

- The initial value of the system response to a step signal of amplitude U_0, i.e. $u(t) = U_0 1(t)$ in the Laplace domain can be evaluated with the aim of the initial value theorem

$$y(0) = \lim_{s\to\infty} sY(s) =$$

$$= \lim_{s\to\infty} sG(s)\frac{U_0}{s} =$$

$$\lim_{s\to\infty} G(s)U_0 = \begin{cases} 0, & \textit{for strictly proper systems, i.e } \boldsymbol{D = 0} \\ \neq 0 & \textit{for proper systems, i.e. } \boldsymbol{D \neq 0} \end{cases}$$

- Applying iteratively the initial value theorem it is possible to evaluate the ***derivatives of the step response for $\mathbf{t = 0}$***.

- The difference between the number of poles and zeros of $G(s)$ indicates the number of null derivatives of $y(t)$ in $t = 0$
  - $n - m = 1 \ \rightarrow y(0) = 0 \ , \dot{y}(0) \neq 0$
  - $n - m = 2 \ \rightarrow y(0) = 0 \ , \dot{y}(0) = 0 \, , \ddot{y}(0) \neq 0$

Initial value of the first derivative

- Indeed, if $m < n$, $y(0) = 0$,

$$\dot{y}(0) = \lim_{s\to\infty} s = \lim_{s\to\infty} s(sY(s) - y(0)) = \lim_{s\to\infty} s^2 G(s)\frac{1}{s}$$

$$= \lim_{s\to\infty} sG(s) = \begin{cases} 0, & m < n-1 \\ \neq 0, & m = n-1 \end{cases}$$

Final value of the step response

- The final value of the step response in the Laplace domain can be evaluated with the aim of the final value theorem (see the properties of the Laplace transform)

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) =$$

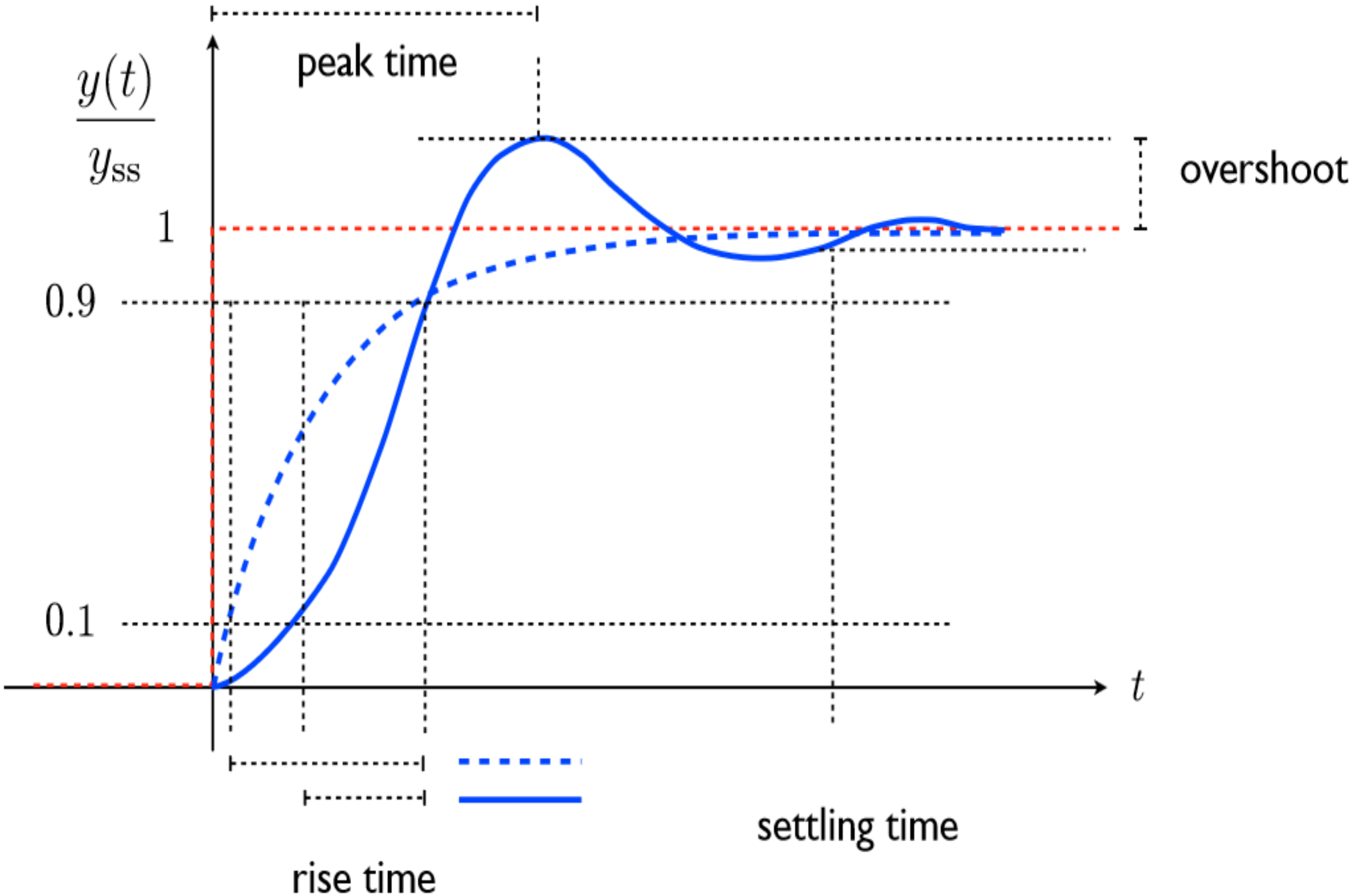
$$= \lim_{s\to 0} sG(s)\frac{U_0}{s} =$$

$$= \lim_{s\to 0} G(s)U_0 = \boldsymbol{G(0)}U_0$$

- The final value theorem can be applied only to asymptotically stable systems and input signal converging to a constant value

- For asymptotically stable system, the value $\boldsymbol{G_0 = G(0)}$ is also said ***Static Gain*** of the system.

Qualitative parameters for the transient



Step response: qualitative parameters

- ***Rise time*** t_r : amount of time required for the signal to go from 10% to 90% of its final value
- ***Steady-state value*** y_{ss} : asymptotic output value (it is constant for the step response and correspond to the final value)
- ***Overshoot*** s: maximum excess of the output w.r.t. the final value (can be defined as a percentage of the final value). In a normalized overshoot is given by the maximum of the normalized output minus one.
- ***Peak time*** t_p : time required for the step response to reach the overshoot
- ***Settling time*** t_s : amount of time required for the step response to stay within 5% ($t_{s\ 5\%}$) or 1% ($t_{s\ 1\%}$) of its final value for all future times

First order system without zeros

- An asymptotically stable first order system without zeros has a transfer function in the form

$$G(s) = \frac{b}{s-p} = \frac{b}{-p\left(\frac{s}{-p}+1\right)} = \frac{G_0}{(1+s\tau)},$$

$$p < 0, G_0 = G(0) = \frac{b}{-p}, \tau = -\frac{1}{p}$$

- The quantitative value of the response to a step signal ($u(t) = U_0 1(t)$) can be evaluated by computing

$$Y(s) = G(s)\frac{U_0}{s} = \frac{bU_0}{s(s-p)} = \frac{A}{s} + \frac{B}{s-p}$$

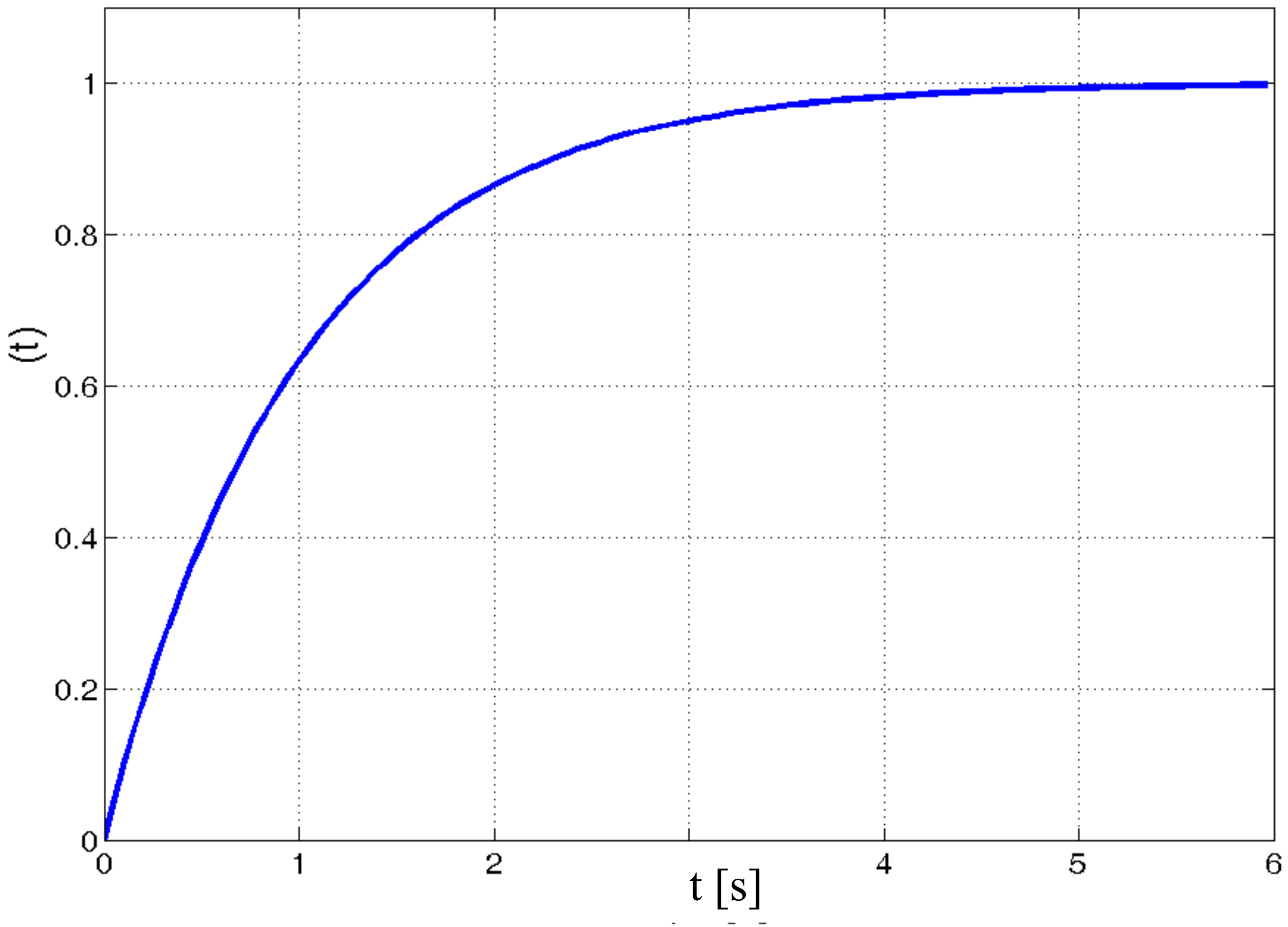
By computing A and B (e.g., by residual method): $\boldsymbol{A} = \frac{bU_0}{-p} = G_0U_0; \boldsymbol{B} = -\mathbf{A}$

By antitransformation

$$y(t) = G_0U_0\left(1 - e^{-\frac{t}{\tau}}\right)1(t)$$

First order system without zeros

Evolution of step resonse for first order system without zeros $(G_0 = 1 \ , \ \tau = 1)$



First order system without zeros: parameters for the qualitative response

- ***Initial value*** $y(0) = 0, \dot{y}(0) = bU_0 = \frac{G_0}{\tau} U_0$
- ***Final value*** $lim_{t\to\infty}\, y(t) = G_0 U_0$
- ***Settling time***
  - $t_{s\ 5\%} = 3\tau$
  - $t_{s\ 1\%} = 4.6\tau$
- ***Rise time*** $t_r \cong 2.2\tau$

Second order system with real poles and no zeros

- An asymptotically stable second order system without zeros has a transfer function in the form

$$G(s) = \frac{b}{s^2 + a_1 s + a_0} = \frac{b}{(s-p_1)(s-p_2)}$$

$$G(0) = G_0 = \frac{b}{a_0} = \frac{1}{p_1 p_2}, p_1 < 0, p_2 < 0$$

- The quantitative value of the response to a step signal ($u(t) = U_0 1(t)$) can be evaluated by computing

$$Y(s) = G(s)\frac{U_0}{s} = \frac{b}{(s-p_1)(s-p_2)}\frac{U_0}{s} =$$

$$= \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \frac{K_3}{s} \qquad K_i = (s-p_i)Y_f(s)|_{s=p_i}$$

$$\xrightarrow{\mathcal{L}^{-1}} y_f(t) = G_0 U_0 \left(1 + \frac{p_2}{p_1 - p_2} e^{p_1 t} - \frac{p_1}{p_1 - p_2} e^{p_2 t}\right) 1(t)$$

Second order system with real poles and no zeros

- The quantitative value of the response to a step signal ($u(t) = U_0 1(t)$) is described by

$$y_f(t) = G_0 U_0 \left(1 + \frac{p_2}{p_1 - p_2} e^{p_1 t} - \frac{p_1}{p_1 - p_2} e^{p_2 t}\right) 1(t)$$

$$p_2 = -{}^{1}/_{\tau_2} \qquad\qquad p_1 = -{}^{1}/_{\tau_1}$$

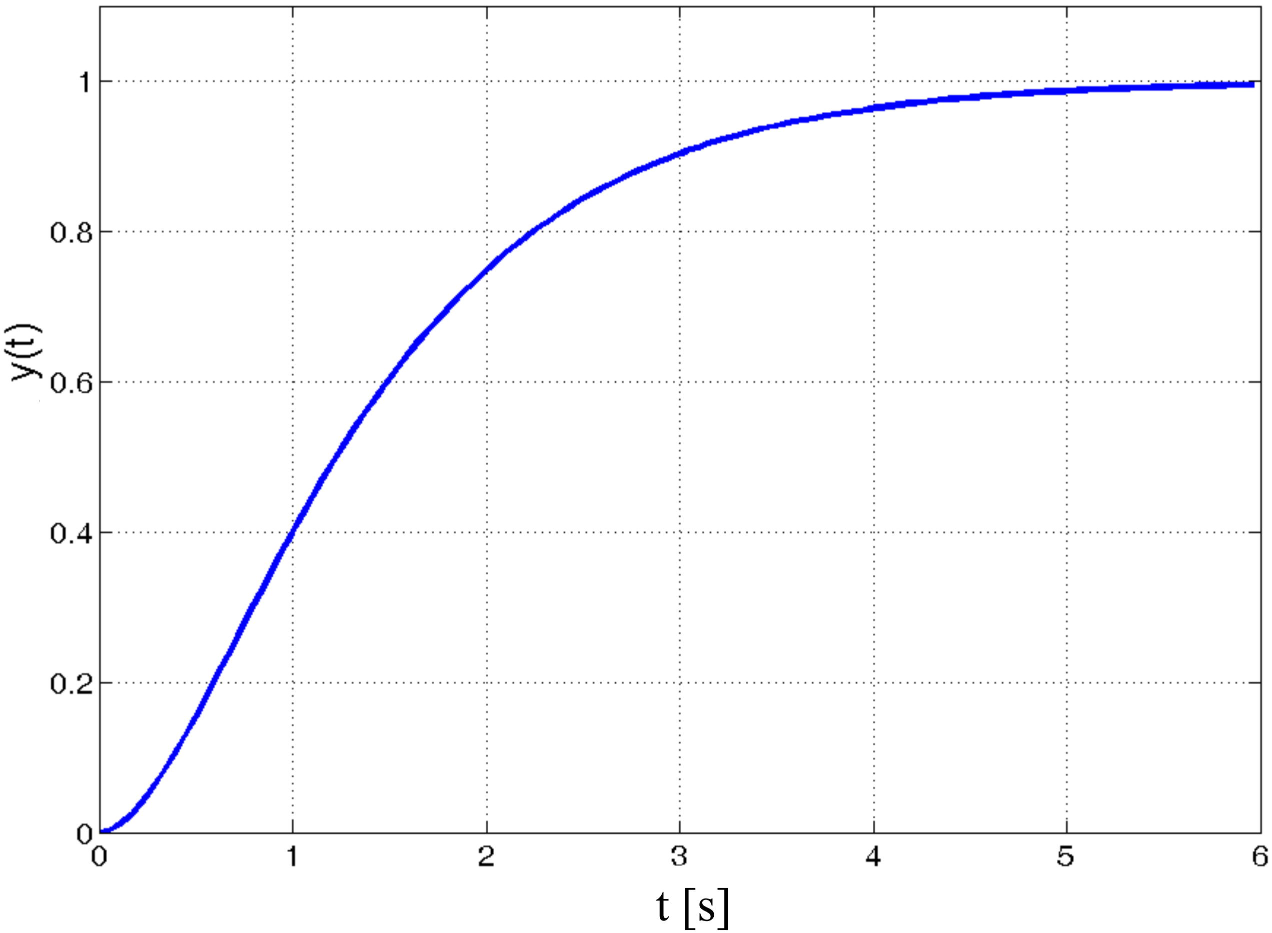
- In terms of time constants,

$$G(s) = \frac{b}{(s - p_1)(s - p_2)} = \frac{b}{(-p_1)\left(\frac{s}{-p_1} + 1\right)(-p_2)\left(\frac{s}{-p_2} + 1\right)} =$$

$$= \frac{G_0}{(1 + s\tau_1)(1 + s\tau_2)}$$

$$\xrightarrow{\mathcal{L}^{-1}} y_f(t) = G_0 U_0 \left(1 - \frac{\tau_1}{\tau_1 - \tau_2} e^{-\frac{t}{\tau_1}} + \frac{\tau_2}{\tau_1 - \tau_2} e^{-\frac{t}{\tau_2}}\right) 1(t)$$

Second order system with real poles and no zeros



Second order system with real poles and no zeros: parameters for the qualitative response

- ***Initial value*** $y(0) = 0 \ , \dot{y}(0) = 0$
- ***Final value*** $lim_{t\to\infty}\, y(t) = G_0 U_0$
- ***Settling time***
  - $t_{s\ 5\%} = 3\tau_{max}$
  - $t_{s\ 1\%} = 4.6\tau_{max}$
- ***Rise time*** $t_r \cong 2.2\tau_{max}$

Second order system with two poles real and coincident

- An asymptotically stable second order system with two poles real and coincident has a transfer function in the form

$$G(s) = \frac{b}{(s-p)^2}, \qquad p < 0$$

$$\Rightarrow \quad Y(s) = \frac{b\,U_0}{(s-p)^2 s} = \frac{K_1}{s-p} + \frac{K_2}{(s-p)^2} + \frac{K_3}{s}$$

$$K_1 = \frac{d\,(s-p)^2 Y_f(s)}{ds}\Big|_{s=p} = \frac{d}{ds}\left[\frac{bU_0}{s}\right]\Big|_{s=p} = -\frac{bU_0}{s^2}\Big|_{s=p} = -\frac{bU_0}{p^2} = -G_0 U_0$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y_f(t) = G_0 U_0 \left(1 - e^{-\frac{t}{\tau}} - \frac{t}{\tau} e^{-\frac{t}{\tau}}\right) 1(t)$$

$$t_{s\,1\%} = 6.6\tau$$

Partial fraction decomposition for multiple poles

- Note, in the case of multiple poles (with multiplicity r_i),

$$F(s) = \frac{N(s)}{(s-p_i)^{r_i}}$$

$$F(s) = \frac{N(s)}{(s-p_i)^{r_i}} = \sum_{l=1}^{r_i} \frac{K_{il}}{(s-p_i)^{r_i-l+1}}$$

- with

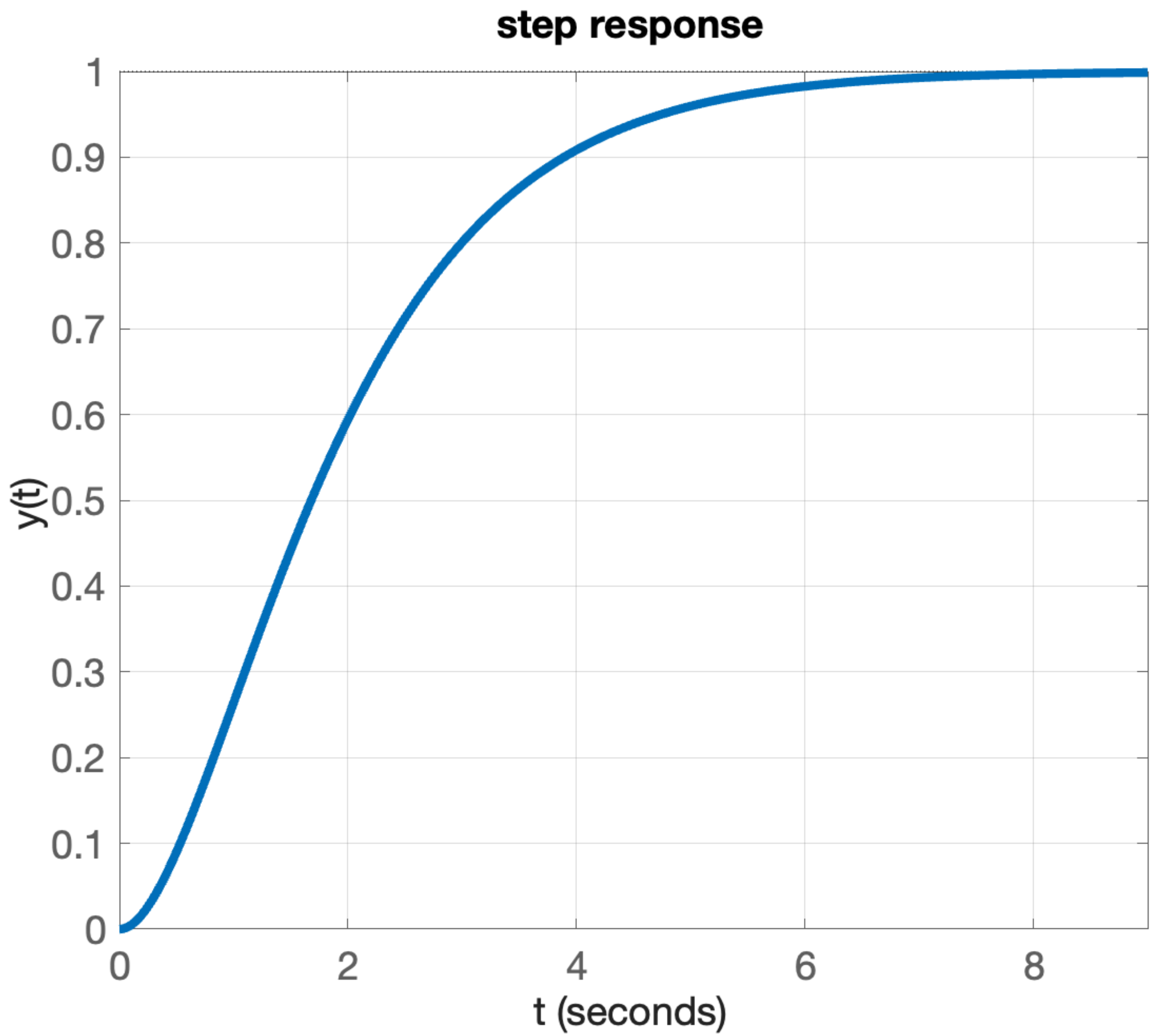
$$K_{i\ell} = \frac{1}{(\ell-1)!} \frac{d^{\ell-1}}{ds^{\ell-1}} (s-p_i)^{r_i} F(s)\Big|_{s=p_i}$$

$\mathcal{L}^{-1}$

$$f(t) = \sum_{l=1}^{r_i} \frac{K_{il}}{(\boldsymbol{r_i - l})!} \boldsymbol{t^{r_i - l} e^{p_i t}}$$

Second order system with two poles real and coincident

$\tau = 1\ s$



- An asymptotically stable second order system with one zero and two real negative poles has a transfer function in the form

$$G_0 = -{}^{bz}/_{(p_1 p_2}$$

$$G(s) = \frac{b(s-z)}{(s-p_1)(s-p_2)} = G_0 \frac{(1+\tau s)}{(1+\tau_1 s)(1+\tau_2 s)},$$

$$p_1 = -{}^1/_{\tau_1}$$
$$p_2 = -{}^1/_{\tau_2}$$
$$z = -{}^1/_{\tau}$$

- The analytic expression of the response to step signal $u(t) = U_0 1(t)$ is given by

$$Y(s) = G(s)U(s) = \frac{b(s-z)}{(s-p_1)(s-p_2)} \frac{U_0}{s} =$$

$$= \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \frac{K_3}{s} \qquad K_i = (s-p_i)Y_f(s)|_{s=p_i}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y_f(t) = G_0 U_0 \left(1 - \frac{\tau_1 - \tau}{\tau_1 - \tau_2} e^{-\frac{t}{\tau_1}} + \frac{\tau_2 - \tau}{\tau_1 - \tau_2} e^{-\frac{t}{\tau_2}}\right) 1(t)$$

Second order system with real poles and one zero

- The behavior of the step response depends on the position of the zero wrt the two poles.

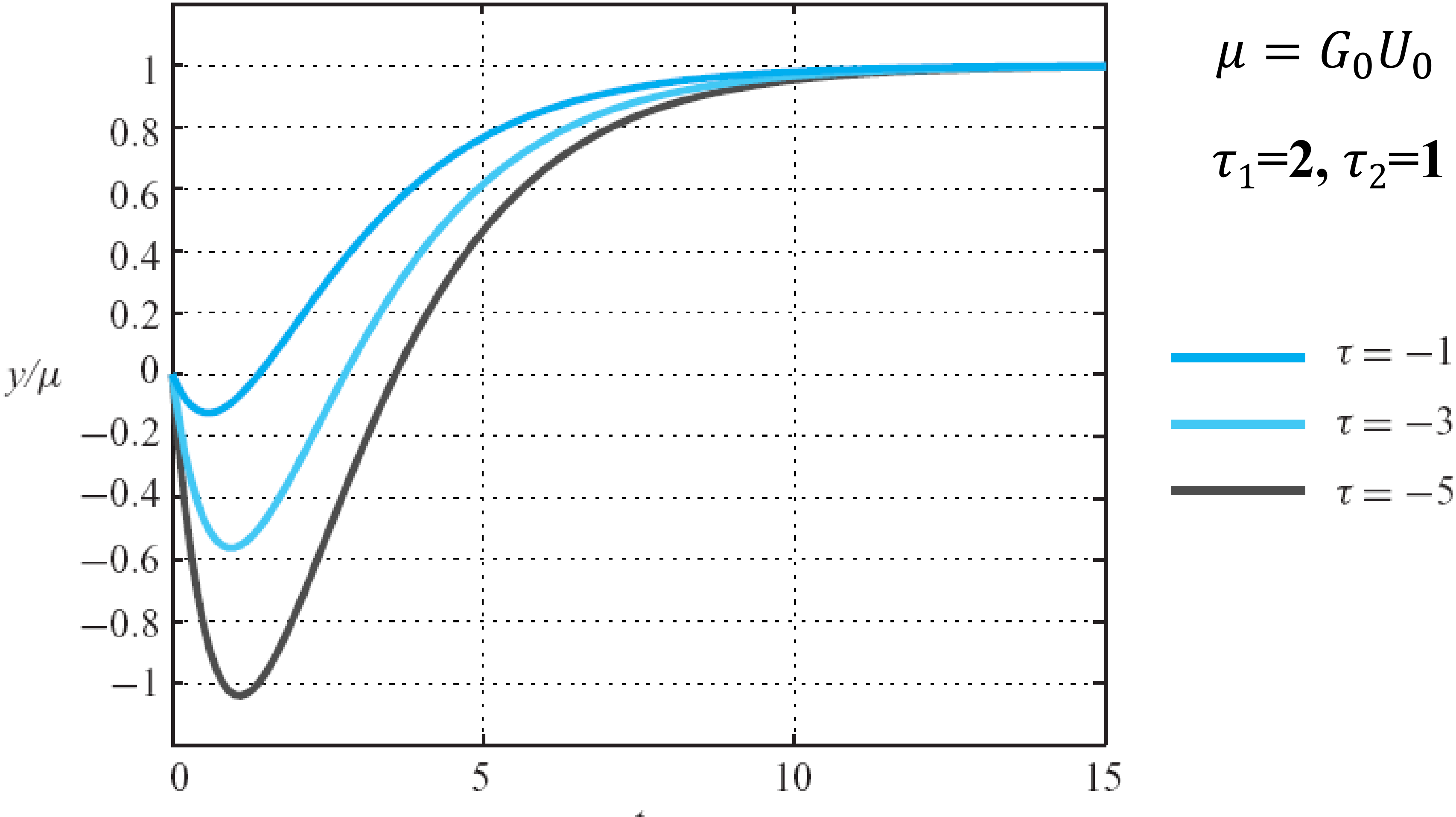
- In the following slide 3 possible cases will be shown:
  - a) Positive zero
  - b) Negative zero in the vicinity of the origin of the complex plane
  - c) Negative zero with an absolute value greater than the absolute values of the two poles

- By exploiting the initial theorem value:

$$y(0) = \lim_{s\to\infty} sY(s) = G(s) = 0$$

$$\dot{y}(0) = \lim_{s\to\infty} s^2Y(s) = sG(s) = G_0U_0\tau/(\tau_1\tau_2)$$

Second order system with real poles and one zero

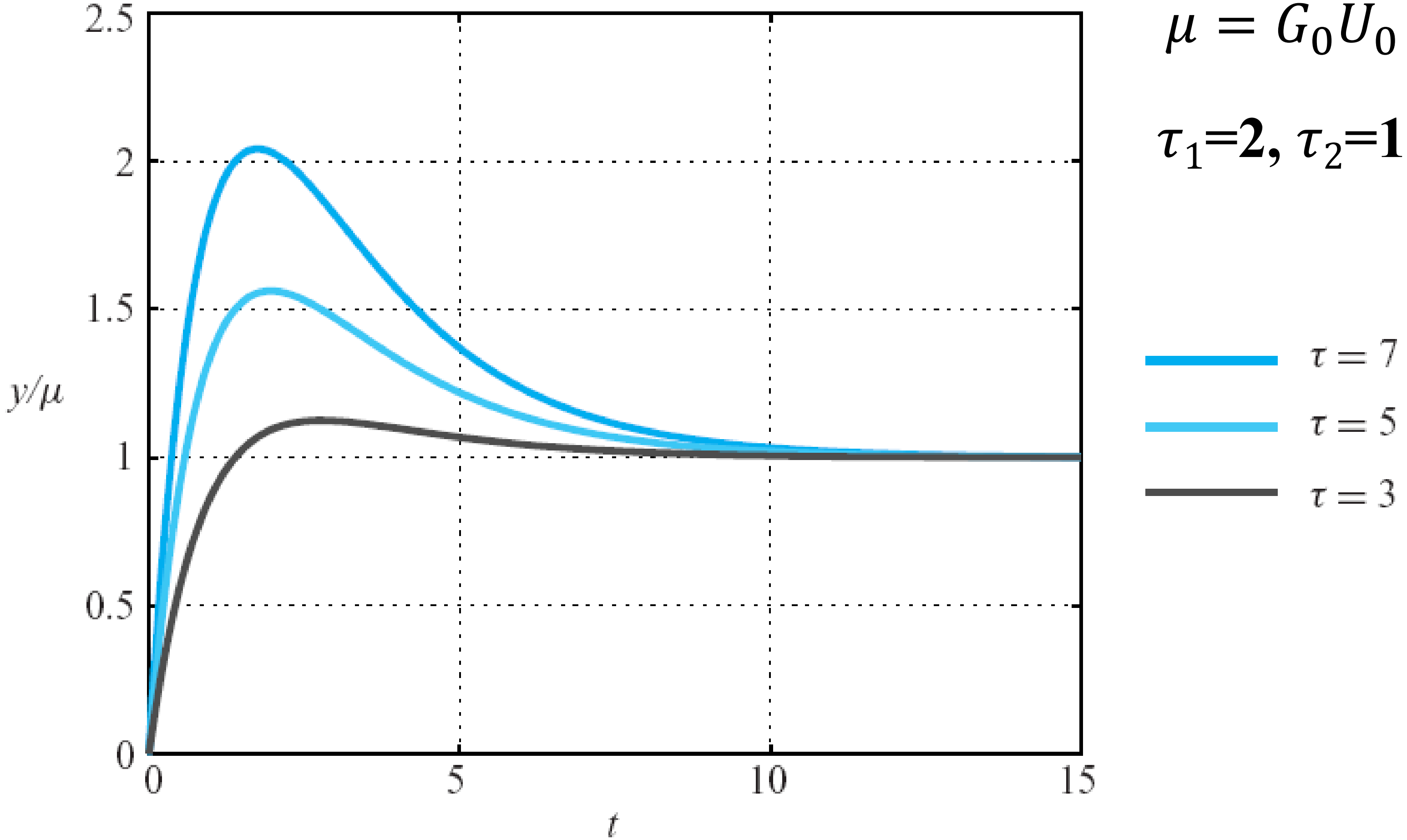


$\mu = G_0 U_0$

τ_1**=2,** τ_2**=1**

Undershoot, more pronounced by increasing the absolute value of $\boldsymbol{\tau}$ **(i.e., positive zero closer to origin)**

Second order system with real poles and one zero



$\mu = G_0 U_0$

τ_1**=2,** τ_2**=1**

Overshoot, more pronounced by increasing the value of $\boldsymbol{\tau}$ (i.e., negative zero closer to origin)

Second order system with real poles and one zero

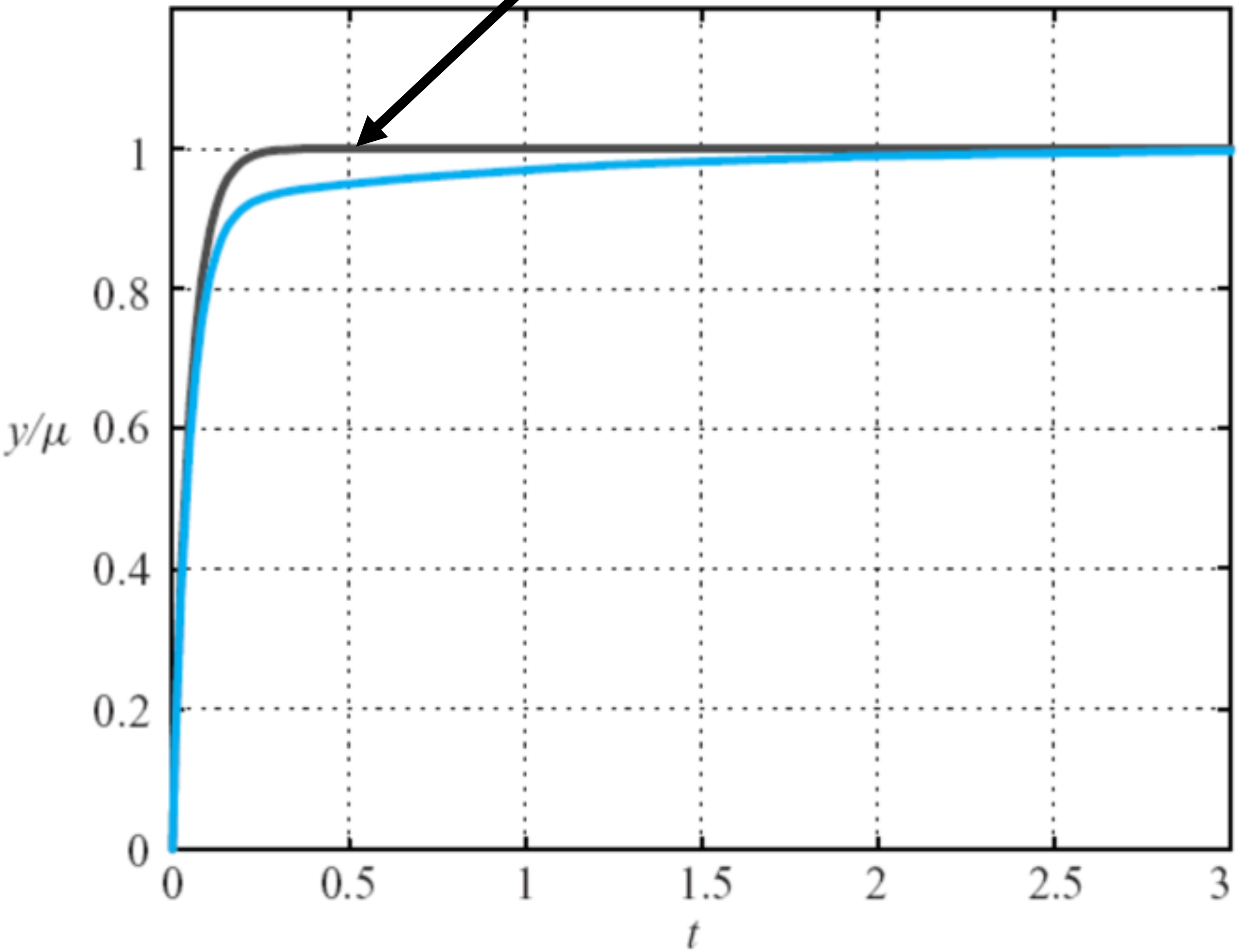
$$y(t) \approx G_0 U_0 \left(1 - e^{-\frac{t}{\tau_2}}\right) 1(t)$$

$\mu = G_0 U_0$

$\tau \approx \tau_1 > \tau_2$

Then,

$$\frac{\tau_1 - \tau}{\tau_1 - \tau_2} \sim 0$$

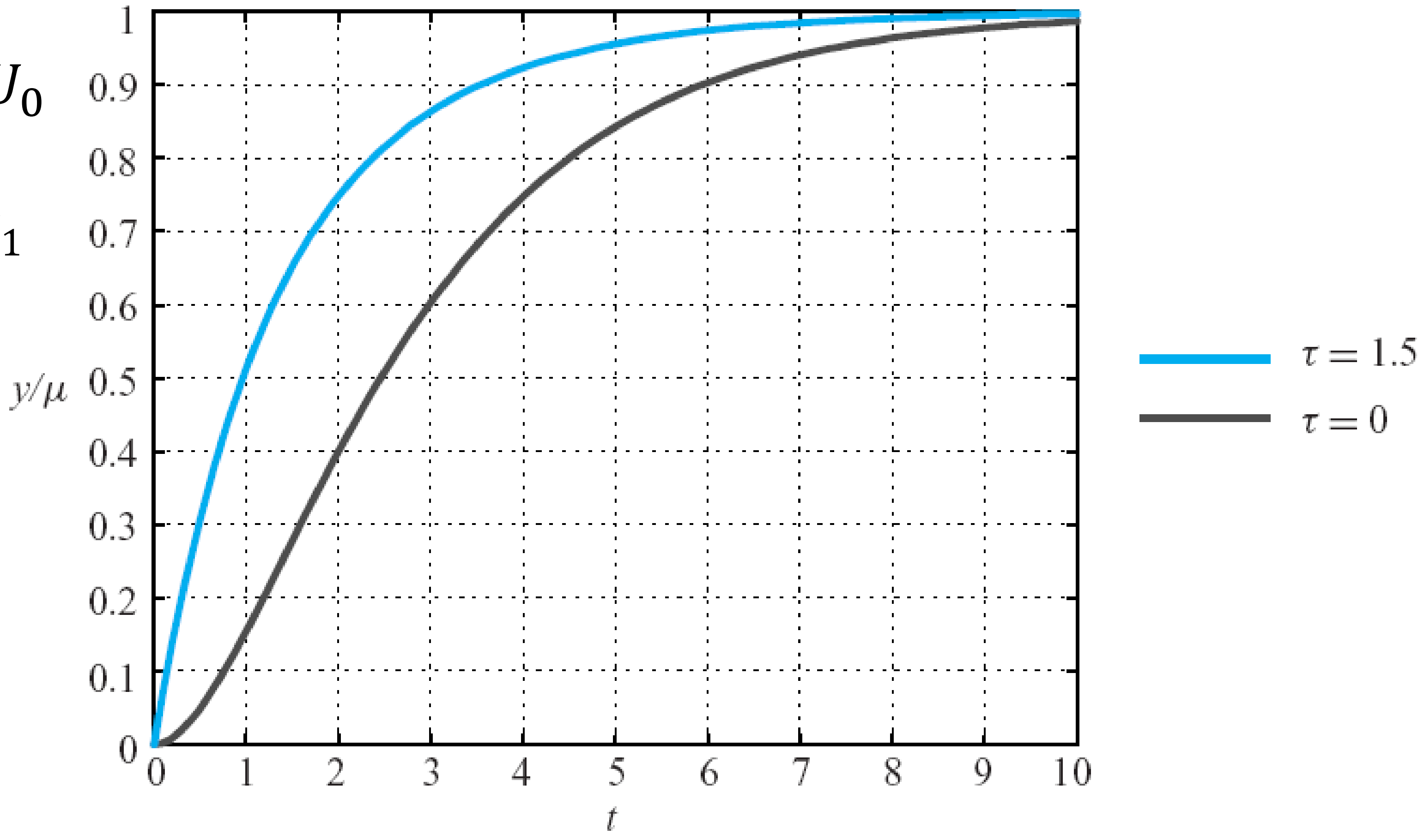


Second order system with real poles and one zero

$\mu = G_0 U_0$

$\tau_2 < \tau < \tau_1$

τ_1**=2**
τ_2**=1**



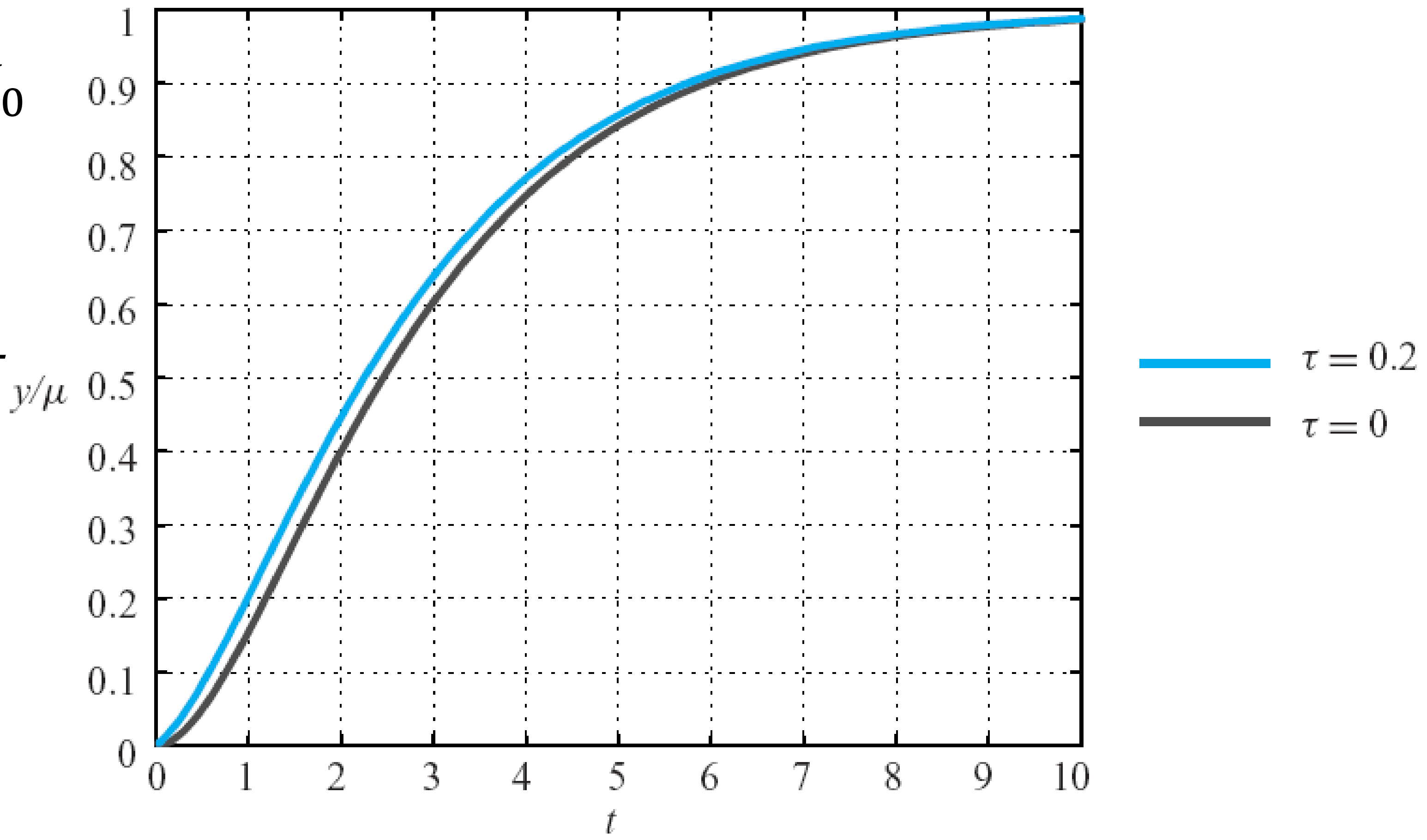
If $\tau \approx \tau_2$**, then** $y(t) \approx G_0 U_0 \left(1 - e^{-\frac{t}{\tau_1}}\right) 1(t)$

Second order system with real poles and one zero

$\mu = G_0 U_0$

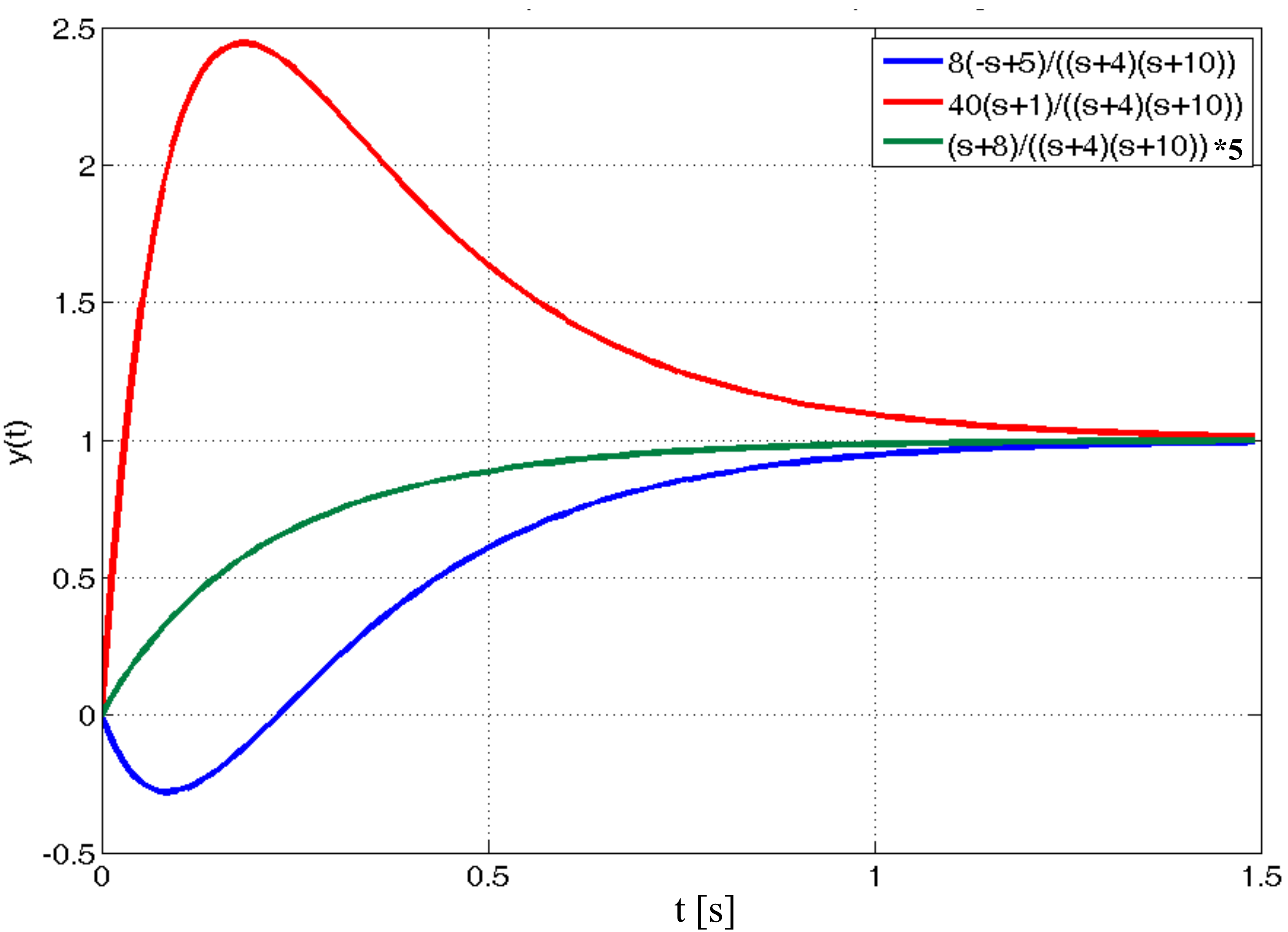
τ_1=**2**
τ_2=**1**

$\tau_1 > \tau_2 > \tau$



By decreasing τ, the response is similar to the one without zero

Second order system with real poles and one zero



Second order system with real poles and one zeros: parameters for the qualitative response

- ***Initial value*** $y(0) = 0$, $\dot{y}(0) \neq 0$
- ***Final value*** $lim_{t\rightarrow\infty}\, y(t) = G_0 U_0$
- ***Settling time***
  - $t_{s\ 5\%} = 3\tau_{max}$
  - $t_{s\ 1\%} = 4.6\tau_{max}$
- ***Rise time*** $t_r \cong 2.2\tau_{max}$

The settling time and the rise time also depend on the location on the zero. See the book for details (drift phenomenon)

Second order system with complex conjugates poles and no zeros

Let us assume an asymptotically stable second order system with t.f.

$$G(s) = \frac{b}{s^2 + a_1 s + a_0}$$

In the case of complex poles, $s^2 + a_1 s + a_0 = 0 \leftrightarrow p = \alpha + j\omega,\ \bar{p} = \alpha - j\omega$ with $\alpha < 0$ (i.e. asymptotically stable system), $W(s)$ can be rewritten by

$$G(s) = \frac{b}{(s-p)(s-\bar{p})} = \frac{b}{s^2 - 2\alpha s + \alpha^2 + \omega^2} = \frac{b}{(s-\alpha)^2 + \omega^2}$$

and the relative response to a step function $u(t) = U_0 1(t)$ is described by

$$Y(s) = G(s)\frac{U_0}{s} = \frac{b}{s^2 - 2\alpha s + \alpha^2 + \omega^2} \cdot \frac{U_0}{s} = \frac{A}{s} + \frac{Bs + C}{s^2 - 2\alpha s + \alpha^2 + \omega^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y(t) = G_0 U_0 \left(1 - \frac{1}{\sin\theta} e^{\alpha t} \sin(\omega t + \theta)\right) 1(t),$$

where $G_0 = \frac{b}{a_0} = \frac{b}{|p|^2} = \frac{b}{\alpha^2 + \omega^2}$, and $\theta = \tan^{-1}\left(-\frac{\omega}{\alpha}\right)$.

(see also the pdf file regarding *Analysis of LTI systems in the time domain* – i.e., *3. Time domain LTI systems analysis*)

The natural frequency is defined by

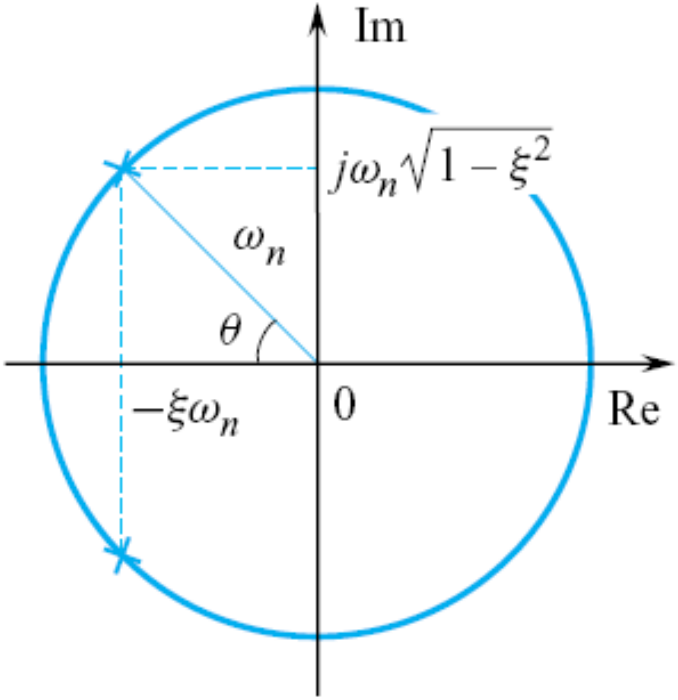
$$\omega_n{}^2 = \alpha^2 + \omega^2$$

and the damping factor

$$\zeta = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

$$\alpha = -\zeta\omega_n,$$

$$\omega = \omega_n\sqrt{1-\zeta^2}$$



$$\omega_n \cos\theta = \zeta\omega_n$$

$$\zeta = \cos\theta$$

Natural frequency and damping factor

- The ***natural frequency*** $\boldsymbol{\omega_n}$ is the oscillation frequency of the pseudo-periodic mode when $\alpha = 0$.

- ***For convergent*** pseudo-periodic modes, the ***damping coefficient*** $\zeta \in (0,1]$ while ***for divergent pseudo-periodic modes*** $\zeta \in [-1,0)$

- ***For convergent*** pseudo-periodic modes, the **damping coefficient** $\boldsymbol{\zeta}$ relates the oscillations of the pseudo-periodic mode to the time before the evolution will extinguish. For $\zeta \ll 1$

$$\zeta = -\frac{\alpha}{\omega_n} \cong -\frac{\alpha}{\omega} = \frac{T}{2\pi\tau} \ll 1$$

where T is the oscillation period. Indeed, the number of the oscillation before the mode will extinguish increases when $\boldsymbol{\zeta}$ becomes small.

$$\zeta = \frac{T}{2\pi\tau} \cong \frac{T}{6\tau} \quad \Longrightarrow \quad \frac{1}{2\zeta} \cong \frac{3\tau}{T}$$

# of oscillations before the mode will extinguish

Second order system with complex conjugates poles and no zeros

- The transfer function can also be rewritten in terms of ζ and ω_n ($0 < \zeta < 1$ for an asymptotic stable system)

$$G(s) = \frac{b}{s^2 - 2\alpha s + \alpha^2 + \omega^2}$$

$$= \frac{G_0 {\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

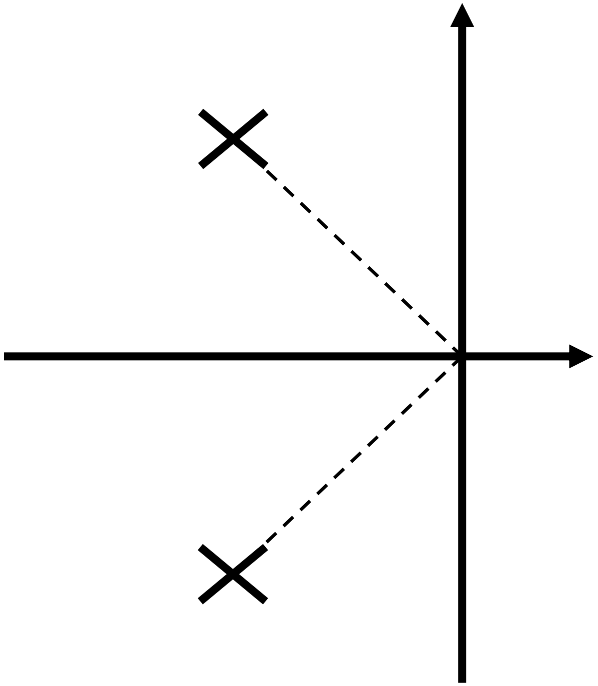
$$= \frac{G_0}{\frac{s^2}{{\omega_n}^2} + \frac{2\zeta}{\omega_n} s + 1}$$

with $G_0 = \frac{b}{\alpha^2+\omega^2} = \frac{b}{\omega_n^2}$, and the analytic expression of the step response, evaluated with the antitransform of $G(s)/s$, is given by ($k = G_0 U_0$)

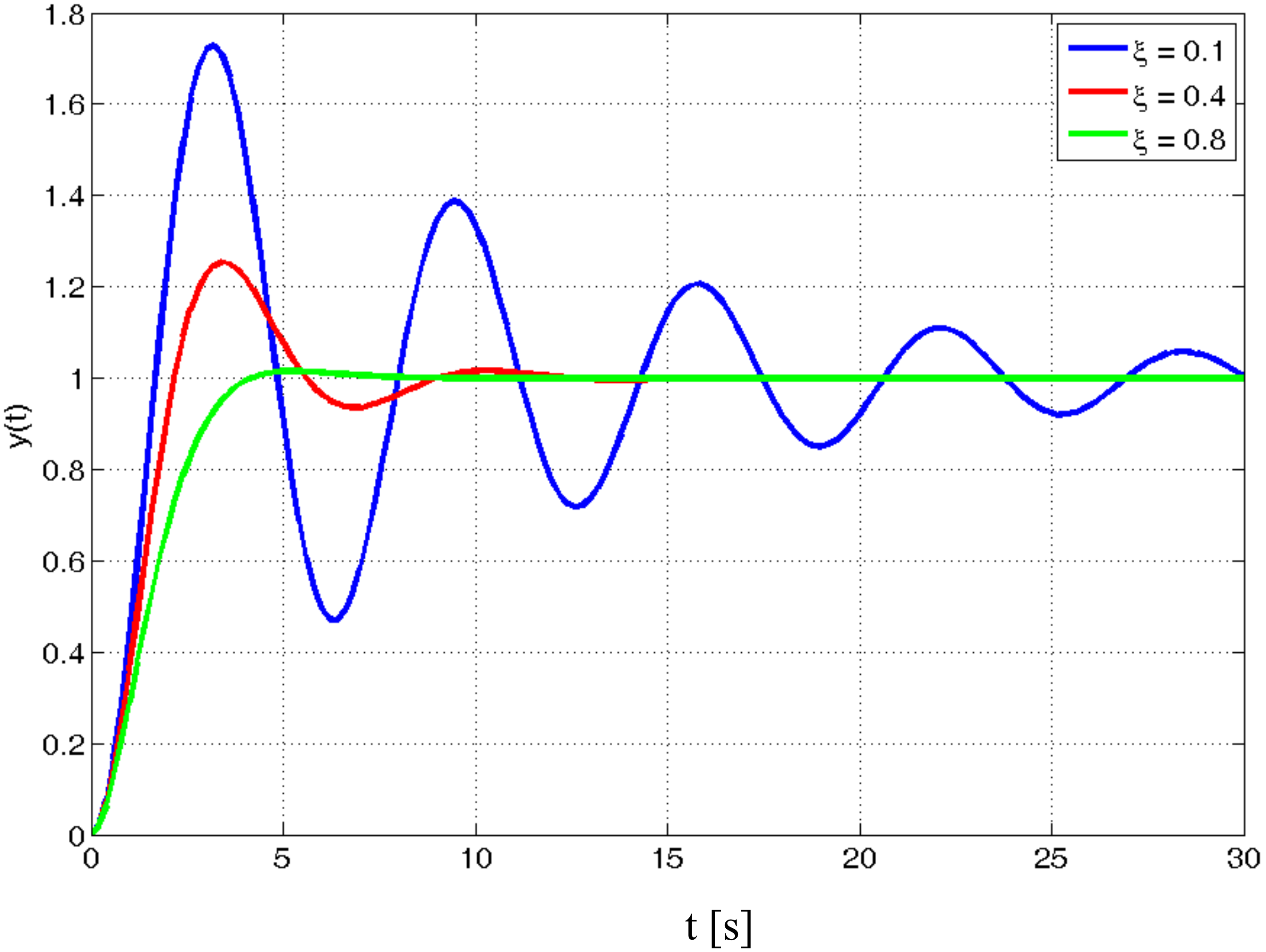
$$y(t) = k\left(1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos\left(\sqrt{1-\xi^2}\,\omega_n t - \arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right)\right) 1(t)$$

Second order system with complex conjugates poles and no zeros

- The behavior of the responce strongly depends of the value of ξ. In the following slide 3 possible cases will be shown:
  - $\xi \ll 1$ ($\xi = 0.1$)
  - $\xi = 0.4$
  - $\xi \cong 1$ ($\xi = 0.9$)



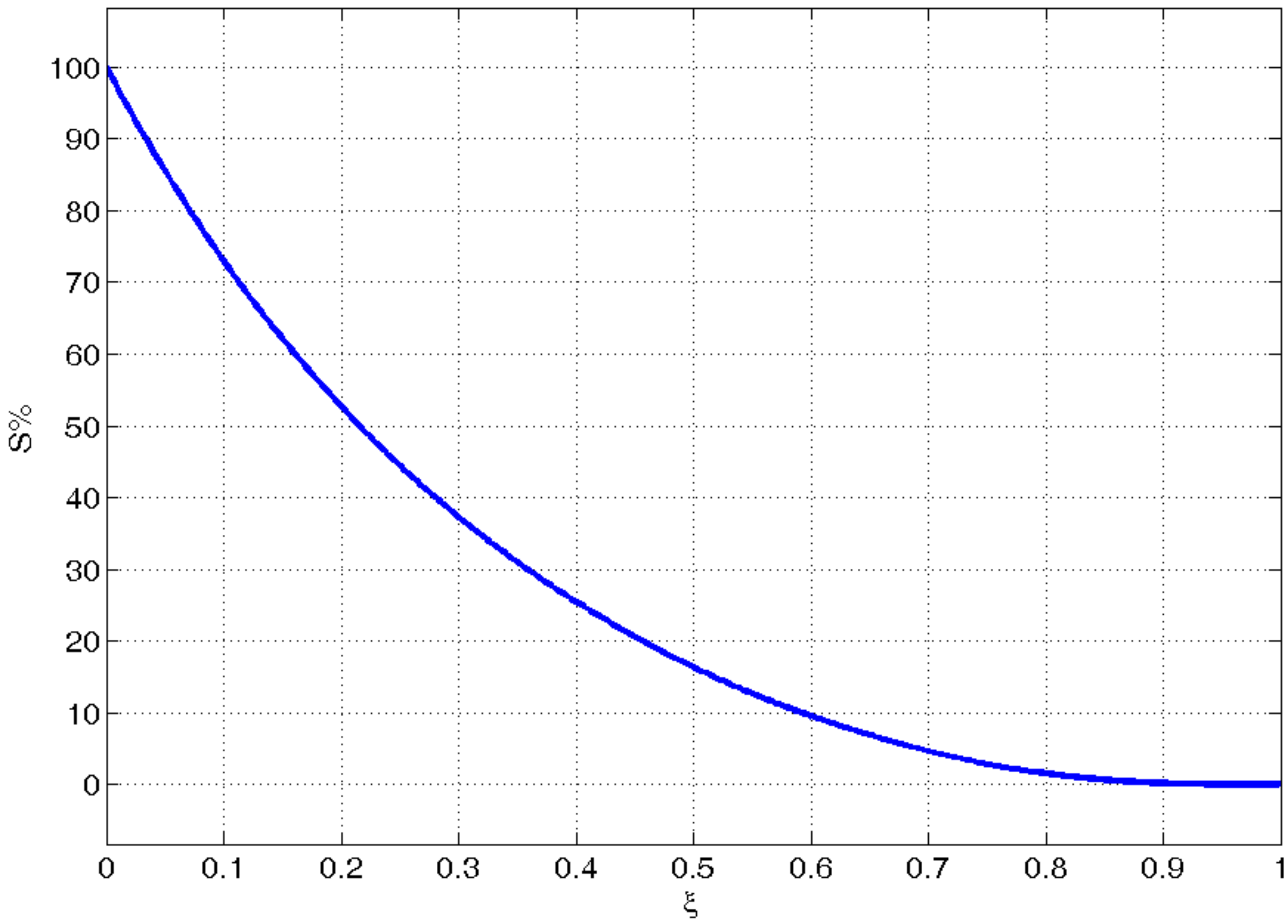
Second order system with complex conjugates poles and no zeros



Second order system with complex conjugates poles and no zeros

- ***Initial value*** $y(0) = 0\ ,\ \dot{y}(0) = 0$
- ***Final value*** $lim_{t\to\infty}\, y(t) = G_0 U_0$
- ***Settling time*** $\begin{cases} t_{s5\%} \cong 3/\zeta\omega_n & \zeta \ll 1 \\ t_{s5\%} \cong 4.75/\omega_n & \zeta \cong 1 \end{cases}$
- ***Rise time*** $\begin{cases} t_r \cong 1/\omega_n & \zeta \ll 1 \\ t_r \cong 3.4/\omega_n & \zeta \cong 1 \end{cases}$
- **Overshoot** $s = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$
- ***Peak time*** $t_p = \pi/\left(\omega_n\sqrt{1-\zeta^2}\right)$
- **Oscillation period** $T = 2\pi/\left(\omega_n\sqrt{1-\zeta^2}\right)$
- **# of oscillations** $= \frac{1}{2\zeta},\ \zeta \ll 1$

Overshoot



Examples

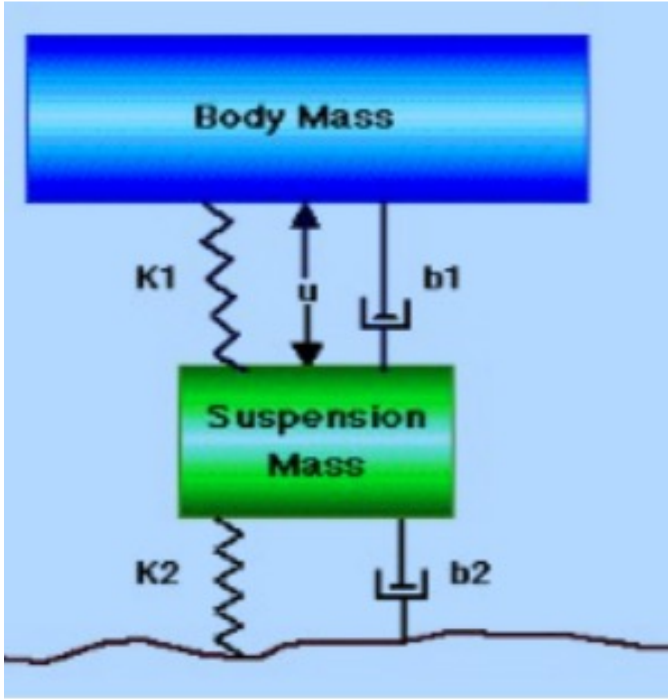
- Plot the qualitative step response of the following systems

$$W(s) = \frac{4}{s^2 + s + 2}$$

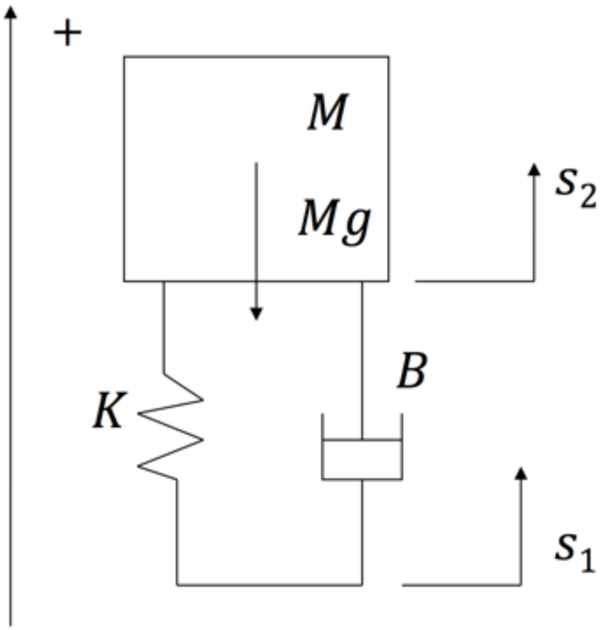
$$W(s) = \frac{4}{s^2 + 2s + 3}$$

$$W(s) = -\frac{4s}{s^2 + s + 2}$$

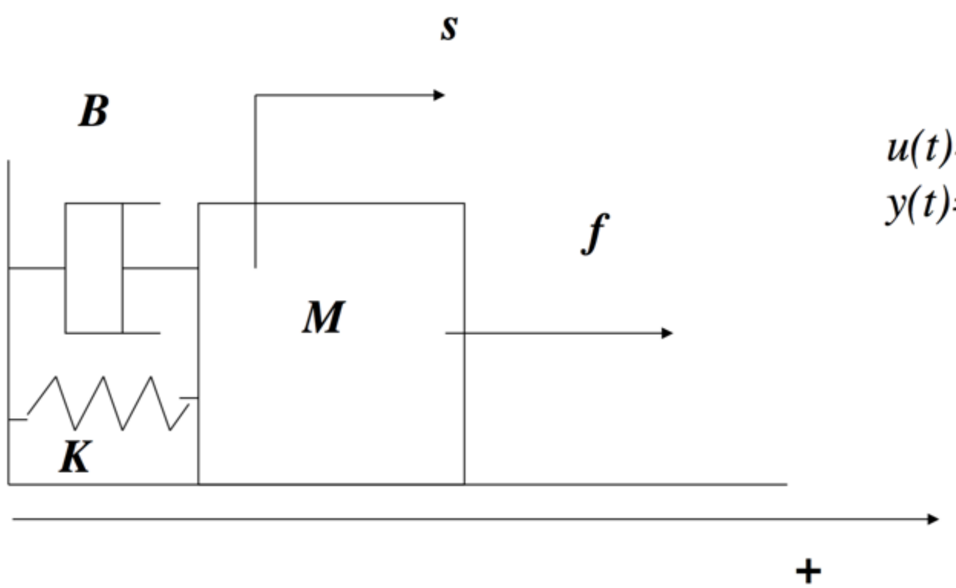
$$W(s) = \frac{(4s + 1)}{(s^2 + s + 2)}$$



Simplified model



Example: mass-spring-damper system



$u(t)=f(t)$
$y(t)=s(t)$

- Input output representation

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t)$$

- State space representation

$x_1 = s$ e $x_2 = v = ds/dt$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ \dot{s} \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/M \end{pmatrix} u,$$

$$\mathbf{y} = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$G(s) = \frac{\frac{1}{M}}{s^2 + \frac{B}{M}s + \frac{K}{M}}$$

In general a second order system…

- **Input output representation**

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = bu(t)$$

- **State space representation**

$x_1 = y$ e $x_2 = \dot{y} = dy/dt$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} u,$$

$$y = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- **Transfer function**

$$G(s) = \frac{b}{s^2 + a_1 s + a_0} = \frac{b}{(s - p_1)(s - p_2)},$$

real poles

complex poles

$G_0 = b/a_0$

$a_0 = |p|^2 = \alpha^2 + \omega^2 = {\omega_n}^2$

$$\frac{b}{(s - p)(s - \bar{p})} = \frac{b}{s^2 - 2\alpha s + \alpha^2 + \omega^2} = \frac{G_0 {\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

In general a second order system…

- Rewriting the characteristic equation $s^2+a_1 s+a_0=0$ in terms of ζ and ω_n,

$s^2+2\zeta\omega_n s+\omega_n{}^2=0,$

$$a_1 = 2\zeta\omega_n \qquad a_0 = \omega_n{}^2 \quad \Rightarrow \quad \zeta = \frac{a_1}{2\sqrt{a_0}} \qquad \omega_n = \sqrt{a_0}$$

- **$|\zeta|<1 \Rightarrow$ *complex conjugates poles (0< ζ<1 underdamped system)***
- **$|\zeta|=1 \Rightarrow$ *real multiple poles (ζ =1 critically damped system)***
- **$|\zeta|>1 \Rightarrow$ *real and distinct poles (ζ >1 overdamped system)***

The geometric interpretation of ζ is valid only for complex conjugates poles.

$\zeta < 0 \Rightarrow$ *unstable system*

In general a second order system…

- **|ζ|>1,**

$$y(t) = k\left(1 - \frac{\tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_2}{\tau_1 - \tau_2} e^{-t/\tau_2}\right) 1(t)$$

- **|ζ|=1,**

$$y(t) = k\left(1 - e^{-t/\tau} - \frac{t}{\tau} t e^{-t/\tau}\right) 1(t)$$

- **|ζ|<1,**

$$y(t) = k\left(1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos\left(\sqrt{1-\xi^2}\,\omega_n t - \arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right)\right) 1(t)$$

Problem 1.a

✧ ***Compute the analytic expression of the step response of the following LTI systems:***

- $G_1(s) = \frac{s+10}{s^2+6s+5}$; $G_2(s) = \frac{s+20}{s^2+s+1}$; $G_3(s) = \frac{-3(s-2)}{(s^2+4s+3)}$;
- $G_4(s) = \frac{s+14}{s^2+10s+30}$; $G_5(s) = \frac{s+24}{s^2+3s+45}$; $G_6(s) = \frac{s+15}{s^2+9s+20}$.

✧ ***Plot the step response for the different LTI systems***

Problem 1.b

✧ *Compute the analytic expression of the step response of the following LTI systems:*

- $G_1(s) = \frac{s}{s^2+6s+5}$; $G_2(s) = \frac{s}{s^2+s+1}$; $G(s) = \frac{20(s+0.1)}{(s^2+21s+20)}$
- $G_4(s) = \frac{10(s+3)}{(s+1/3)(s+9)}$; $G_5(s) = \frac{(1-10s)}{(s^2+3s+2)}$;
- $G_6\,(s) = \frac{10(10s+1)}{(s^2+101s+100)}$

✧ *Plot the step response for the different LTI systems*

Problem 2

- *Compute the transfer function of the following LTI system:*

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ \boldsymbol{a} & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \,,$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

- *Discuss the stability by varying a* $\in (-\infty, +\infty)$.

- *Compute the free evolution for the LTI system with a = -1 and* $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- *Plot the step response for the LTI system with a = -4.*

Problem 3

- ***Given the LTI system defined by the following state-space representation,***

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1/2 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u \,,$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

  - ***Compute the transfer function.***
  - ***Compute the analytic expression of the step response.***

- ***Plot the step response of the following LTI system defined by the following transfer function***

$$\boldsymbol{G(s) = \frac{-4(s-3)}{(s^2+5s+4)}}.$$

Problem 4

✧ ***Discuss the stability of the following LTI systems:***

1. $\dot{x} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$;

2. $G(s) = \frac{4(s+1)}{s^2(5s+1)}$;

3. $G(s) = \frac{-10}{s(s^2+3s+1)}$;

4. $$\begin{aligned} \dot{x} &= -kx + u \\ y &= x \end{aligned}$$

5. $G(s) = \frac{(s-k)}{(9s^2+2s+1)}$

For the systems at 4. and 5. discuss the stability by varying $k \in (-\infty, +\infty)$