

Course of "Automatic Control Systems" 2024/25

Analysis of LTI systems in the time domain

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▲ *Linear time invariant (LTI) systems* in the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t), \quad x(t_0) = x_0$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, where x(t) is the state vector, u(t) is the input vector and y(t) is the output vector of the system.



Lagrange Formula

▲ Let us consider a *Linear Time Invariant* (*LTI*) system in the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0$$

 $y(t) = Cx(t) + Du(t)$
(1)

The solution of the linear differential equation (1) defines the *time evolution of the state variables* and it is given by the Lagrange Formula

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau, \quad t \ge t_0$$
⁽²⁾

▲ The *time evolution of the outputs* turns out to be

$$y(t) = C e^{A(t-t_0)} x_0 + C \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau + D u(t), \quad t \ge t_0$$
(3)



Lagrange Formula

▲ Taking into account that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau) d\tau = f(t,b(t))\frac{db(t)}{dt} - f(t,a(t))\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{d}{dt}f(t,\tau) d\tau$$

▲ Lagrange formula (2) can be easily verified by derivation (assuming $t_0 = 0$)

$$\dot{x}(t) = \frac{d}{dt} (e^{At} x_0) + e^{A(t-t)} Bu(t) + \int_0^t \frac{d}{dt} \left[e^{A(t-\tau)} Bu(\tau) \right] d\tau$$
$$= A e^{At} x_0 + Bu(t) + \int_0^t A e^{A(t-\tau)} Bu(\tau) d\tau$$
$$= A \left[e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \right] + Bu(t) = Ax(t) + Bu(t)$$



▲ The *time evolution of the state and output variables* can be conceptually divided in two parts,



- A The *free evolution* indicate the evolution of state and output vectors that would be obtained in the absence of input (u(t) = 0).
- A The *forced evolution* indicate the evolution of state and output vectors that would be obtained in the presence of input and null initial conditions ($x_0 = 0$)

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Free evolution: matrix **A** diagonalizable

The free evolution of an LTI system in the time domain is defined by the matrix exponential e^{At} . Generalizing the Taylor expansion of an exponential to the matrix case, we have

$$e^{M} = \sum_{i=0}^{\infty} \frac{1}{i!} M^{i} = I_{n} + M + \frac{M^{2}}{2!} + \cdots$$

In case the matrix A has real and distinct eigenvalues, λ_i , i=1,...,n, it is diagonalizable: it is possible to find a matrix T_D , such that $\hat{A} = T_D A T_D^{-1} = \text{diag}\{\lambda_1, ..., \lambda_n\}$. Then $A = T_D^{-1} \hat{A} T_D$ and e^{At} turns out to be

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} (A t)^{i} = T_{D}^{-1} \sum_{i=0}^{\infty} \frac{1}{i!} (A t)^{i} T_{D}$$
$$= T_{D}^{-1} \operatorname{diag} \left\{ \sum_{i=0}^{\infty} \frac{(\lambda_{1} t)^{i}}{i!}, \sum_{i=0}^{\infty} \frac{(\lambda_{2} t)^{i}}{i!}, \cdots, \sum_{i=0}^{\infty} \frac{(\lambda_{n} t)^{i}}{i!} \right\}$$
$$= T_{D}^{-1} \operatorname{diag} \left\{ e^{\lambda_{1} t}, e^{\lambda_{2} t} \dots, e^{\lambda_{n} t} \right\} T_{D}.$$

Remark: $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigenvalues of the A matrix, T_D is the transformation (*i.e.*, $\hat{x} = T_D x$) that allows achieving a diagonal \hat{A} matrix. Prof. Francesco Montefusco



 \checkmark The exponential terms,

 $e^{\lambda_i t}$

are the modes of the system, named *aperiodic modes*

A In case of *complex conjugate eigenvalues* $\lambda_i = \alpha_i + j\omega_i$ e $\overline{\lambda_i} = \alpha_i - j\omega_i$, the corresponding complex exponential function determines terms as follows:

$$e^{lpha_i t} \sin(oldsymbol{\omega}_i t)$$
 , $e^{lpha_i t} \sin(oldsymbol{\omega}_i t)$

These latter modes are named *pseudo-periodic modes*.



In case of a non-diagonalizable A matrix (eigenvalues with multiplicity (η) greater than one) we must resort to the Jordan form (see text for further information): the matrix \hat{A} has an almost diagonal structure, with the elements on the diagonal corresponding to the eigenvalues, with the addition of superdiagonal elements, Jordan miniblocks, which determine terms of the type

$$t^{\eta-1}e^{\lambda_i t}$$
, if $\lambda_i \in \mathbb{R}$,

or

$t^{\eta-1}e^{\alpha_i t}\sin(\omega_i t+\varphi_i)$ if $\lambda_i \in \mathbb{C}$

where η is an integer between 1 and the maximum size of the Jordan miniblocks associated with λ_i .



- Analysis in the time domain
 - $\begin{aligned} x_l(t) &= e^{At} x_0 \\ y_l(t) &= C e^{At} x_0 \end{aligned}$
- $x_1(t)$ (and then $y_1(t)$) is given by a combination of terms as
 - $\geq e^{\lambda_i t}$, in the case of real and distinct eigenvalues;
 - *e*^{α_it} sin($ω_it + φ_i$), in the case of complex conjugate eigenvalues of multiplicity one;
 - > $t^{\eta-1}e^{\lambda_i t}$, in the case of real eigenvalues with multiplicity $\eta > 1$;
 - > $t^{\eta-1}e^{\alpha_i t} \sin(\omega_i t + \varphi_i)$ in the case of complex conjugate eigenvalues of multiplicity $\eta > 1$.



An aperiodic mode is an evolution mode of a linear system related to a real eigenvalue of the matrix A of multiplicity 1. It can be written in the form

$e^{\lambda_i t}$

- A Depending on the sign of the eigenvalue λ_i , an aperiod evolution mode can be
 - * convergent ($\lambda_i < 0$)
 - * constant $(\lambda_i = 0)$
 - * divergent ($\lambda_i > 0$)



Aperiodic evolution modes (2/3)



♦ Constant aperiodic mode

Divergent aperiodic mode





▲ When the evolution mode is convergent it is possible to introduce a new parameter said *time constant of the mode* defined as

$$\tau_i = -\frac{1}{\lambda_i}$$

- ▲ The time constant gives us an information about the time needed before the convergent mode will be extinguished.
- ▲ It is straightforward to verify that
 - * After a time $\bar{t} = 3\tau$ the magnitude of the mode will be reduced to the 5% of the initial value
 - * After a time $\overline{t} = 4.6\tau$ the magnitude of the mode will be reduced to the 1% of the initial value



Pseudo-periodic evolution modes (1/5)

 $e^{\alpha_i t} \sin(\omega_i t + \varphi_i)$

Convergent pseudo-periodic mode

Constant pseudo-periodic mode

Divergent pseudo-periodic mode





- A The pseudo-periodic mode is completely characterized by the pair of parameters (α_i , ω_i) that represent the real part and the imaginary part of the complex conjugate eigenvalues.
- A These parameters give direct information both on the exponential law that envelops the oscillation peaks (parameter α_i) and on the angular frequency of the oscillations (parameter ω_i).
- ▲ Frequently, instead of using these parameters, other pairs of parameters related to (α_i, ω_i) through simple relations are used, and that provide information more directly related to other characteristics of the response, especially in the case of convergent oscillatory motion.



▲ For convergent pseudo-periodic mode, the *time constant* is defined as

$$\tau_i = -\frac{1}{\alpha_i}$$

A The parameter $\boldsymbol{\omega}_i$ is called the angular frequency of the system, while T, related to $\boldsymbol{\omega}_i$ by the relation $T=2\pi/\boldsymbol{\omega}_i$, is called the oscillation period of the system.

▲ Sometimes, instead of the period T, the frequency f=1/T is specified.



Evolution modes with distinct eigenvalues $(\eta = 1)$





Evolution modes with multiple eigenvalues $(\eta = 2)$





Natural frequency and damping factor

Other important parameters for pseudo-periodic mode are the natural frequency ω_n and the damping coefficient ζ .

The natural frequency is defined by



$$\alpha = -\zeta \omega_n,$$
$$\omega = \omega_n \sqrt{1 - \zeta^2}$$

$$\omega_n \cos \theta = \zeta \omega_n$$

 $\zeta = \cos \theta$



- A The *natural frequency* ω_n is the oscillation frequency of the pseudoperiodic mode when $\alpha = 0$.
- ▲ For convergent pseudo-periodic modes the damping coefficient $\zeta \in (0,1]$ while for divergent pseudo-periodic modes $\zeta \in [-1,0)$
- ▲ For convergent pseudo-periodic modes, the damping coefficient ζ relates the oscillations of the pseudo-periodic mode to the time before the evolution will extinguish. For $ζ \ll 1$

$$\zeta = -rac{lpha}{\omega_n} \cong -rac{lpha}{\omega} = rac{T}{2\pi au} \ll 1$$

where T is the oscillation period. Indeed, the number of the oscillation before the mode will extinguish increases when $\boldsymbol{\zeta}$ becomes small.

$$\zeta = \frac{T}{2\pi\tau} \cong \frac{T}{6\tau} \implies \frac{1}{2\zeta} \cong \frac{3\tau}{T} \stackrel{\# \text{ of oscillations}}{\text{before the mode will extinguish}}$$



▲ Let us consider the forced response of an LTI system in the output $(x_0 = 0)$

$$y_f(t) = C \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + D u(t), \ t \ge t_0$$

- ▲ The evaluation of the forced response in the time domain is demanding due to the presence of the convolution product.
- A Only in some particular case, such as the *step response* $u(t) = \overline{u} \cdot \mathbf{1}(t)$, it becomes straightforward

$$v_f(t) = C \int_0^t e^{A(t-\tau)} B \,\overline{u} \, d\tau + D \,\overline{u}$$

$$= \left[-CA^{-1}e^{A(t-\tau)}B\overline{u}\right]_0^t + D\overline{u}$$
$$= CA^{-1}e^{At}B\overline{u} + \left[-CA^{-1}B + D\right]\overline{u}$$

▲ In the other cases the forced response is evaluated in the *Laplace domain*

1(*t***)**



▲ For a first-oder LTI system in the form

$$\dot{x}(t) = ax(t) + bu(t)$$

 $y(t) = cx(t) + du(t)$, $x(t_0 = 0) = x_0$

the *time evolution of the state variable* in case of $u(t) = \overline{u} \cdot \mathbf{1}(t)$ is given $x_{l} = e^{at}x_{0} + \frac{1}{a}e^{at}b\overline{u} - \frac{1}{a}b\overline{u}, t \ge 0$

▲ The *time evolution of the output* turns out to be

$$y(t) = ce^{at}x_0 + c\frac{1}{a}e^{at}b\overline{u} - c\frac{1}{a}b\overline{u} + d\overline{u},$$

$$= ce^{at}x_0 - c\frac{b}{a}\overline{u}(1 - e^{at}) + d\overline{u}$$

$$y_l \qquad y_f$$



Example of first order LTI system:



▲ For a first orer LTI, the *time evolution of the output for* $u(t) = \overline{u} \cdot \mathbf{1}(t)$ *is given by*





Example of second-order LTI system: massspring-damper system



Problem:

For the mass-spring-damper system reported in figure, assuming M=1Kg, K=16 N/m, evaluate the evolution modes by varying B values [Ns/m] in the interval [0 20].



Example of second-order LTI system: RLC circuit



$$u(t) = v(t), \quad y(t) = v_c(t)$$

$$\mathbf{x}_1(t) = \mathbf{v}_c(t) \quad \mathbf{x}_2(t) = \mathbf{i}_L(t)$$

Problem:

For the RLC circuit in series configuration, compute the input-output and state space representations.

Assuming the capacity value C=1e-6 μ F, and the inductance value L=1e-3 mH, compute the values of **R** for which the system exhibits aperiodic and pseudo-periodic modes.





$$u(t) = v(t), \quad y(t) = v_c(t)$$

Input-output representation:

$$LC\ddot{y}(t) + RC\dot{y}(t) + y(t) = u(t)$$

State space representation:
$$\mathbf{x}_{I}(t) = \boldsymbol{v}_{c}(t) \quad \mathbf{x}_{2}(t) = \boldsymbol{i}_{L}(t)$$

$$\dot{x}_1 = \frac{1}{C} x_2$$

$$\dot{x}_2 = -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/C \\ -1/L & -R/L \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/L \end{pmatrix} u$$

$$y = (1 \quad 0) x$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$y = x_1$$



Appendix 1

INVERSE OF A MATRIX N×N



▲ Given a quadratic and invertible matrix

$$A = \begin{pmatrix} x_{1,1} & \dots & x_{1,j} \\ \vdots & \ddots & \vdots \\ x_{i,1} & \dots & x_{i,j} \end{pmatrix}$$

its inverse is defined as

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \operatorname{cof}(A, x_{1,1}) & \dots & \operatorname{cof}(A, x_{1,j}) \\ \vdots & \ddots & \vdots \\ \operatorname{cof}(A, x_{i,1}) & \dots & \operatorname{cof}(A, x_{i,j}) \end{pmatrix}^T$$

where the cofactor is

$$\operatorname{cof}(A, i, j) = (-1)^{i+j} \operatorname{det}(\operatorname{minor}(A, i, j))$$

and the minor (i, j) is the determinant of the matrix obtained excluding the row i and the column j.



Inverse of a 2×2 matrix

 \checkmark Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$



Inverse of a 3×3 matrix

▲ Given a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

its inverse is

$$\frac{1}{\det(A)} \begin{pmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & - \begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & + \begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix}} \\ - \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & - \begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\ + \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} & - \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} & + \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$



Appendix 2

EIGENVALUES AND EIGENVECTORS



▲ Given a matrix $A \in \mathbb{R}^{n \times n}$, a scalar $\lambda \in \mathbb{C}$ is said *eigenvalue* of the matrix A if there exists a vector $v \in \mathbb{C}^n$, said *eigenvector*, such that

$$Av = \lambda v$$

Taking into account account that eigenvalues and eigenvectors of a matrix verify the equation

$$(A-\lambda I)v=0.$$

The eigenvalues can be found evaluating the roots of the *characteristic polynomial* $p(\lambda)$ defined as

$$p(\lambda) = det(A - \lambda I).$$





$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

 $p(\lambda) = \lambda^2 + 3\lambda - 10$

$$p(\lambda) = \lambda^2 + 2\lambda + 1$$

Eigenvalues

Eigenvalues

$$\lambda_1=2, \ \lambda_2=-5$$

$$\lambda_1 = \lambda_2 = -1$$