Course of
"Industrial Control System Security"
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Harmonic response function

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Team code**: 09tkpu5**

Transient and Steady-state

- Let us consider an ***asymptotically stable LTI system.***

- Given an input signal $u(t)$ and an initial condition $x(0)$, we define

  - ***steady-state response*** $\boldsymbol{y_{ss}(t)}$, the regular behavior of the total response $y(t)$ *(if exist)* after an infinite time from the application of the input.

  - ***transient response*** $\boldsymbol{y_t(t)}$, the difference between the total response of the system and the steady-state response $\boldsymbol{y_t(t) = y(t) - y_{ss}(t)}$.

Transient and Steady-state

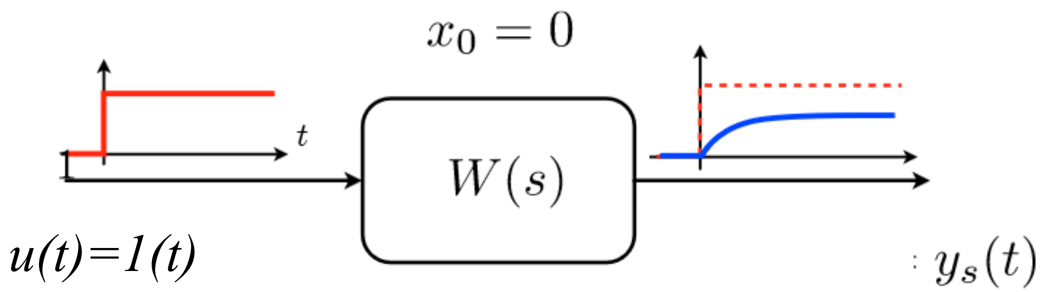
- ***The steady-state response*** of asymptotically stable system is independent from the initial condition.

- It depends on the particular input applied to the system

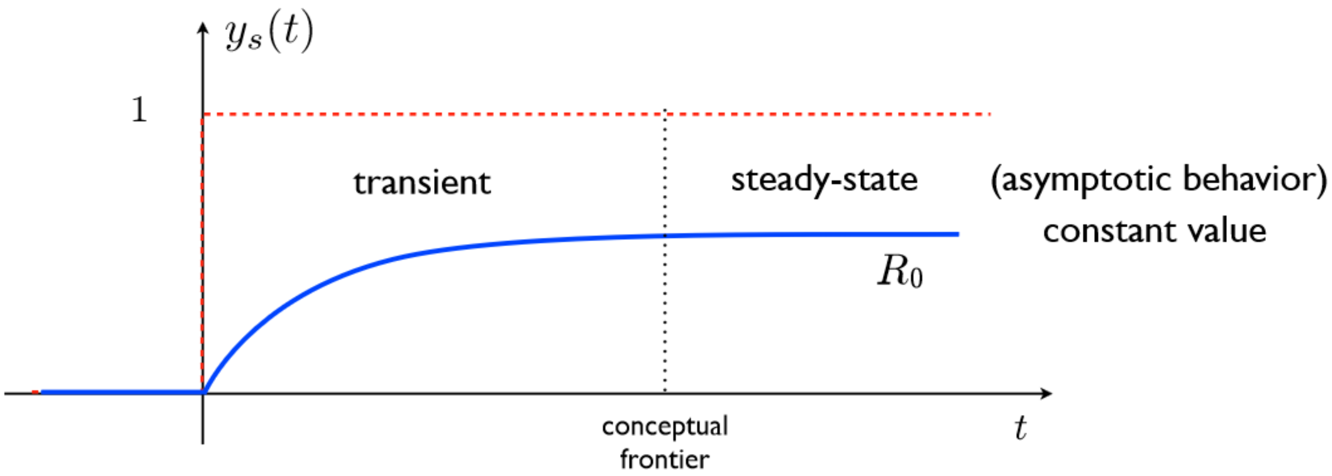
polynomial inputs → polynomial steady state

sinusoidal inputs → sinusoidal steady state

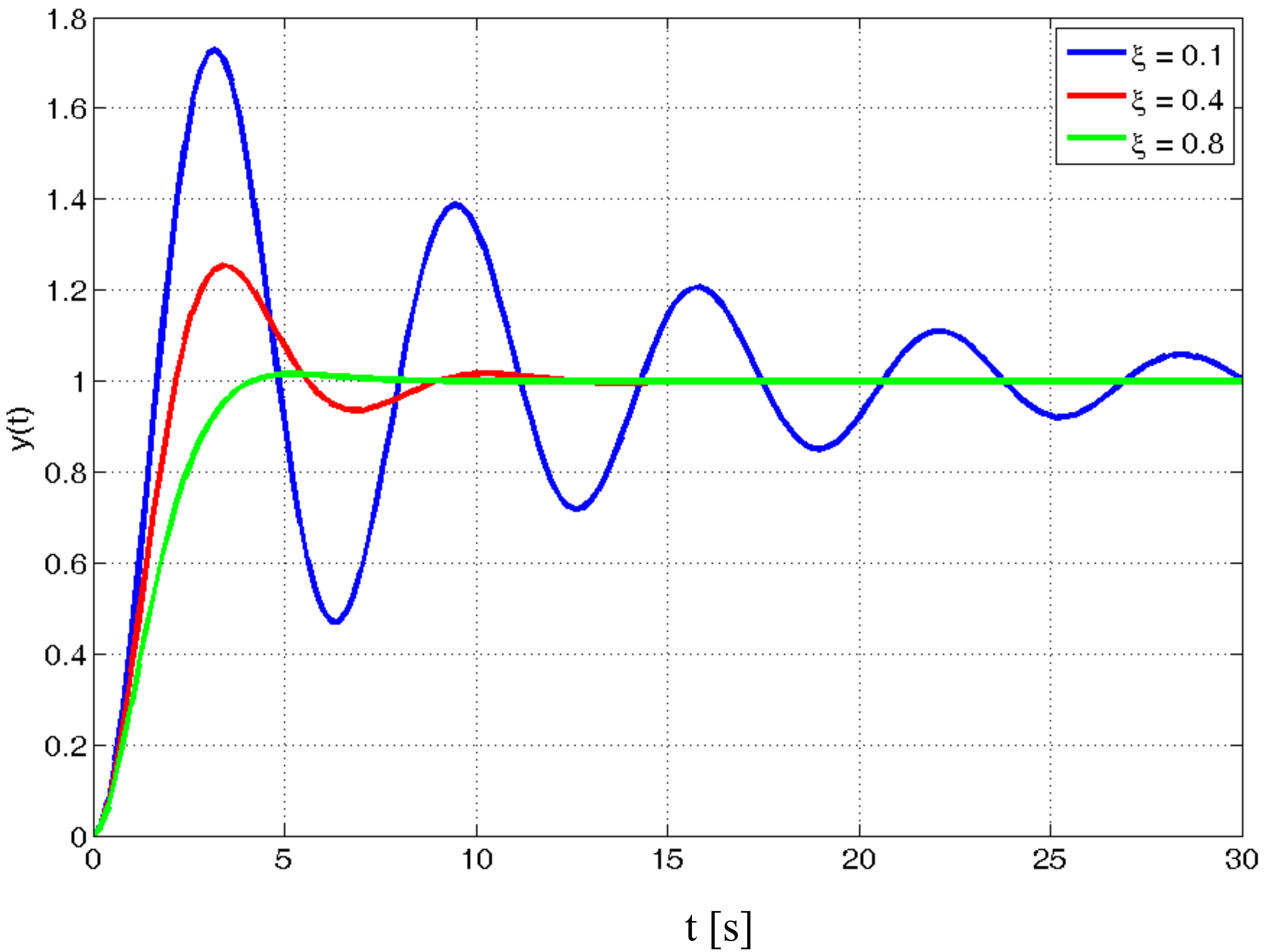
- The step response is characterized by "decaying" exponential functions related to the system evolution modes and a constant value



- The "decaying" exponential functions determine ***the transient*** part of the response while the constant term is the ***steady-state*** value.



- Different evolution modes determine different transient responses.

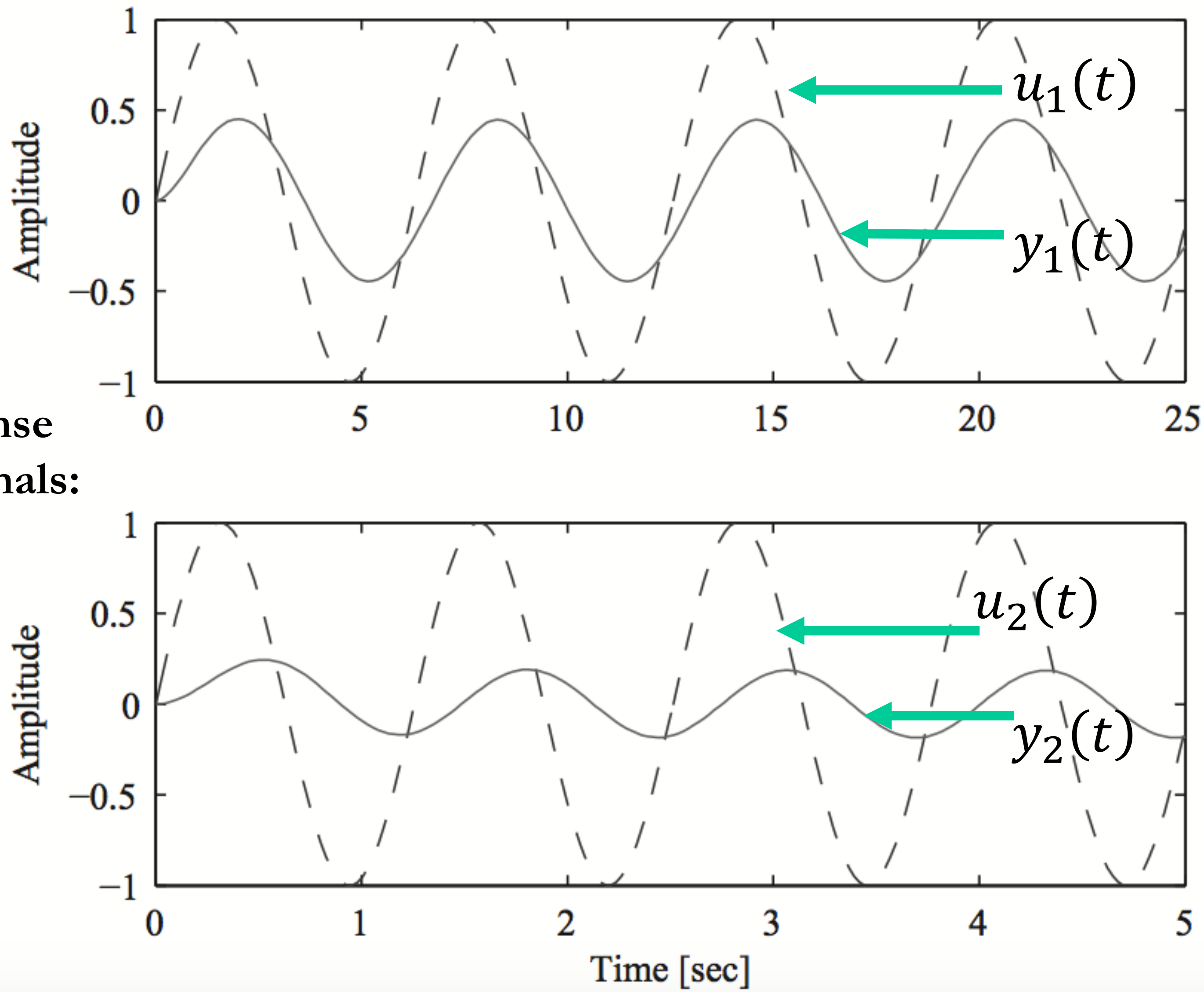


Let assume a first order LTI system:

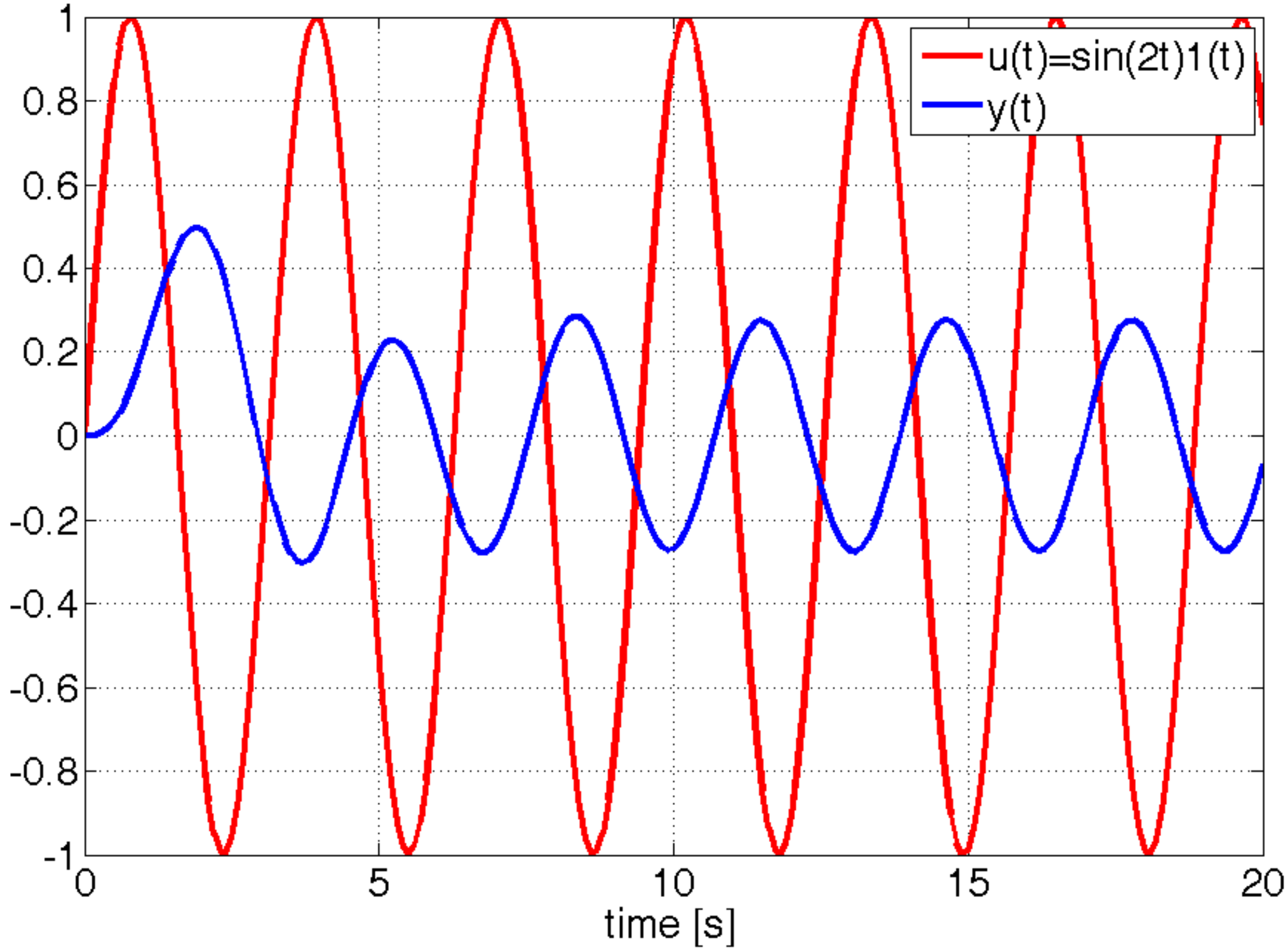
$$\dot{y}(t) + 2\,y(t) = u(t)$$

Compute the response to the following signals:

- $u_1(t) = \sin t$
- $u_2(t) = \sin(5t)$



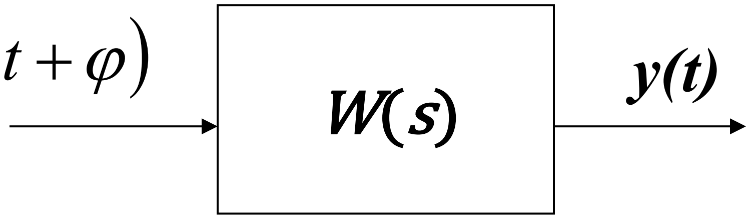
Total response of system $W(s) = 1/(s^2 + s + 1)$ to the input $u(t) = \sin(2t) \cdot 1(t)$.



Steady state response at sinusoidal inputs

- Let us consider an asymptotically stable LTI system with a transfer function $W(s)$ subject to a sinusoidal input signal

$$u(t) = U_0 \sin(\omega_0 t + \varphi)$$



- The evaluation of the steady state response of LTI system to sinusoidal inputs is very interest taking into account that ***any periodic signal,*** $f(t) = f(t+T)$***, with period T*** ($\omega_0 = \frac{2\pi}{T}$)***, can be decomposed in the sum of a finite or infinite sinusoids by means of the Fourier series, as***

$$f(t) = F_0 + \sum_{n=1}^{\infty} \left[F_{cn} \cos(n\omega_0 t) + F_{sn} \sin(n\omega_0 t) \right]$$

- In this case, the frequency spectrum (i.e., the coefficients of the Fourier series) of the signal is discrete (i.e., it is defined only a certain frequencies)

- ***An aperiodic signal*** can be analysed in the frequency domain by applying the Fourier transform, defined as $\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt$. The spectrum becomes a continuous function of ω (i.e. defined for all the frequency values).

Steady state response at sinusoidal inputs

- It is possible to prove that the steady state response of an LTI system with transfer function $W(s)$ to a sinusoidal inputs $u(t) = \mathrm{U}_0 \sin(\omega_0 t + \phi)$ can be written in the time domain as

$$y_{ss}(t) = U_0 \left| W(s) \right|_{s=j\omega_0} \sin\left(\omega_0 t + \varphi + \angle W(s)_{s=j\omega_0}\right)$$

where

- $|W(s)|_{s=j\omega_0}$ is the magnitude of the Laplace transform of $W(s)$ evaluated in $s = j\omega_0$.
- $\angle W(s)|_{s=j\omega_0}$ is the phase of the Laplace transform of $W(s)$ evaluated in $s = j\omega_0$.

- The proposed result can be summarized as follows:
  - The magnitude of a sinusoidal input signal $u(t) = \sin(\omega_0 t + \phi)$ is ***amplified or reduced*** by a linear system depending on the value of $|W(s)|_{s=j\omega_0}$.
  - An input signal $u(t) = \sin(\omega_0 t + \phi)$ is ***phase shifted*** by a linear system depending on the value of $\angle W(s)|_{s=j\omega_0}$.
- In other terms, ***a linear system can be designed as a filter*** able to amplify without distortion a certain set of input signals Ω_1 and reduce or eliminate another signals.
- Possible structures of filters will be discussed in the following lessons.

Harmonic response function

- This result underlines the importance of the function $W(j\omega)$ for the analysis of the forced response of LTI systems.

- The function $W(j\omega)$ is called ***harmonic response function*** of the system.

- In the following we present a method able to rapidly evaluate the magnitude and the phase $W(j\omega)$ as a function of ω.

Bode diagrams

- ***Bode diagrams*** allows to extract the magnitude and the phase of $W(j\omega)$ as a function of ω

- Bode diagrams are a main tool for the closed loop control design

$W(j\omega)$ general form

- Given an asymptotically stable LTI system, the ***harmonic response function*** $W(j\omega)$ is given by the ratio of polynomial with real and complex conjugate roots

$$W(j\omega) = W(s)\Big|_{s=j\omega} = K \left. \frac{s^{\nu} \prod_i (1+\sigma_i s)^{m_i} \prod_q \left(1 + \frac{2\xi_q}{\omega_{nq}} s + \frac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_j (1+\tau_j s)^{n_j} \prod_p \left(1 + \frac{2\zeta_p}{\omega_{np}} s + \frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}} \right|_{s=j\omega}$$

Bode diagrams definition

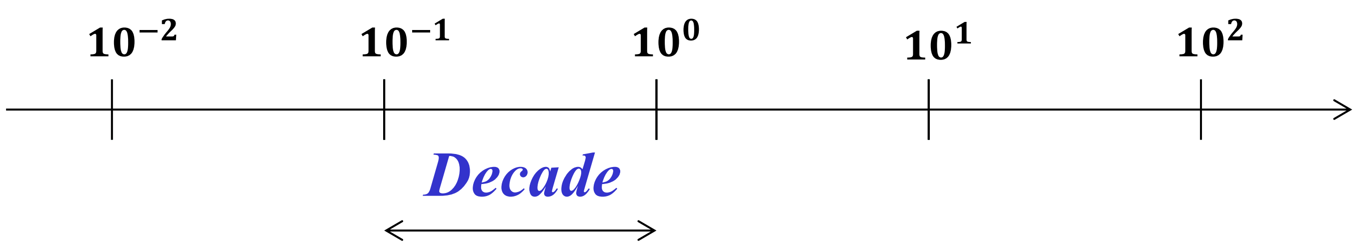
- In the Bode diagrams magnitude and phase of W(jω) are represented on two different Cartesian planes.

- ***The x-axis of both magnitude and phase Bode diagrams are in a logarithmic scale ($\log_{10}\omega$)***

On a logarithmic scale, the distance between two frequencies ω_1 and ω_2 depends on the difference of the logarithms and hence on the ratio on the frequencies

$$\log(\omega_2) - \log(\omega_1) = \log\left(\frac{\omega_2}{\omega_1}\right)$$

A decade is defined as the distance between two frequencies whose ratio is 10.



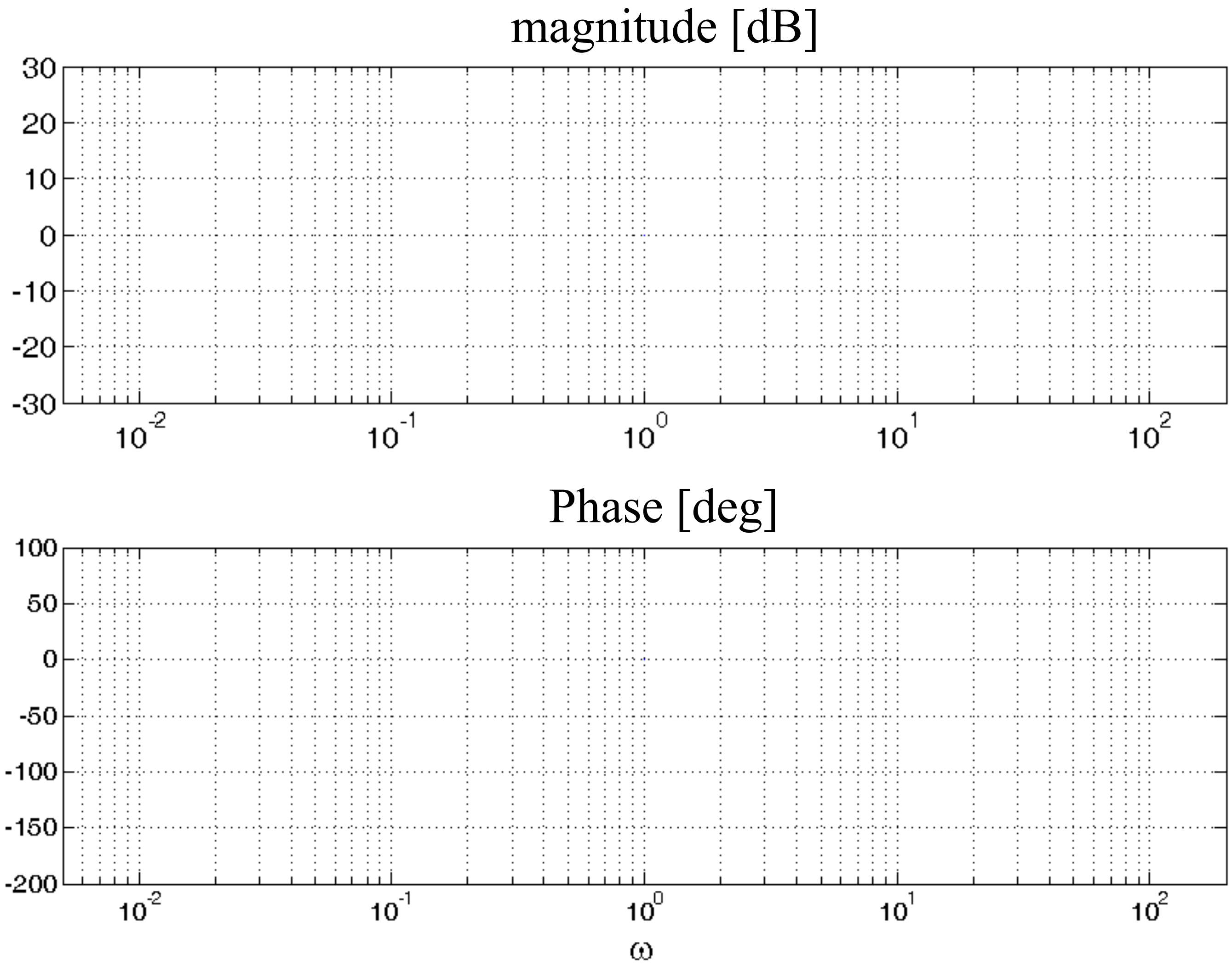
Bode diagrams definition

- The y-axis of the magnitude and phase Bode diagrams indicate respectively
  - ***the magnitude of the transfer function in dB (decibel)***

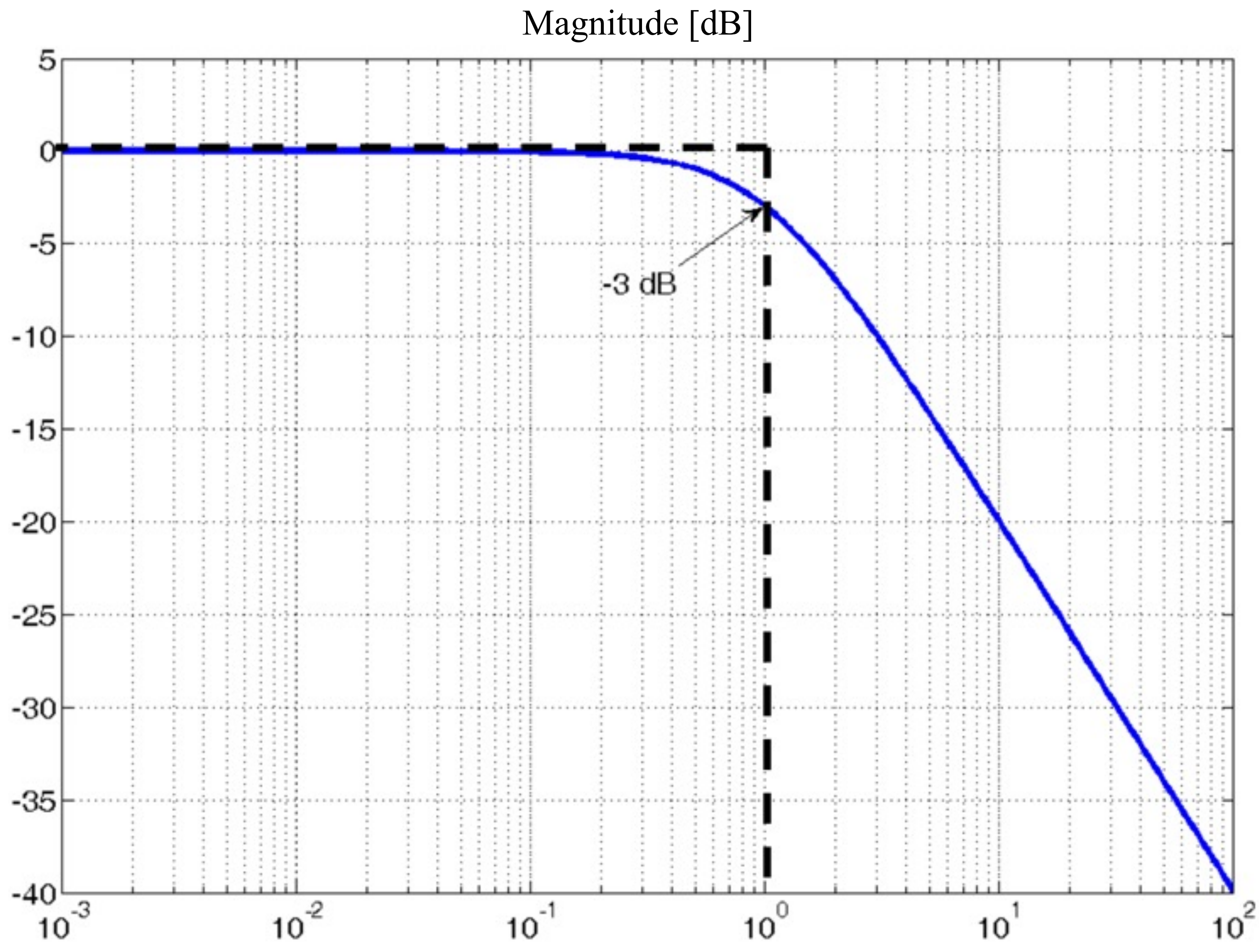
$$|W(j\omega)|_{db} = 20\log_{10}|W(j\omega)|$$

  - ***the phase of the transfer function in degrees or radians***

$$\angle W(j\omega)$$

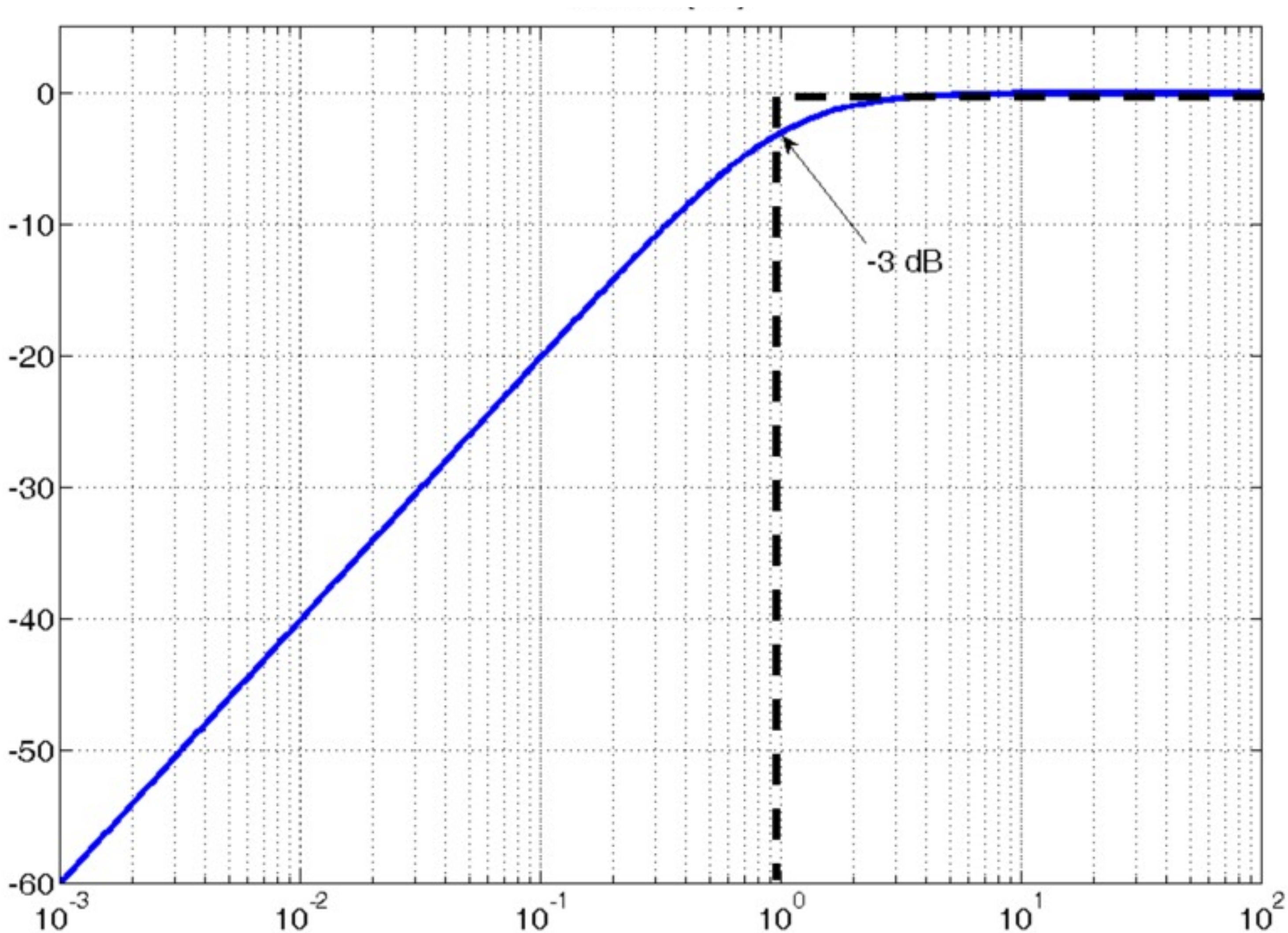

magnitude [dB]
30
20
10
0
-10
-20
-30
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Phase [deg]
100
50
0
-50
-100
-150
-200
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
ω

Low-pass filter

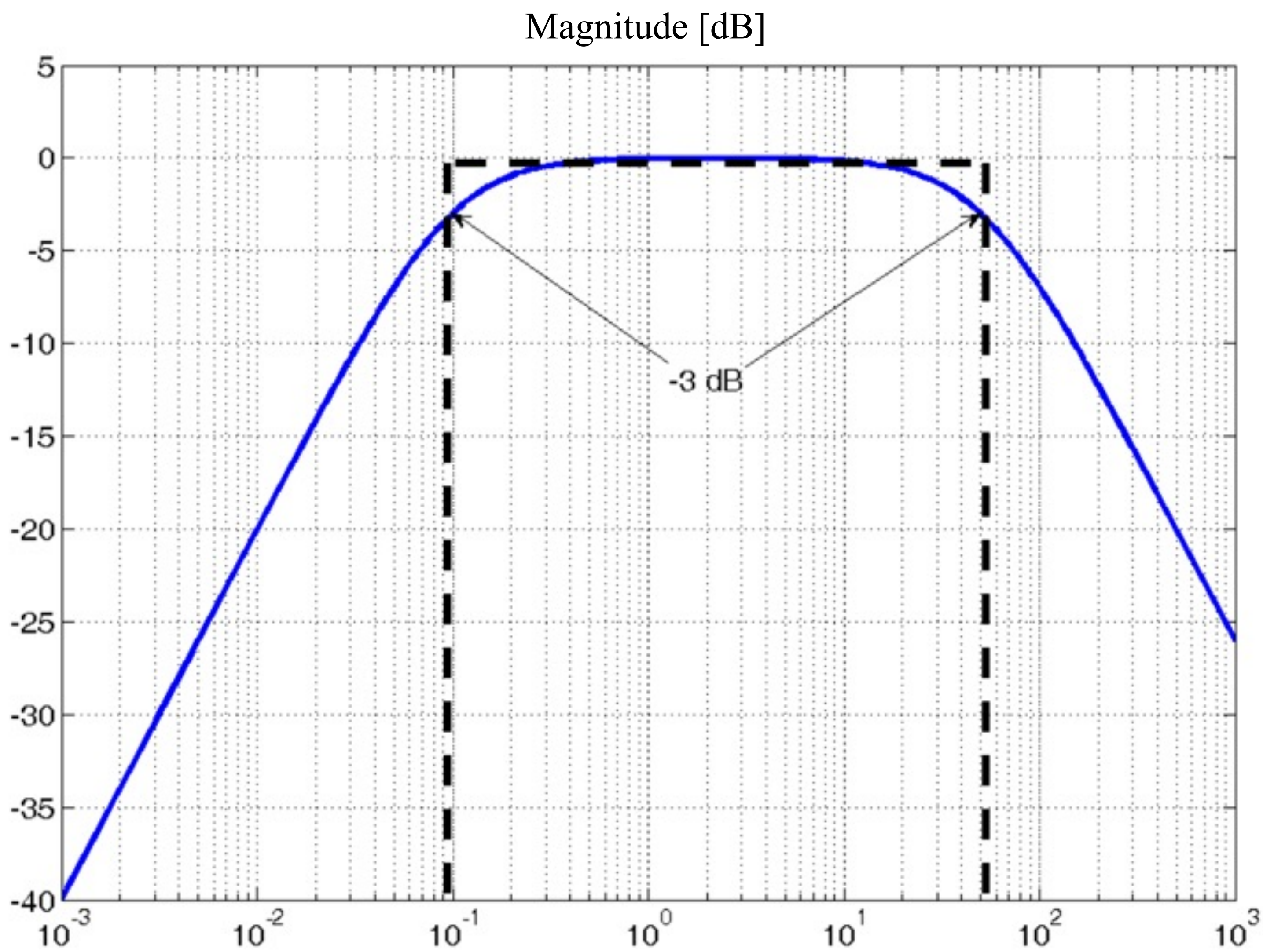


High-pass filter

Magnitude [dB]



Band-pass filter



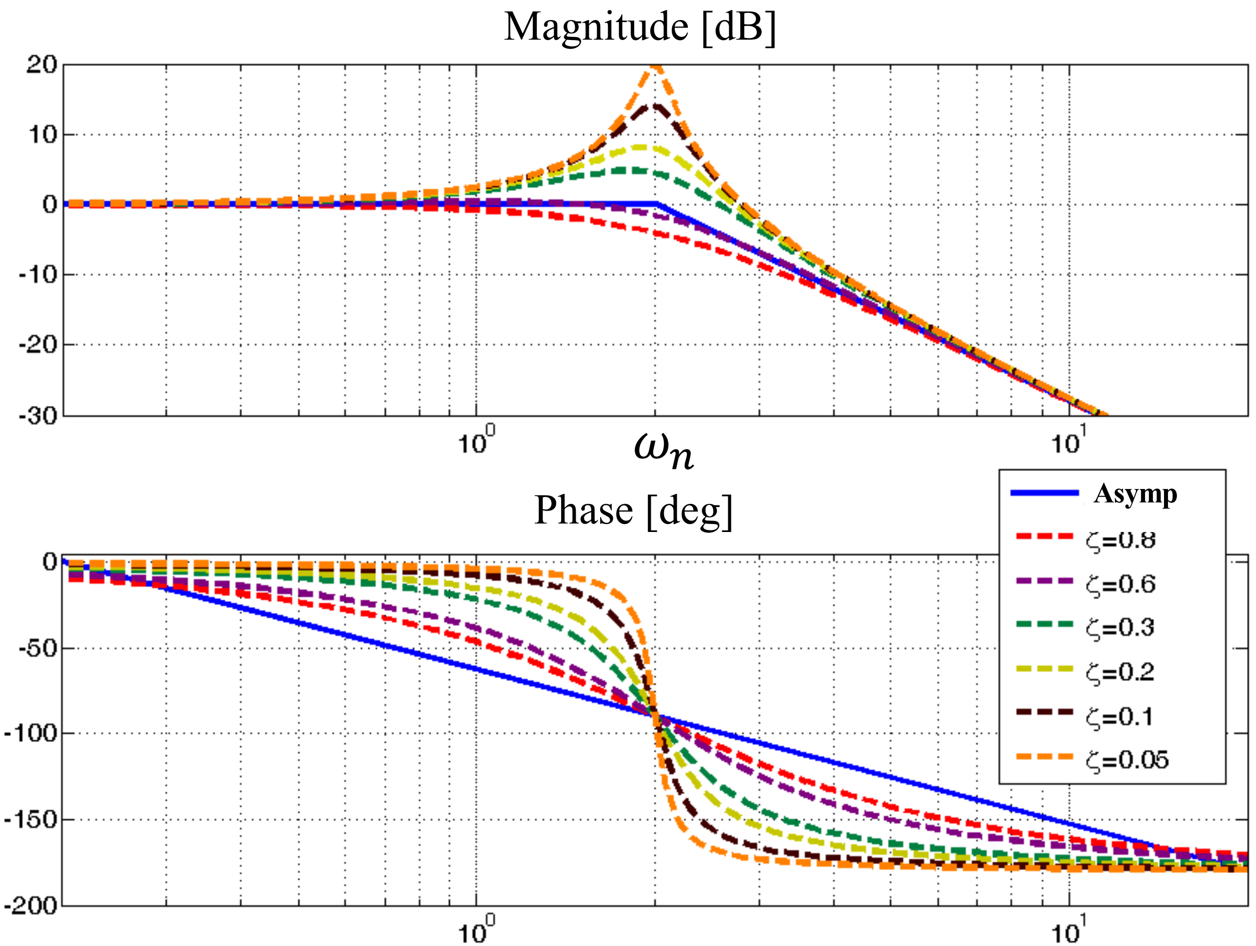
Complex conjugate poles of multiplicity one $W(s) = \left(1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}\right)^{-1}$

Peak module

$$M_p = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$$

Peak frequency

$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$



This result can be easily generalized to a generic trinomial term

Response to sinusoidal inputs of a second order LTI system

Let assume a second order LTI system described by the following t.f.:

$$G(s) = \frac{9}{s^2+s+9}$$