Course of
"Automatic Control Systems"
2024/25

Laplace transform, transfer function, Laplace domain analysis of LTIs

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

Università degli studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code: **tz3jpwb**

Contents

Laplace transform definition

- ***The Laplace transform*** of a function ***f(t)*** is defined as

or $\mathcal{L}(f(t))$

$$f(t) \to F(s) = L(f(t)) = \int_{0}^{+\infty} f(t)e^{-st}dt$$

where $t \in R$ is a real variable, while $s = \alpha + j\omega \in C$ is a complex variable.

- Vice versa, given a function $F(s)$ in the Laplace domain, the original function in the time domain can be obtained using the ***Laplace anti-transformation***

$$F(s) \to f(t) = \lim_{\omega \to \infty} \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds$$

- ***The Laplace transform is a bilateral only if the function f(t) is null for $t < 0$***

Laplace transform main properties (1/2)

- ***Linearity***

$$L\left(af(t)+bg(t)\right)=aF(s)+bG(s)$$

- ***Translation in the Laplace domain***

$$L\left(e^{\alpha t}f(t)\right)=F(s-\alpha)$$

- ***Translation in the time domain***

$$L\left(f(t-T)\right)=F(s)e^{-sT}$$

Laplace transform main properties (2/2)

- ***Time domain derivation***

$$L\left(\frac{df(t)}{dt}\right)=sF(s)-f(0)$$

- ***Time domain integration***

$$L\left(\int_0^t f(\tau)\,d\tau\right)=\frac{1}{s}F(s)$$

- ***Time domain convolution***

$$L\big(f(t)*g(t)\big)=F(s)G(s)$$

Additional properties useful in control theory

- ***Initial value theorem***

$$f(0)=\lim_{s\to\infty} sF(s)$$

- ***Final value theorem***

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$$

- ***Initial value theorem of the derivate of the function***

$$\left.\frac{df(t)}{dt}\right|_{t=0}=\lim_{s\to\infty} s^2F(s)-sf(0)$$

Selected Laplace transforms

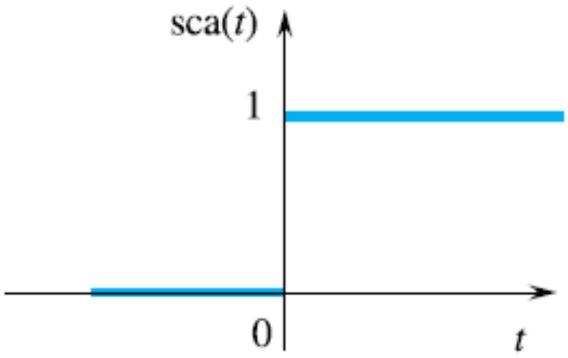
- In the system theory, we will mainly use the Laplace transform for the evaluation of the forced response of LTI systems to selected sets of input :

  - ***Polynomial inputs*** $\boldsymbol{u(t) = t^n 1(t)}$

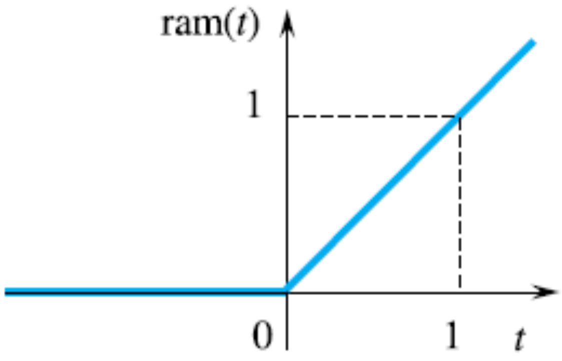
  - ***Sinusoidal inputs*** $\boldsymbol{u(t) = sin(\omega t)\, 1(t)}$

    $\boldsymbol{u(t) = cos(\omega t) 1(t)}$

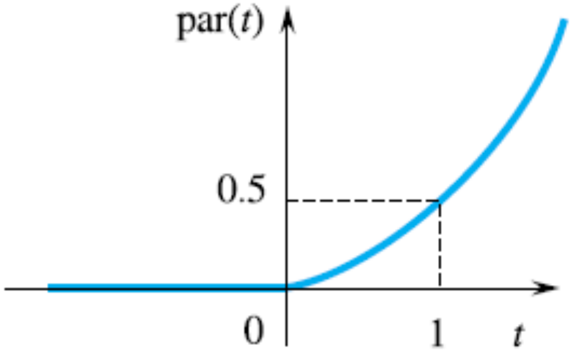
Selected Laplace transforms



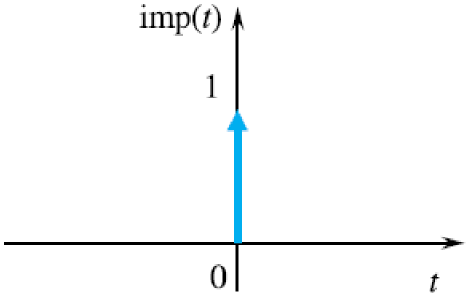
a)

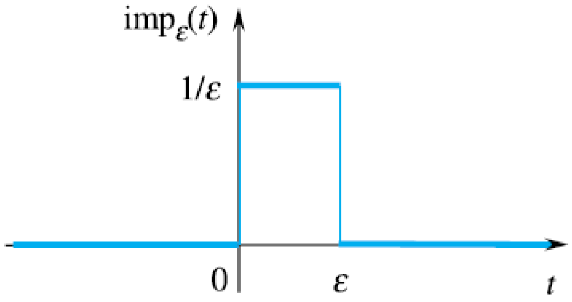


b)



c)





Selected Laplace transforms: polynomial signals

- In order to evaluate the Laplace transform of polynomial signals, let us firstly consider the Laplace transform of the impulse

  - ***Impulse $\delta(t)$*** $\longrightarrow$ $L(\delta(t)) = 1$ *(from the Laplace transform definition)*

- Then, using the ***time domain integration property***, we have

  - ***Step 1(t)*** $\longrightarrow$ $L(1(t)) = \dfrac{1}{s}$

  - ***Ramp $t \cdot 1(t)$*** $\longrightarrow$ $L(t \cdot 1(t)) = \dfrac{1}{s^2}$

  - ***Polinomial function $t^n \cdot 1(t)$*** $\longrightarrow$ $L(t^n \cdot 1(t)) = \dfrac{n!}{s^{n+1}}$

Selected Laplace transforms: sinusoidal signals

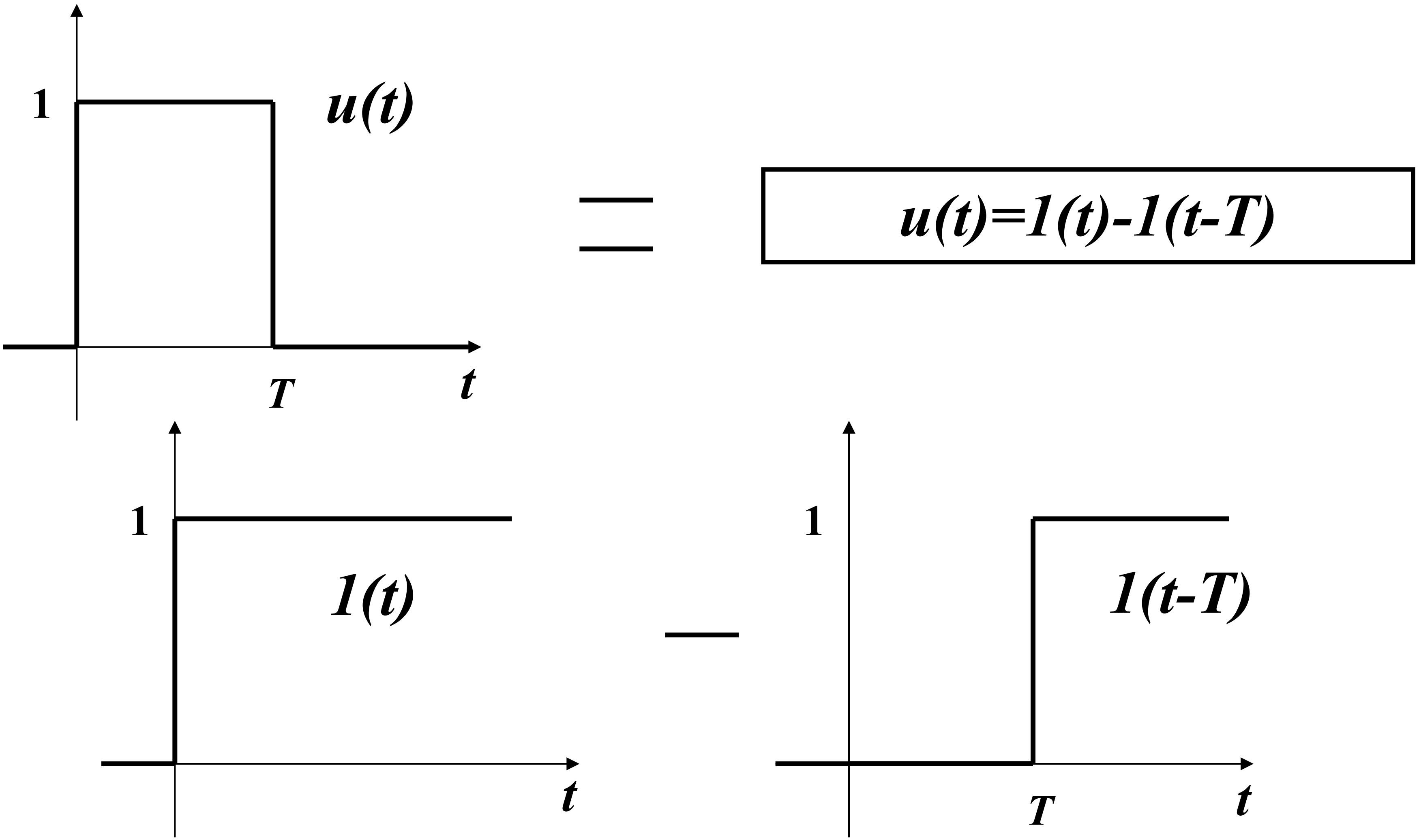
- The Laplace transform of sinusoidal functions
  - ***Sine*** $\boldsymbol{\sin(\omega t)1(t)}$ $\longrightarrow$ $L\left(\sin(\omega t)\cdot 1(t)\right)=\dfrac{\omega}{s^2+\omega^2}$
  - ***Cosine*** $\boldsymbol{\cos(\omega t)1(t)}$ $\longrightarrow$ $L(\cos(\omega t)\cdot 1(t))=\dfrac{s}{s^2+\omega^2}$
- Finally, in the control theory the following transformations are of interest for the definition of the Laplace domain of the evolution modes of LTI systems

$$L(e^{\alpha t}1(t))=\frac{1}{s-\alpha}$$

$$L\left(e^{\alpha t}\cos(\omega t)\cdot 1(t)\right)=\frac{s-\alpha}{(s-\alpha)^2+\omega^2}$$

$$L\left(e^{\alpha t}sen(\omega t)\cdot 1(t)\right)=\frac{\omega}{(s-\alpha)^2+\omega^2}$$

Example: Laplace transform of a window signal



$$\boxed{u(t)=1(t)-1(t-T)}$$

Example: Laplace transform of a window signal

- The Laplace transform of a window signal can be evaluated from the Laplace transforms of two steps.

$$
\begin{aligned}
L(u(t)) &= L(1(t) - 1(t-T)) \\
&= L(1(t)) - L(1(t-T)) \\
&= \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1-e^{-sT}}{s}
\end{aligned}
$$

Solution of first order linear differential equation

Let us consider a first order differential equation, linear with constant coefficients,

$$\dot{y}(t) + a_0 y(t) = b_0 u(t), \quad y(t_0) = y_0$$

By applying Laplace trasform, assuming a step input signal, ***$u(t)=U_0 1(t)$*****,** with amplitude U_0

$$L\big(\dot{y}(t) + a_0 y(t)\big) = L(b_0 U_0 1(t))$$

$$Y(s) = L(y(t))$$

$$L(bU_0 1(t)) = \frac{b_0 U_0}{s}$$

$$\Rightarrow \quad sY(s) - y_0 + a_0 Y(s) = \frac{b_0 U_0}{s} \Rightarrow Y(s) = \underbrace{\frac{y_0}{s + a_0}}_{Y_{free}} + \underbrace{\frac{b_0 U_0}{s(s + a_0)}}_{Y_{forced}}$$

Solution of first order linear differential equation

$$Y_{free}(s) = \frac{y_0}{s + a_0} \xrightarrow{\mathcal{L}^{-1}} y_{free}(t) = e^{-\boldsymbol{a_0 t}} y_0 1(t)$$

$$Y_{forced}(s) = \frac{b_0 U_0}{s(s + a_0)} = \frac{A}{s} + \frac{B}{s + a_0}$$

Compute ***A*** and ***B*** by equating:

$$\boldsymbol{Y_{forced}}(s) = \frac{A(s + a_0) + Bs}{s(s + a_0)}$$

$$= \frac{(A + B)s + Aa_0}{s(s + a_0)}$$

$$\begin{cases} A + B = 0 \\ Aa_0 = bU_0 \end{cases} \Rightarrow \quad \boldsymbol{A} = \frac{b_0 U_0}{a_0} \quad \boldsymbol{B} = -\frac{b_0 U_0}{a_0}$$

Or by residual method:

$$A = (s - 0) Y_{forced}(s)|_{s=0}$$

$$= \frac{b_0 U_0}{s + a_0}|_{s=0} = \frac{b_0 U_0}{a_0}.$$

$$B = (s - (-a_0)) Y_f(s)|_{s=-a_0}$$

$$= \frac{b_0 U_0}{s}|_{s=-a_0} = -\frac{b_0 U_0}{a_0}.$$

Solution of first order linear differential equation

$$Y_{forced}(s)=\frac{b_0U_0}{s(s+a_0)}=\frac{A}{s}+\frac{B}{s+a_0}=\frac{\frac{b_0U_0}{a_0}}{s}+\frac{-\frac{bU_0}{a_0}}{s+a_0}$$

$$\mathcal{L}^{-1}$$

$$y_{forced}(t)=\frac{b_0U_0}{a_0}1(t)-\frac{b_0U_0}{a_0}e^{-\boldsymbol{a_0t}}1(t)=\frac{b_0U_0}{a_0}(1-e^{-\boldsymbol{a_0t}})1(t)$$

Then,

$$y(t)=y_{free}(t)+y_{forced}(t)=\left(e^{-\boldsymbol{a_0t}}y_0+\frac{b_0U_0}{a_0}(1-e^{-\boldsymbol{a_0t}})\right)1(t)$$

Laplace Transform and Transfer function

- The analysis of LTI system is simplified by using Laplace transform.
- By exploiting the important property of the Laplace transform of the derivative of a signal $f(t)$ (with zero initial conditions, i.e. $f(0) = 0$)

$$\mathcal{L}\left(\dot{f}(t)\right) = sF(s),$$

- Given the differential equation of a linear system, it is possible to find the transfer function, $G(s)$, of that system, defined by

$$G(s) = \frac{Y(s)}{U(s)}$$

- Then for a LTI system of first order described by

$$\dot{y}(t) + a_0 y(t) = b_0 u(t), y(0) = y_0 = 0$$

$$\Big\downarrow \mathcal{L}$$

$$sY(s) + a_0 Y(s) = b_0 U(s)$$

$$Y(s)(s + a_0) = b_0 U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s + a_0}$$

Laplace Transform and Transfer function

- Then, for a LTI system of second order described by

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t), y(0) = 0, \dot{y}(0) = 0$$

$\mathcal{L}$

$$s^2Y(s) + a_1 sY(s) + a_0 Y(s) = b_0 U(s)$$

$$Y(s)(s^2 + a_1 s + a_0) = b_0 U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^2 + a_1 s + a_0}$$

- Therefore, given the transfer function $G(s)$ and the input $u(t)$ with transfer function $U(s)$, the output is the product

$$Y(s) = G(s)U(s)$$

- Using Laplace transforms, the output $Y(s)$ can be expanded into its elementary terms (i.e., the sum of well-known transforms of selected/common signals/functions) by partial fraction expansion: therefore, it is possible to rewrite $Y(s)$ as a sum of terms for which is known the anti-transformation; the total time function $y(t)$ is given by the sum of these anti-transformation terms.

Laplace Transform and Transfer function

- Using Laplace transforms, the output $Y(s)$ can be expanded into its elementary terms (i.e., the sum of well-known transforms of selected/common signals/functions) by partial fraction expansion: therefore, it is possible to rewrite $Y(s)$ as a sum of terms for which is known the anti-transformation; the total time function $y(t)$ is given by the sum of these anti-transformation terms.

$$\begin{tikzcd}
Y(s) = \arrow[d, bend right, "\mathcal{L}^{-1}"'] & Y_1(s) + \arrow[d, "\mathcal{L}^{-1}"'] & Y_2(s) + \cdots \arrow[d, "\mathcal{L}^{-1}"'] \\
y(t) = & y_1(t) + & y_2(t) + \cdots
\end{tikzcd}$$

Y_i as,

$$Y_i(s) = \frac{A}{s} \xrightarrow{\mathcal{L}^{-1}} y_i(t) = A \cdot 1(t); \qquad Y_i(s) = \frac{A}{s-\alpha} \xrightarrow{\mathcal{L}^{-1}} y_i(t) = A \cdot e^{\alpha t} 1(t)$$

$$Y_i(s) = \frac{\omega}{(s-\alpha)^2+\omega^2} \xrightarrow{\mathcal{L}^{-1}} y_i(t) = e^{\alpha t} \sin(\omega t)\, 1(t)$$

$$Y_i(s) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2} \xrightarrow{\mathcal{L}^{-1}} y_i(t) = e^{\alpha t} \cos(\omega t)\, 1(t)$$

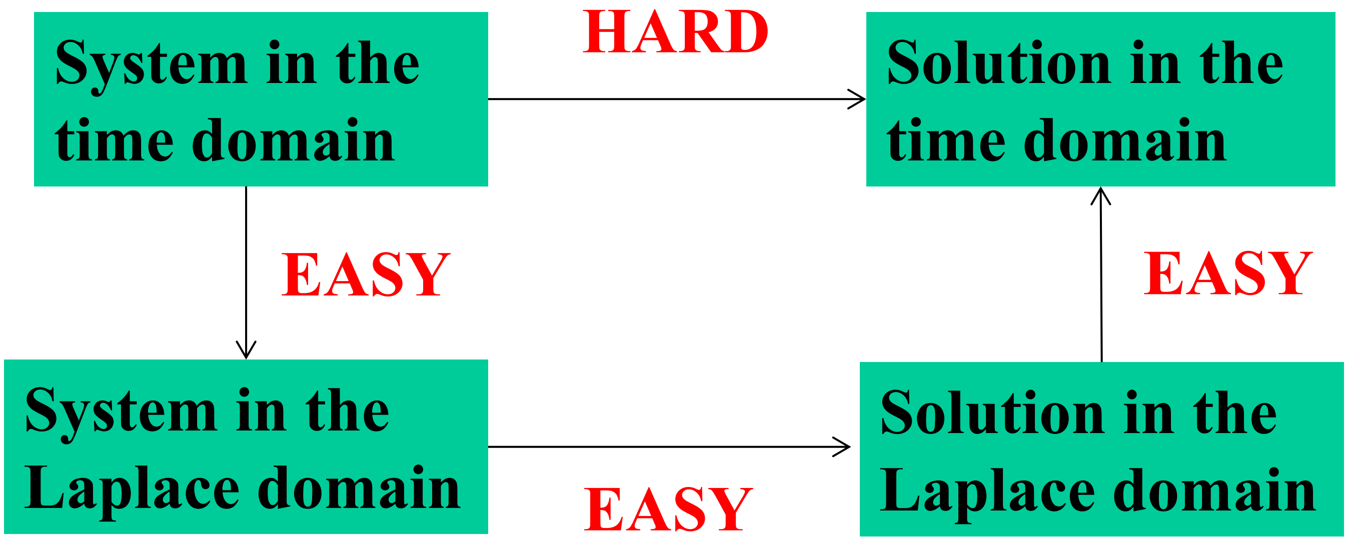
Evaluation of an LTI system response

- Let us consider a Linear Time Invariant (LTI) system in the state space form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \tag{1.a}$$

$$y(t) = Cx(t) + Du(t) \tag{1.b}$$

- The Evaluation of an LTI system response in a transformed domain is convenient only if



System in the time domain

HARD

Solution in the time domain

EASY

EASY

System in the Laplace domain

EASY

Solution in the Laplace domain

LTI systems in the Laplace domain

- Let us indicate with ***$X(s), U(s)$ and $Y(s)$ the Laplace transforms of the signals $x(t), u(t)$ and $y(t)$.***

Transforming both the sides of the equation (1a),

$$L(\dot{x}(t)) = L\big(Ax(t) + Bu(t)\big)$$

using the time domain derivation property of the Laplace transform, a linear system in the Laplace domain can be written has

$$sX(s) - x_0 = AX(s) + BU(s)$$

we have

$$(sI - A)X(s) = x_0 + BU(s)$$

and

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

The matrix function $\Phi(s) = (sI - A)^{-1}$is called ***Transition matrix,*** then

$$X(s) = \Phi(s)x_0 + \Phi(s)BU(s)$$

Recall: Inverse of a matrix

- Given a quadratic and invertible matrix

$$A = \begin{pmatrix} x_{1,1} & \dots & x_{1,j} \\ \vdots & \ddots & \vdots \\ x_{i,1} & \dots & x_{i,j} \end{pmatrix}$$

its inverse is defined as

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \mathrm{cof}(A, x_{1,1}) & \dots & \mathrm{cof}(A, x_{1,j}) \\ \vdots & \ddots & \vdots \\ \mathrm{cof}(A, x_{i,1}) & \dots & \mathrm{cof}(A, x_{i,j}) \end{pmatrix}^T$$

where the cofactor is

$$\mathrm{cof}(A, i, j) = (-1)^{i+j} \det(\mathrm{minor}(A, i, j))$$

and the minor (i, j) is the determinant of the matrix obtained excluding the row i and the column j.

Inverse of a 2×2 matrix

- Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a 3×3 matrix

- Given a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

its inverse is

$$\frac{1}{\det(A)} \begin{pmatrix} +\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & +\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\ \\ -\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & +\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\ \\ +\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} & +\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \end{pmatrix}$$

Transition matrix

- For the ***Transition matrix*** $\Phi(s) = (sI - A)^{-1}$
  - Each element is a rational function in ***s*** variable*:*
    - denominator of degree ***n*** given by $\mathbf{det}(\boldsymbol{sI} - \boldsymbol{A}) = \boldsymbol{p_A}(\boldsymbol{s})$, whose roots are the **eigenvalues** of **A**.
    - numerator of element (i,j) corresponds to the algebraic complement of element (j,i) which by construction is a degree at most ***n***-1

⇨ $$\boldsymbol{\Phi_{ij}(s) = \frac{N(s)}{D(s)}}$$

$\boldsymbol{N(s)}$ of degree at most ***n*-1**

$\boldsymbol{D(s)}$ of degree **n,**

LTI systems in the Laplace domain

- Transforming both the sides of the equation (1b), we have

$$L(y(t)) = L\big(Cx(t) + Du(t)\big) \Leftrightarrow Y(s) = CX(s) + DU(s)$$

- and by substituting the previous equation, $X(s) = \Phi(s)x_0 + \Phi(s)BU(s)$

$$Y(s) = C\Phi(s)x_0 + C\Phi(s)BU(s) + DU(s)$$

$$Y(s) = C\Phi(s)x_0 + (C\Phi(s)B + D)U(s)$$

- The matrix function $G(s) = C\Phi(s)B + D = C(sI - A)^{-1}B$ is called ***transfer function,*** therefore

$$Y(s) = C\Phi(s)x_0 + G(s)U(s)$$

- For ***Single Input Single Output (SISO)*** systems the transfer function $G(s)$ is a scalar function;

- For ***Multiple Input Multiple Output (MIMO)*** systems the transfer function $G(s)$ is a matrix whose element $G(s)_{ij}$ will connect the output i with the input j.

Transfer function

- For the *Transfer function* $\boldsymbol{G(s)} = \boldsymbol{C}\Phi(s)B + D = C(sI - A)^{-1}B + D$

$$\boldsymbol{\Phi_{ij}(s)} = \frac{\boldsymbol{N(s)}}{\boldsymbol{D(s)}}$$

$\boldsymbol{N(s)}$ of degree at most ***n*-1**

$\boldsymbol{D(s)}$ of degree **n,**

$\boldsymbol{G(s)}$ is a rational function in ***s*** variable*:*

$$G(s) = \frac{a_m\, s^m + a_{m-1}\, s^{m-1} + \dots + a_1\, s + a_0}{b_n\, s^n + a_{n-1}\, s^{n-1} + \dots + b_1\, s + b_0}$$

- Since the multiplication on the left of $\boldsymbol{\Phi(s)}$ by ***C*** and the one on the right by ***B*** correspond to a linear combination of $\boldsymbol{\Phi(s)}$ elements, all with the same denominator, i.e. $\det(sI - A)$, then all the elements of $\boldsymbol{C}\Phi(s)B$ are rational functions in ***s*** with a denominator polynomial of degree ***n*** and a numerator of degree $\boldsymbol{m} \leq n - 1$:

- **If *D*=0, *m* < *n*,** ***the system is said strictly proper.***

- **If *D*≠ 0, *m* = *n*,** ***the system is said proper.***

Transfer function calculation: examples

$$\dot{x} = \begin{pmatrix} -2 & -1.5 \\ 2 & 0 \end{pmatrix} x + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} x$$

$$\Downarrow$$

$$G(s) = \frac{s+2}{s^2+2s+3}$$

$$\dot{x} = \begin{pmatrix} -4 & -2.5 \\ 2 & 0 \end{pmatrix} x + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} -1.5 & -1.25 \end{pmatrix} x + u$$

$$\Downarrow$$

$$G(s) = \frac{s^2+s}{s^2+4s+5}$$

Transfer function

- Given a ***transfer function***

$$G(s) = \frac{N(s)}{D(s)} = \frac{a_m\, s^m + a_{m-1}\, s^{m-1} + \cdots + a_1\, s + a_0}{b_n\, s^n + a_{n-1}\, s^{n-1} + \cdots + b_1\, s + b_0}$$

- The roots of the $\mathrm{N(s)}$ are said ***zeros.***

- The roots of the $\mathrm{D(s)}$ are said ***poles.***

- The polynomial $\mathrm{D(s)}$ is defined as $D(s) = \det(sI - A)$, hence
  - ***D(s) coincides with the characteristic polynomial of the system***
  - ***the poles coincide with the eigenvalues of the system*** except for possible pole-zero cancellation

LTI systems in the Laplace domain

Then, for a LTI system, by Laplace transform the state equation:

$$L(\dot{x}(t)) = L\big(Ax(t) + Bu(t)\big), \quad x(t_0) = x_0 \Rightarrow$$

$$\Rightarrow sX(s) - x_0 = AX(s) + BU(s) \Rightarrow (sI - A)X(s) = x_0 + BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

$$X(s) = \boxed{\Phi(s)}x_0 + \Phi(s)BU(s)$$

Transition matrix

By Laplace transform the output equation: $L(y(t)) = L\big(Cx(t) + Du(t)\big)$

$$Y(s) = CX(s) + DU(s)$$

$$\Rightarrow = C\Phi(s)x_0 + C\Phi(s)BU(s) + DU(s)$$

$$= C\Phi(s)x_0 + \boxed{\big(C\Phi(s)B + D\big)}U(s)$$

G(s): transfer function

$$Y(s) = \overbrace{C\Phi(s)x_0}^{Y_{free}(s)} + \underbrace{G(s)U(s)}_{Y_{forced}(s)}$$

Laplace antitransform

- For SISO systems ***the free evolution in the Laplace domain is given by the ratio of polynomial functions***

$$Y_{free}(s) = C\Phi(s)x_0$$

- ***This is also true for the forced evolution in case we restrict our attention to the case of polynomial and sinusoidal inputs***

$$Y_{forced}(s) = G(s)U(s)$$

- It is convenient to antitransform $Y(s)$ by reducing the ratio of high degree polynomial functions to the sum of selected signals transform such as

$$L\left(e^{\alpha t}\cos\left(\omega t\right)\cdot 1(t)\right) = \frac{s-\alpha}{\left(s-\alpha\right)^2+\omega^2} \qquad L\left(e^{\alpha t} sen\left(\omega t\right)\cdot 1(t)\right) = \frac{\omega}{\left(s-\alpha\right)^2+\omega^2}$$

$$L(e^{\alpha t}1(t)) = \frac{1}{s-\alpha}$$

Laplace antitransform

- Different methods can be used to reduce the ratio of high degree polynomial functions to the sum polynomial functions of degree one or two, such as ***the residual method*** (see the book for details).

Residual method for real and distinct eigenvalues

(see the book for the other cases)

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_{i=1}^{n}(s-p_i)} \qquad p_i \neq p_j \text{ for } i \neq j$$

- In case of real and distinct eigenvalues, $F(s)$ can be also written as

$$F(s) = \sum_{i=1}^{n} \frac{A_i}{s-p_i}$$

where $A_k = \lim_{s \to p_k} (s-p_k)F(s)$. Hence

$$f(t) = \sum_{i=1}^{n} A_i \, e^{p_i t}$$

LTI system, first order, strictly proper (d=0)

Consider a first order differential equation, linear with constant coefficients,

$$\dot{y}(t) + a_0 y(t) = b_0 u(t), \quad y(t_0) = y_0$$

that can be described by LTI system as

$$\dot{x}(t) = ax(t) + bu(t), \quad x(t_0) = x_0$$

$$y(t) = x(t)$$

Note that $b = b_0, a = -a_0, x_0 \ = y_0$

By Laplace transform

$$Y(s) = X(s) = \Phi(s)x_0 + G(s)U(s),$$

Where

$$\Phi(s) = \frac{1}{s-a}, G(s) = \Phi(s)b = \frac{b}{s-a}$$

Then,

$$Y(s) = \overbrace{\frac{y_0}{s-a}}^{\boldsymbol{Y(s)_{free}}} + \underbrace{\frac{b}{s-a}U(s)}_{\boldsymbol{Y(s)_{forced}}}$$

LTI system, first order, strictly proper: free and force responses

$$\tau = -\frac{1}{a}$$

$$Y_{free}(s) = \frac{y_0}{s-a} \xrightarrow{\mathcal{L}^{-1}} y_{free}(t) = e^{\boldsymbol{at}} y_0 1(t) = e^{-\frac{t}{\tau}} y_0 1(t)$$

$$u(t) = U_0 1(t)$$

$$Y_{forced}(s) = \frac{bU_0}{s(s-a)} = \frac{A}{s} + \frac{B}{s-a}$$

Compute ***A*** and ***B*** by imposing the equality :

$$Y_{\boldsymbol{forced}}(s) = \frac{A(s-a) + Bs}{s(s-a)}$$

$$= \frac{(A+B)s - Aa}{s(s-a)}$$

$$\begin{cases} A + B = 0 \\ -Aa = bU_0 \end{cases} \Rightarrow \quad \boldsymbol{A} = \frac{bU_0}{-a} \quad \boldsymbol{B} = \frac{bU_0}{a}$$

Or by residual method:

$$A = (s-0)Y_{forced}(s)|_{s=0} = \frac{bU_0}{s-a}|_{s=0} = \frac{bU_0}{-a}.$$

$$B = (s-a)Y_f(s)|_{s=a} = \frac{bU_0}{s}|_{s=a} = \frac{bU_0}{a}.$$

LTI system, first order, strictly proper: free and force responses

$$Y_{forced}(s) = \frac{bU_0}{s(s-a)} = \frac{A}{s} + \frac{B}{s-a} = \frac{\frac{bU_0}{-a}}{s} + \frac{\frac{bU_0}{a}}{s-a}$$

By denoting with $G_0 = \frac{b}{-a}, \tau = -\frac{1}{a}$

$$Y_{forced}(s) = \frac{G_0U_0}{s} - \frac{G_0U_0}{s-a}$$

$$\xRightarrow{\mathcal{L}^{-1}} \quad y_{forced}(t) = G_0U_0\left(1 - e^{-\frac{t}{\tau}}\right)1(t),$$

Then,

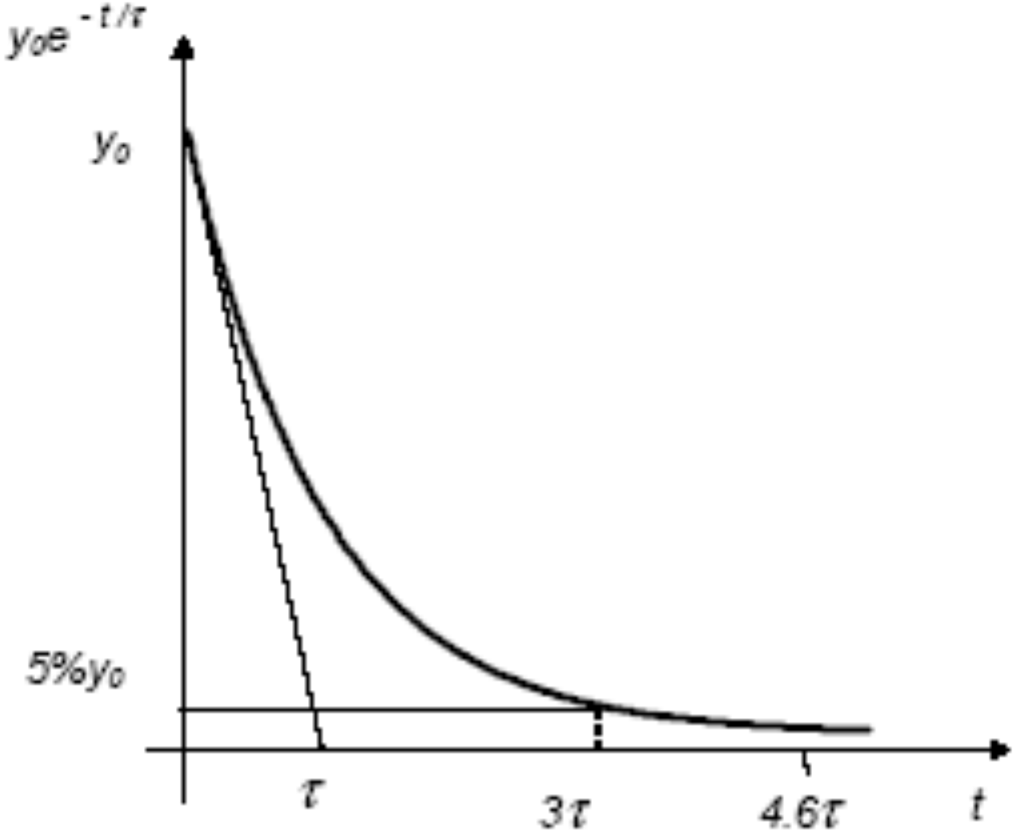
$$y(t) = y_{free}(t) + y_{forced}(t) = \left(e^{-\frac{t}{\tau}}y_0 + G_0U_0\left(1 - e^{-\frac{t}{\tau}}\right)\right)1(t)$$

LTI system, first order, strictly proper: free response

$$\dot{x}(t) = ax(t) + bu(t)$$
$$y(t) = x(t)$$
$$x(t_0) = \boldsymbol{x_0}$$

$$u(t) = \mathbf{0}$$

$$y_{free}(t) = e^{-\frac{t}{\tau}} y_0$$



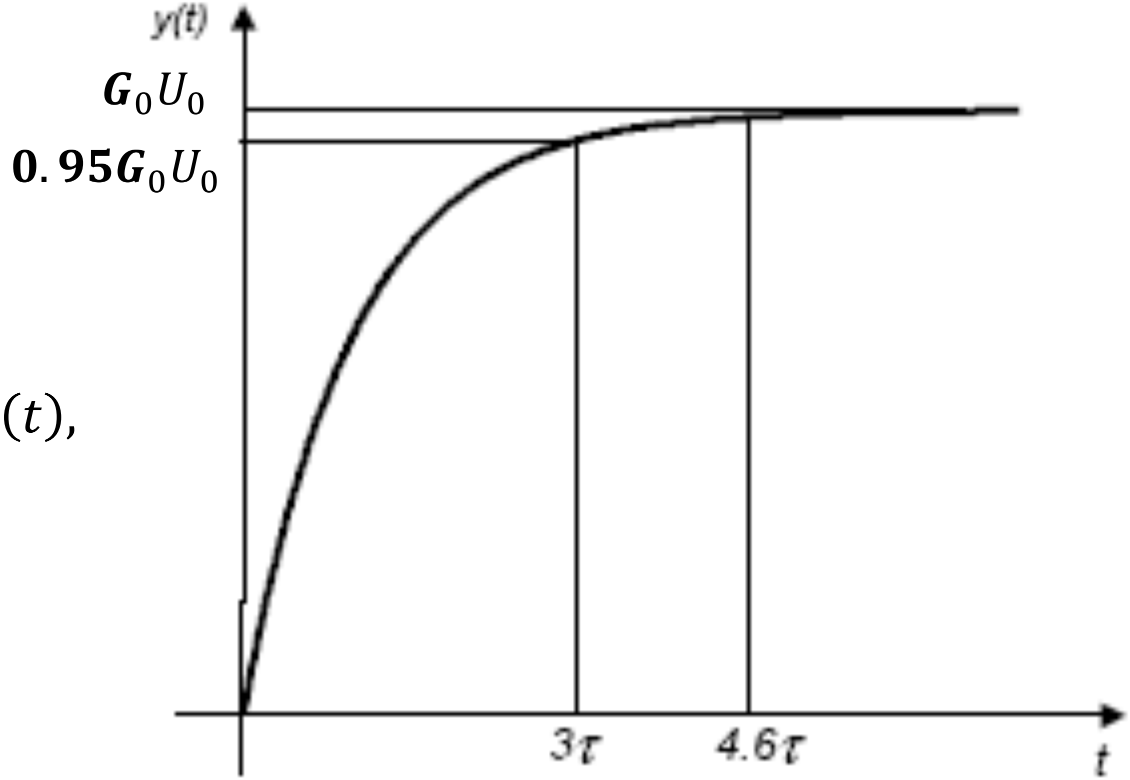
LTI system, first order, strictly proper: step response

$$\dot{x}(t) = ax(t) + bu(t)$$
$$y(t) = x(t)$$
$$x(t_0) = \mathbf{0}$$

$$u(t) = U_0 1(t)$$

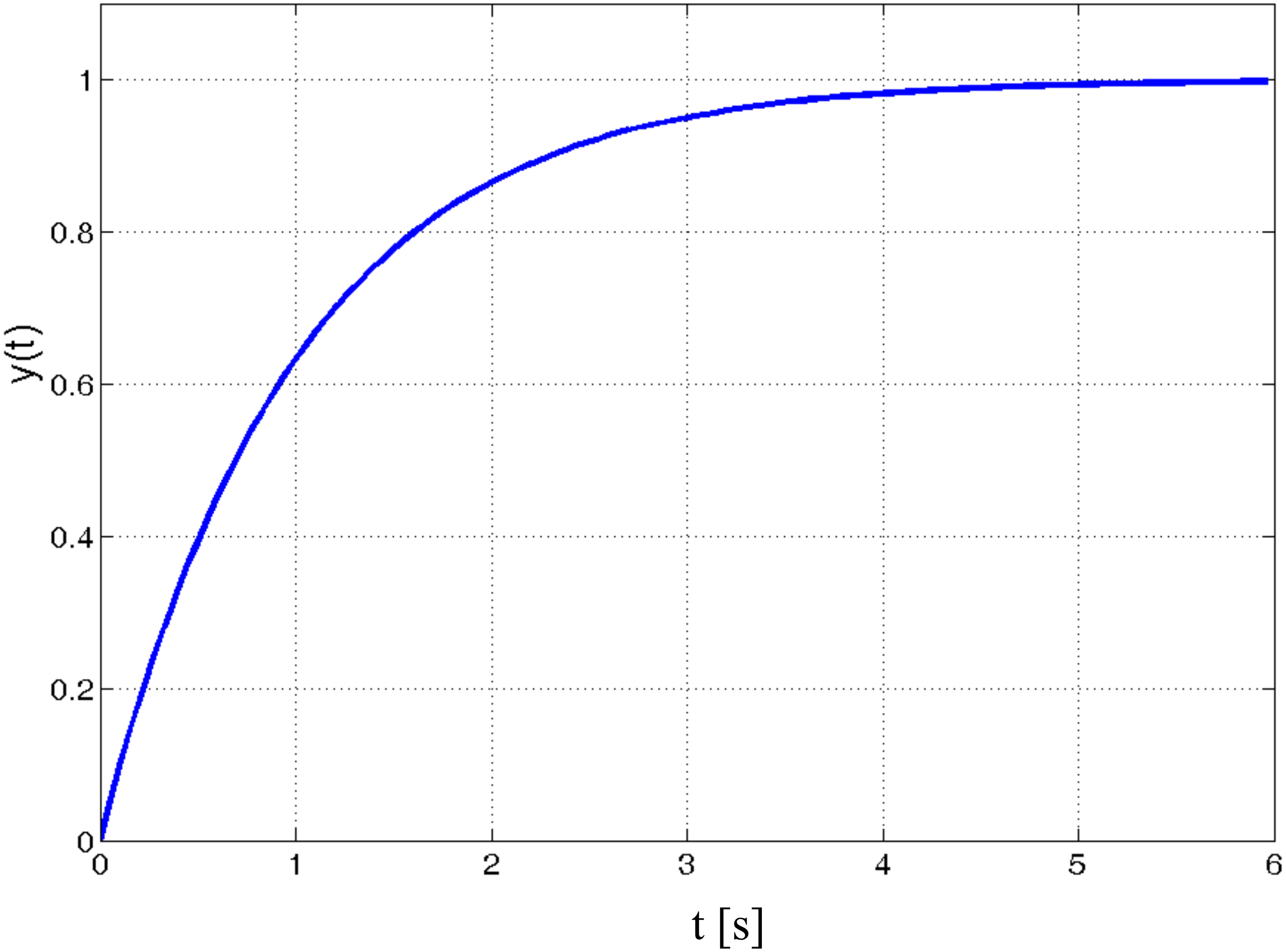
$$y_{step}(t) = \boldsymbol{G}_0 U_0 \left(1 - e^{-\frac{t}{\tau}}\right) 1(t),$$

$$\boldsymbol{G}_0 = \frac{b}{-a}, \tau = -\frac{1}{a}$$



LTI system, first order, strictly proper: step response

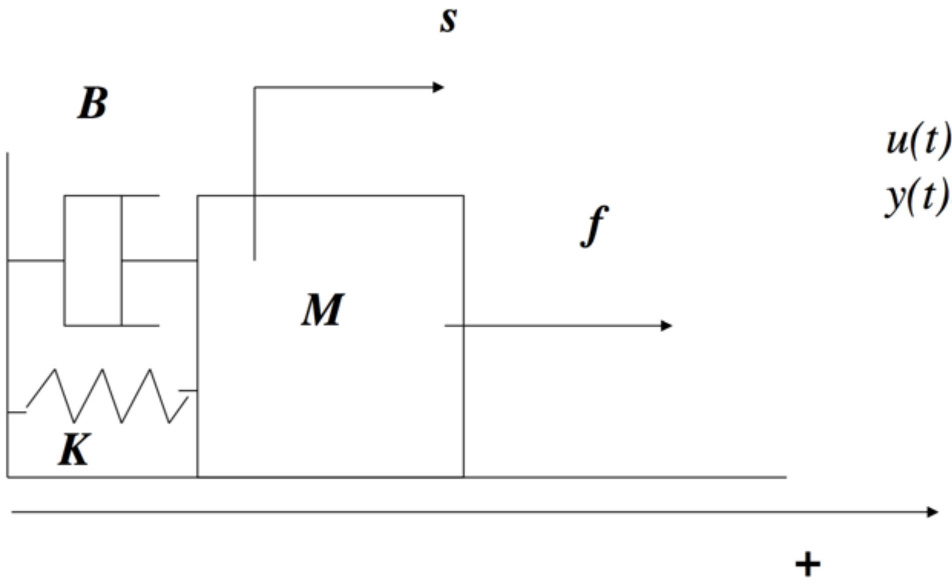
Response to a step input for a first order, strictly proper system ($\boldsymbol{G_0} = 1$, $\tau = 1$)



LTI system, first order, strictly proper: parameters for the qualitative step response

- ***Initial value*** $y(0) = 0$
- ***Final value*** $lim_{t\to\infty}\, y(t) = G_0U_0$
- ***Settling time***
  - $t_{s\ 5\%} = 3\tau$
  - $t_{s\ 1\%} = 4.6\tau$
- ***Rise time*** $t_r \cong 2.2\tau$

Example: mass-spring-damper system



$u(t)=f(t)$
$y(t)=s(t)$

- **Input output representation**

$$M\ddot{y}(t)+B\dot{y}(t)+Ky(t)=u(t)$$

- **State space representation**

$x_1=s$ e $x_2=v=ds/dt$

$$\Downarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ \dot{s} \end{pmatrix}$$

$$\dot{x}=\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+\begin{pmatrix} 0 \\ 1/M \end{pmatrix}u,$$

$$\mathrm{y}=\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

In general a second order system…as a mass-spring-damper system

- Input output representation

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = bu(t)$$

- State space representation

  $x_1 = y$ e $x_2 = \dot{y} = dy/dt$ ⇩ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

⇩

- Transition matrix

$$\Phi(s) = \frac{\begin{pmatrix} s + a_1 & 1 \\ -a_0 & s \end{pmatrix}}{s^2 + a_1 s + a_0}$$

- Transfer function

$$G(s) = \frac{b}{s^2 + a_1 s + a_0}$$

$$Y(s) = C\Phi(s)x_0 + G(s)U(s)$$

In general a second order system… as a mass-spring-damper system

$$\boldsymbol{Y(s) = \underbrace{C\Phi(s)x_0}_{Y(s)_{free}} + \underbrace{G(s)U(s)}_{Y(s)_{forced}}}$$

$$\boldsymbol{x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}}$$

$$Y(s)_{free} = \frac{(s+a_1)x_{10} + x_{20}}{s^2 + a_1 s + a_0}$$

$$U(s) = \frac{U_0}{s}, \quad Y(s)_{step} = \frac{b}{s^2 + a_1 s + a_0} \cdot \frac{U_0}{s}$$

Characteristic equation …

- The characteristic equation, $\boldsymbol{s^2 + a_1 s + a_0 = 0}$, determines the evolution modes

Three cases:

- ***real and distinct poles***
- ***real multiple poles***
- ***complex conjugates poles***

Laplace Transform and Transfer function

- ***Real and distinct poles,*** **terms as**

$$\frac{1}{s-a}$$

corresponding to a real pole/real eigenvalue of the dynamic matrix

- ***Real multiple poles, a term as***

$$\frac{1}{(s-a)^2}$$

corresponding to a real multiple pole/eigenvalue of the dynamic matrix

- ***Complex conjugates poles,*** **terms as**

$$\frac{\omega}{(s-\alpha)^2+\omega^2} \quad \textbf{or} \quad \frac{s-\alpha}{(s-\alpha)^2+\omega^2}$$

corresponding to a pair of complex conjugate poles/eigenvalues $a \pm j\omega$ of the dynamic matrix

Laplace antitransform: example 1

CASE 1: real and distinct eigenvalues/poles

$$Y_{free}(s) = C\Phi(s)x_0 = \frac{s-10}{s^2+7s+10} = \frac{s-10}{(s+2)(s+5)}$$

- Appling the residual method we have

$$Y_{free}(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+5)}$$

with

$$A_1 = \lim_{s\to -2}(s+2)Y_{free}(s) = \lim_{s\to -2}\frac{s-10}{s+5} = -4$$

$$A_2 = \lim_{s\to -5}(s+5)Y_{free}(s) = \lim_{s\to -5}\frac{s-10}{s+2} = 5$$

Hence

$$y_{free}(t) = (-4e^{-2t} + 5e^{-5t})\cdot 1(t)$$

Laplace antitransform: example 2

CASE 2: real multiple eigenvalues/poles

$$Y_{forced}(s) = G(s)U(s) = \frac{18}{s^2+6s+9}U(s) \qquad \text{with } u(t) = 1(t)$$

- This function can be written as the sum of three terms

$$Y_{forced}(s) = \frac{18}{s(s+3)^2} = \frac{A_1}{s} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+3)^2}$$

- The residual method can be applied to evaluate A_1 and A_3, while A_2 can be evaluated by substitution

$$A_1 = \lim_{s\to 0} sY_{forced}(s) = 2 \qquad A_3 = \lim_{s\to -3} (s+3)^2\, Y_{forced}(s) = -6$$

while $A_2 = -2$.

Hence, $$y_{forced}(t) = (2 - 2e^{-3t} - 6te^{-3t}) \cdot 1(t)$$

Partial fraction decomposition for multiple poles

- Note, in the case of multiple poles (with multiplicity r_i),

$$F(s) = \frac{N(s)}{(s-p_i)^{r_i}}$$

$$\Rightarrow \qquad F(s) = \frac{N(s)}{(s-p_i)^{r_i}} = \sum_{l=1}^{r_i} \frac{K_{il}}{(s-p_i)^{r_i-l+1}}$$

- with

$$K_{i\ell} = \frac{1}{(\ell-1)!}\frac{d^{\ell-1}}{ds^{\ell-1}}(s-p_i)^{r_i}F(s)\big|_{s=p_i}$$

$$\xrightarrow{\mathcal{L}^{-1}} \qquad f(t) = \sum_{l=1}^{r_i} \frac{K_{il}}{\boldsymbol{(r_i-l)!}}\boldsymbol{t^{r_i-l}e^{p_i t}}$$

Laplace antitransform: example 2

…CASE 2: real multiple eigenvalues/poles

For

$$Y_{forced}(s) = \frac{18}{s(s+3)^2} = \frac{A_1}{s} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+3)^2}$$

A_2 can be computed through residual method by

$$A_2 = \lim_{s \to 0} \frac{d}{ds}(s+3)^2 Y_{forced}(s) = \frac{d}{ds}(s+3)^2 Y_{forced}(s)|_{s=-3} =$$

$$\frac{d}{ds}\frac{18}{s}|_{s=3} = -\frac{18}{s^2}|_{s=3} = -2$$

Laplace antitransform: example 3

CASE 3: complex conjugate eigenvalues/poles

$$Y_{free}\,(s) = C\Phi(s)x_0 = \frac{s+3}{(s^2+4s+13)}$$

- This function can be written as the sum of two terms

$$Y_{free}\,(s) = \frac{s+3}{(s^2+4s+4+9)} = \frac{s+2-2+3}{((s+2)^2+3^2)} =$$

- Hence, $Y_{free}\,(s) = \frac{s+2}{(s+2)^2+3^2} + \frac{1}{3}\frac{3\cdot 1}{(s+2)^2+3^2}$

$$y_{free}(t) = \left(e^{-2t}\cos(3t) + \frac{1}{3}e^{-2t}\sin(3t)\right)\cdot 1(t)$$

$$y_{free}(t) = e^{-2t}\left(\cos(3t) + \frac{1}{3}\sin(3t)\right)\cdot 1(t)$$

Stability

- A linear system is said ***stable*** if no evolution mode is divergent (only convergent and constant evolution modes).

- It happens if all the eigenvalues of the matrix A (pole of $G(s)$) have a negative or null real part and the eigenvalues with null real part have multiplicity 1.

- In a stable system
  - ***the free evolution doesn't tend to infinity***
  - ***the free evolution doesn't converge to zero*** if the constant evolution mode is excited

Asymptotic stability

- A linear system is said ***asymptotically stable*** if all evolution modes are convergent.

- It happens if all the eigenvalues of the matrix A (pole of $G(s)$ have negative real part

- In an ***asymptotically stable*** system
  - ***the free evolution converges to zero***

Unstability

- A linear system is said ***unstable*** if there is a divergent evolution mode.

- It happens if an eigenvalues of the matrix A (pole of $G(s)$) have a real part positive or an eigenvalue (pole) with null real part with multiplicity >1.

- In an ***unstable*** system
  - ***the free evolution tends to infinity***

Stability analysis - Example 1

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In order to evaluate the eigenvalues of the matrix A, we can calculate the roots of the characteristic polynomial
- In Matlab, it is possible to use the command ***eig(A)***
- In this example we have $p_1 = p_2 = -1$.
- This system is ***asymptotically stable*** because it has all eigenvalues with negative real part

Stability analysis - Example 2

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In this example we have $p_1 = p_2 = 1$.

- The system is ***unstable*** because it has two eigenvalues with positive real part

Stability analysis - Example 3

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In this example we have $p_1 = 0$, $p_2 = -1.$

- The system is ***stable*** because it has
  - a null eigenvalue
  - an eigenvalue with negative real part

Stability analysis - Example 4

- Let us consider the transfer function of an LTI system

$$G(s) = \frac{s+1}{s^2(s+5)}$$

- This system is ***unstable*** because it has two null poles.

Routh-Hurwitz criterion

- Routh-Hurwitz criterion is used to study the sign of the real part of a polynomial roots.
- It is particularly useful in case of high order polynomials or polynomials with uncertain parameters where the evaluation of the roots can be difficult.
- Routh-Hurwitz criterion is of interest to study the stability of LTI systems both in the state-space form and in the Laplace domain

$$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

where the poles of $W(s)$ coincide with the eigenvalues of the matrix A

Necessary condition

- Let us consider a polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$

and without loss of generality let us assume that

- $a_n > 0$
- $a_0 \neq 0$

- **Stodola criterion (Necessary condition):**

A necessary condition for the roots of the polynomial D(s) to have negative real parts is that

$$\boldsymbol{sign(a_0) = sign(a_1) = \cdots = sign(a_n).}$$

This condition is also sufficient for polynomials of degree $\boldsymbol{n = 1\ , n = 2}$.

Routh table

- Let us consider the polynomial

$$D(s)=a_n s^n + a_{n-1}s^{n-1}+\ldots+a_0$$

- ***The Routh table*** is defined as follows

$\boldsymbol{n}$	$\boldsymbol{a_n}$	$\boldsymbol{a_{n-2}}$	$\boldsymbol{a_{n-4}}$	**…**
$\boldsymbol{n-1}$	$\boldsymbol{a_{n-1}}$	$\boldsymbol{a_{n-3}}$	$\boldsymbol{a_{n-5}}$	**…**
$\boldsymbol{n-2}$	$\boldsymbol{b_{n-2}}$	$\boldsymbol{b_{n-4}}$	$\boldsymbol{b_{n-6}}$	**…**
$\boldsymbol{n-3}$	$\boldsymbol{c_{n-3}}$	$\boldsymbol{c_{n-5}}$	**…**	**…**
…	**…**	**…**	**…**	

$$b_{n-2}=-\frac{1}{a_{n-1}}\det\begin{pmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{pmatrix}$$

$$b_{n-4}=-\frac{1}{a_{n-1}}\det\begin{pmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{pmatrix}$$

$$c_{n-3}=-\frac{1}{b_{n-2}}\det\begin{pmatrix} a_{n-1} & a_{n-3} \\ b_{n-2} & b_{n-4} \end{pmatrix}$$

Routh table: Example

- Routh table, $n+1$ rows and the last row has 1 element different from zero.

- Let us define the Routh table of the function

$$f(s) = s^4 + 2s^3 + 3s^2 + 5s + 10$$

⬇

4	*1*	*3*	*10*
3	*2*	*5*	*0*
2	*0.5*	*10*	*0*
1	*-35*	*0*	*0*
0	*10*	*0*	*0*

Routh criterion

- Let us consider the Routh table of the polynomial $D(s)$

n	a_n	a_{n-2}	a_{n-4}	…
$n-1$	a_{n-1}	a_{n-3}	a_{n-5}	…
$n-2$	b_{n-2}	b_{n-4}	b_{n-6}	…
$n-3$	c_{n-3}	c_{n-5}	…	…
…	…	…	…	

- ***The roots of the polynomial $D(s)$ have all negative real parts iff the elements of the first column of the Routh table are all positive.***

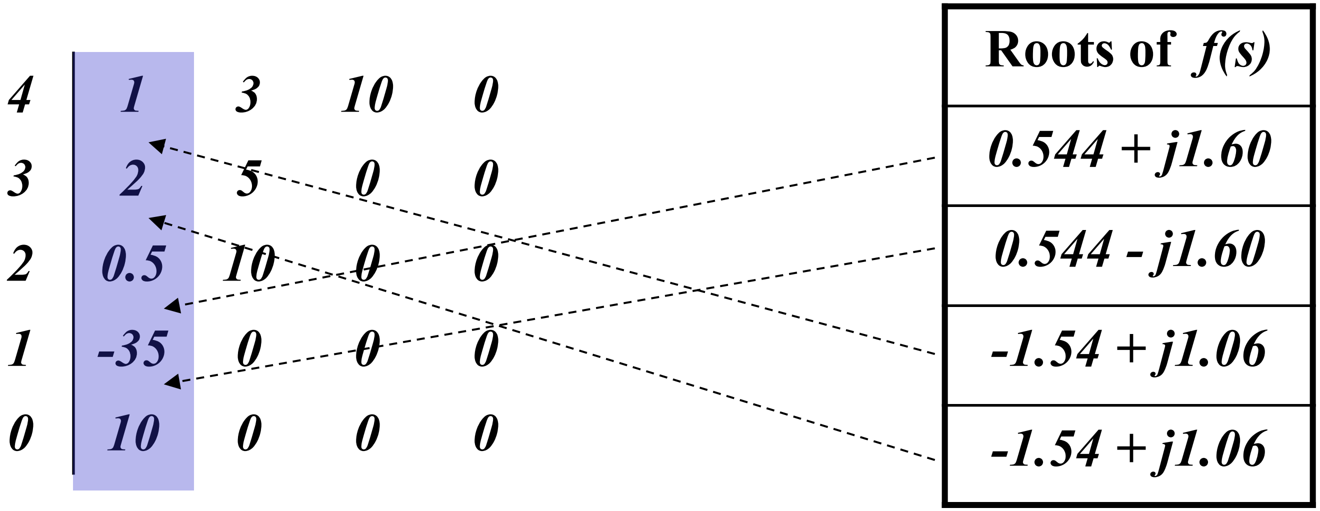
- **Each sign variation of the element of the first column of the Routh table correspond to a root of $D(s)$ *with a positive real part.***

Routh criterion: example

- Let us consider the polynomial

$$f(s) = s^4 + 2s^3 + 3s^2 + 5s + 10$$

- The Routh table of $f(s)$ is



4	1	3	10	0
3	2	5	0	0
2	0.5	10	0	0
1	-35	0	0	0
0	10	0	0	0

Roots of *f(s)*
$0.544 + j1.60$
$0.544 - j1.60$
$-1.54 + j1.06$
$-1.54 + j1.06$

Routh Criterion: uncertain parameters

- Let us consider a transfer function $W(s)$ of an LTI system where the poles of $W(s)$ depends on un uncertain parameter p,

$$W(s)=\frac{s+1}{2s^3+5ps^2+(3+p)s+1}$$

- From the Routh table we have that

3	**2**	**3+p**
2	**5p**	**1**
1	$\frac{5p^2+15p-2}{5p}$	**0**
0	**1**	**0**

$$\begin{cases} 5p>0 \\ 5p^2+15p-2>0 \end{cases}$$

$$\begin{cases} p>0 \\ p<-3.13 \wedge p>0.128 \end{cases}$$

$$p>0.128$$

Routh criterion: Singular Cases

- In the design of the Routh table two singular cases can be found

  a) *The first term of a row is null*

  b) *All the terms of a row are null*

- In these cases, some mathematical manipulations of the Routh table can be adopted. However, it is not of interest for this course.