Course of
"Industrial Control System Security"
2024/25

Laplace transform, transfer function, Laplace domain analysis of LTIs

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

Università degli studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code: **09tkpu5**

Contents

Laplace transform definition

- ***The Laplace transform*** of a function ***f(t)*** is defined as

or $\mathcal{L}(f(t))$

$$f(t) \rightarrow F(s) = L(f(t)) = \int_{0}^{+\infty} f(t)e^{-st}dt$$

where $t \in R$ is a real variable, while $s = \alpha + j\omega \in C$ is a complex variable.

- Vice versa, given a function $F(s)$ in the Laplace domain, the original function in the time domain can be obtained using the ***Laplace anti-transformation***

$$F(s) \rightarrow f(t) = \lim_{\omega \to \infty} \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds$$

- ***The Laplace transform is a bilateral only if the function f(t) is null for*** $t < 0$

Laplace transform main properties (1/2)

- ***Linearity***

$$L\left(af(t)+bg(t)\right)=aF(s)+bG(s)$$

- ***Translation in the Laplace domain***

$$L\left(e^{\alpha t}f(t)\right)=F(s-\alpha)$$

- ***Translation in the time domain***

$$L\left(f(t-T)\right)=F(s)e^{-sT}$$

Laplace transform main properties (2/2)

- ***Time domain derivation***

$$L\left(\frac{df(t)}{dt}\right)=sF(s)-f(0)$$

- ***Time domain integration***

$$L\left(\int_0^t f(\tau)\,d\tau\right)=\frac{1}{s}F(s)$$

- ***Time domain convolution***

$$L\big(f(t)*g(t)\big)=F(s)G(s)$$

Additional properties useful in control theory

- ***Initial value theorem***

$$f(0) = \lim_{s\to\infty} sF(s)$$

- ***Final value theorem***

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- ***Initial value theorem of the derivate of the function***

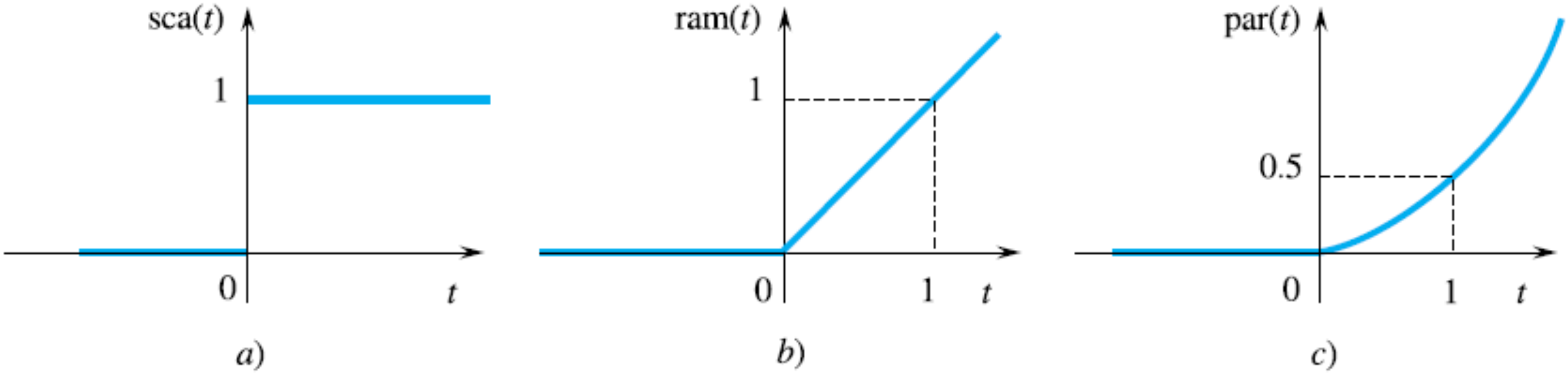
$$\left.\frac{df(t)}{dt}\right|_{t=0} = \lim_{s\to\infty} s^2F(s) - sf(0)$$

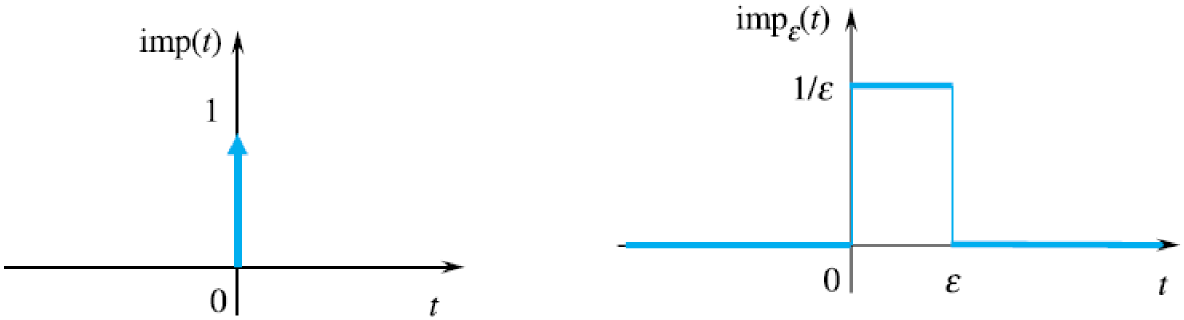
Selected Laplace transforms

- In the system theory, we will mainly use the Laplace transform for the evaluation of the forced response of LTI systems to selected sets of input :
  - ***Polynomial inputs*** $\boldsymbol{u(t) = t^n 1(t)}$
  - ***Sinusoidal inputs*** $\boldsymbol{u(t) = sin(\omega t)\, 1(t)}$

    $\boldsymbol{u(t) = cos(\omega t) 1(t)}$

Selected Laplace transforms





Selected Laplace transforms: polynomial signals

- In order to evaluate the Laplace transform of polynomial signals, let us firstly consider the Laplace transform of the impulse

  - ***Impulse $\delta(t)$*** $\longrightarrow$ $L(\delta(t)) = 1$ *(from the Laplace transform definition)*

- Then, using the ***time domain integration property***, we have

  - ***Step 1(t)*** $\longrightarrow$ $L(1(t)) = \dfrac{1}{s}$

  - ***Ramp t·1(t)*** $\longrightarrow$ $L(t \cdot 1(t)) = \dfrac{1}{s^2}$

  - ***Polinomial function $t^n \cdot 1(t)$*** $\longrightarrow$ $L(t^n \cdot 1(t)) = \dfrac{n!}{s^{n+1}}$

Selected Laplace transforms: sinusoidal signals

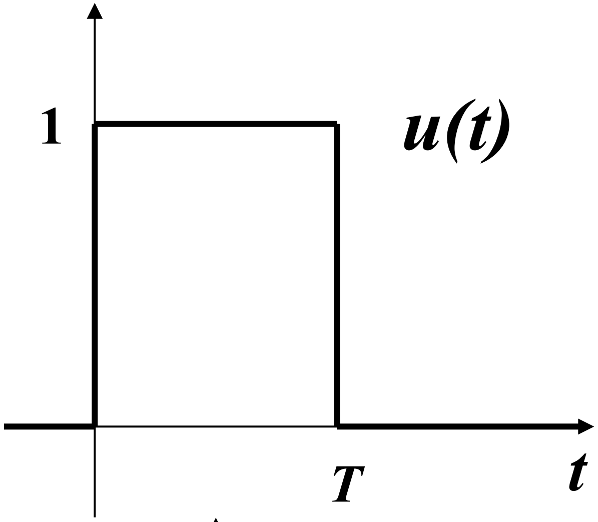
- The Laplace transform of sinusoidal functions
  - ***Sine*** $\boldsymbol{\sin(\omega t)1(t)}$ ⟶ $L(\sin(\omega t)\cdot 1(t)) = \dfrac{\omega}{s^2+\omega^2}$
  - ***Cosine*** $\boldsymbol{\cos(\omega t)1(t)}$ ⟶ $L(\cos(\omega t)\cdot 1(t)) = \dfrac{s}{s^2+\omega^2}$
- Finally, in the control theory the following transformations are of interest for the definition of the Laplace domain of the evolution modes of LTI systems

$$L(e^{\alpha t}1(t)) = \frac{1}{s-\alpha}$$

$$L\left(e^{\alpha t}\cos(\omega t)\cdot 1(t)\right) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2}$$

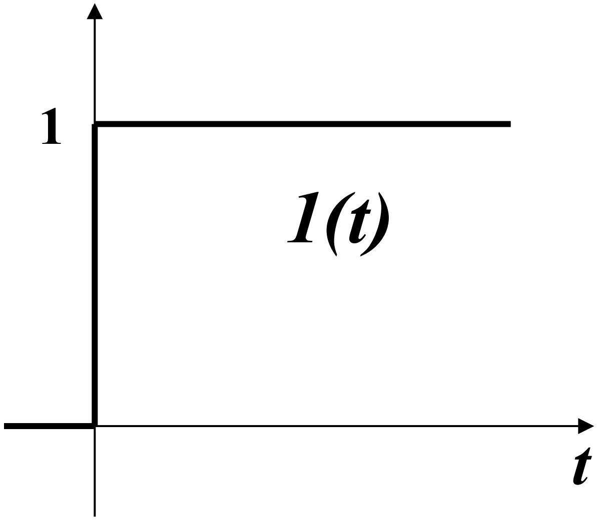
$$L\left(e^{\alpha t} sen(\omega t)\cdot 1(t)\right) = \frac{\omega}{(s-\alpha)^2+\omega^2}$$

Example: Laplace transform of a window signal

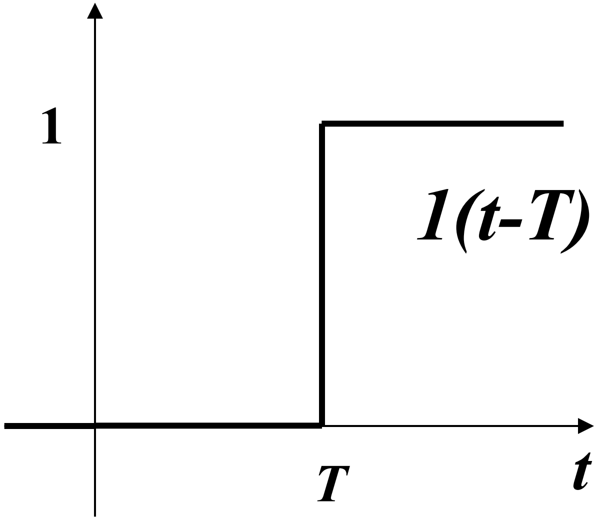


=

$$\boxed{u(t)=1(t)-1(t-T)}$$



−



Example: Laplace transform of a window signal

- The Laplace transform of a window signal can be evaluated from the Laplace transforms of two steps.

$$
\begin{aligned}
L(u(t)) &= L(1(t)-1(t-T)) \\
&= L(1(t))-L(1(t-T)) \\
&= \frac{1}{s}-\frac{e^{-sT}}{s}=\frac{1-e^{-sT}}{s}
\end{aligned}
$$

Solution of first order linear differential equation

Let us consider a first order differential equation, linear with constant coefficients,

$$\dot{y}(t) + a_0 y(t) = b_0 u(t), \quad y(t_0) = y_0$$

By applying Laplace trasform, assuming a step input signal, ***u*(*t*)=*U*$_0$1(*t*),** with amplitude U_0

$$L\big(\dot{y}(t) + a_0 y(t)\big) = L(b_0 U_0 1(t))$$

$$Y(s) = L(y(t))$$

$$L(bU_0 1(t)) = \frac{b_0 U_0}{s}$$

$$\Rightarrow \quad sY(s) - y_0 + a_0 Y(s) = \frac{b_0 U_0}{s} \Rightarrow \quad Y(s) = \underbrace{\frac{y_0}{s + a_0}}_{Y_{free}} + \underbrace{\frac{b_0 U_0}{s(s + a_0)}}_{Y_{forced}}$$

Solution of first order linear differential equation

$$Y_{free}(s) = \frac{y_0}{s + a_0} \xrightarrow{\mathcal{L}^{-1}} y_{free}(t) = e^{-\boldsymbol{a_0 t}} y_0 1(t)$$

$$Y_{forced}(s) = \frac{b_0 U_0}{s(s + a_0)} = \frac{A}{s} + \frac{B}{s + a_0}$$

Compute ***A*** and ***B*** by substitution:

$$Y_{\boldsymbol{forced}}(s) = \frac{A(s + a_0) + Bs}{s(s + a_0)}$$

$$= \frac{(A + B)s + Aa_0}{s(s + a_0)}$$

$$\begin{cases} A + B = 0 \\ Aa_0 = bU_0 \end{cases} \Rightarrow \boldsymbol{A} = \frac{b_0 U_0}{a_0} \quad \boldsymbol{B} = -\frac{b_0 U_0}{a_0}$$

Or by residual method:

$$A = (s - 0) Y_{forced}(s)|_{s=0}$$

$$= \frac{b_0 U_0}{s + a_0}|_{s=0} = \frac{b_0 U_0}{a_0}.$$

$$B = (s - (-a_0)) Y_f(s)|_{s=-a_0}$$

$$= \frac{b_0 U_0}{s}|_{s=-a_0} = -\frac{b_0 U_0}{a_0}.$$

Solution of first order linear differential equation

$$Y_{forced}(s) = \frac{b_0 U_0}{s(s+a_0)} = \frac{A}{s} + \frac{B}{s+a_0} = \frac{\frac{b_0 U_0}{a_0}}{s} + \frac{-\frac{bU_0}{a_0}}{s+a_0}$$

$$\mathcal{L}^{-1}$$

$$y_{forced}(t) = \frac{b_0 U_0}{a_0} 1(t) - \frac{b_0 U_0}{a_0} e^{\boldsymbol{-a_0 t}} 1(t) = \frac{b_0 U_0}{a_0} (1 - e^{\boldsymbol{-a_0 t}}) 1(t)$$

Then,

$$y(t) = y_{free}(t) + y_{forced}(t) = \left(e^{\boldsymbol{-a_0 t}} y_0 + \frac{b_0 U_0}{a_0} (1 - e^{\boldsymbol{-a_0 t}}) \right) 1(t)$$

Laplace Transform and Transfer function

- The analysis of LTI system is simplified by using Laplace transform.

- By exploiting the important property of the Laplace transform of the derivative of a signal $f(t)$ (with zero initial conditions, i.e. $f(0) = 0$)

$$\mathcal{L}\left(\dot{f}(t)\right) = sF(s),$$

- Given the differential equation of a linear system, it is possible to find the transfer function, $G(s)$, of that system, defined by

$$G(s) = \frac{Y(s)}{U(s)}$$

- Then for a LTI system of first order described by

$$\dot{y}(t) + a_0 y(t) = b_0 u(t), y(0) = y_0 = 0$$

$$\xrightarrow{\mathcal{L}} \quad sY(s) + a_0 Y(s) = b_0 U(s)$$

$$Y(s)(s + a_0) = b_0 U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s + a_0}$$

Laplace Transform and Transfer function

- Then, for a LTI system of second order described by

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t), y(0) = 0, \dot{y}(0) = 0$$

$$\mathcal{L} \Big\downarrow$$

$$s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_0 U(s)$$

$$Y(s)(s^2 + a_1 s + a_0) = b_0 U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^2 + a_1 s + a_0}$$

- Therefore, given the transfer function $G(s)$ and the input $u(t)$ with transfer function $U(s)$, the output is the product

$$Y(s) = G(s)U(s)$$

- Using Laplace transforms, the output $Y(s)$ can be expanded into its elementary terms (i.e., the sum of well-known transforms of selected/common signals/functions) by partial fraction expansion: therefore, it is possible to rewrite $Y(s)$ as a sum of terms for which is known the anti-transformation; the total time function $y(t)$ is given by the sum of these anti-transformation terms.

Laplace Transform and Transfer function

- Using Laplace transforms, the output $Y(s)$ can be expanded into its elementary terms (i.e., the sum of well-known transforms of selected/common signals/functions) by partial fraction expansion: therefore, it is possible to rewrite $Y(s)$ as a sum of terms for which is known the anti-transformation; the total time function $y(t)$ is given by the sum of these anti-transformation terms.

$$\begin{tikzcd}
Y(s) = \arrow[d, bend right=60, "\mathcal{L}^{-1}"'] & Y_1(s) + \arrow[d, "\mathcal{L}^{-1}"'] & Y_2(s) + \cdots \arrow[d, "\mathcal{L}^{-1}"'] \\
y(t) = & y_1(t) + & y_2(t) + \cdots
\end{tikzcd}$$

Y_i as,

$$Y_i(s) = \frac{A}{s} \xrightarrow{\mathcal{L}^{-1}} y_i(t) = A \cdot 1(t); \qquad Y_i(s) = \frac{A}{s - \alpha} \xrightarrow{\mathcal{L}^{-1}} y_i(t) = A \cdot e^{\alpha t} 1(t)$$

$$Y_i(s) = \frac{\omega}{(s-\alpha)^2 + \omega^2} \xrightarrow{\mathcal{L}^{-1}} y_i(t) = e^{\alpha t} \sin(\omega t)\, 1(t)$$

$$Y_i(s) = \frac{s - \alpha}{(s - \alpha)^2 + \omega^2} \xrightarrow{\mathcal{L}^{-1}} y_i(t) = e^{\alpha t} \cos(\omega t)\, 1(t)$$

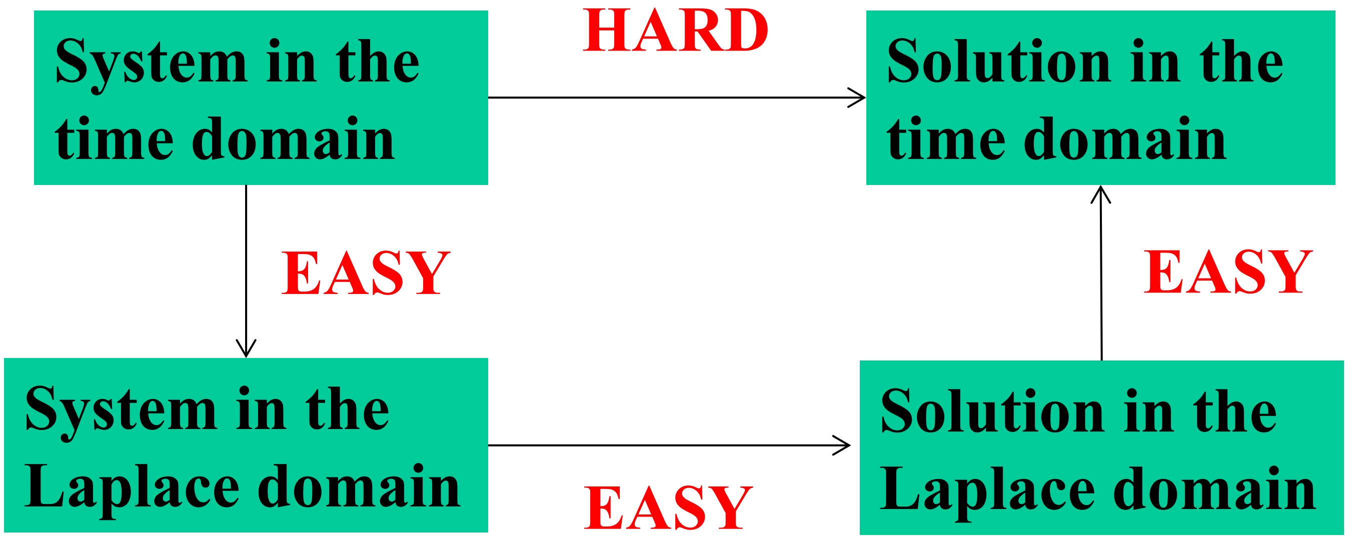
Evaluation of an LTI system response

- Let us consider a Linear Time Invariant (LTI) system in the state space form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \tag{1.a}$$

$$y(t) = Cx(t) + Du(t) \tag{1.b}$$

- The Evaluation of an LTI system response in a transformed domain is convenient only if



LTI systems in the Laplace domain

- Let us indicate with ***$X(s), U(s)$ and $Y(s)$ the Laplace transforms of the signals $x(t), u(t)$ and $y(t)$.***

Transforming both the sides of the equation (1a),

$$L(\dot{x}(t)) = L\big(Ax(t) + Bu(t)\big)$$

using the time domain derivation property of the Laplace transform, a linear system in the Laplace domain can be written has

$$sX(s) - x_0 = AX(s) + BU(s)$$

we have

$$\left(sI - A\right)X(s) = x_0 + BU(s)$$

and

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

The matrix function $\Phi(s) = (sI - A)^{-1}$is called ***Transition matrix,*** then

$$X(s) = \Phi(s)x_0 + \Phi(s)BU(s)$$

Recall: Inverse of a matrix

- Given a quadratic and invertible matrix

$$A = \begin{pmatrix} x_{1,1} & \dots & x_{1,j} \\ \vdots & \ddots & \vdots \\ x_{i,1} & \dots & x_{i,j} \end{pmatrix}$$

its inverse is defined as

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \mathrm{cof}(A, x_{1,1}) & \dots & \mathrm{cof}(A, x_{1,j}) \\ \vdots & \ddots & \vdots \\ \mathrm{cof}(A, x_{i,1}) & \dots & \mathrm{cof}(A, x_{i,j}) \end{pmatrix}^T$$

where the cofactor is

$$\mathrm{cof}(A, i, j) = (-1)^{i+j} \det(\mathrm{minor}(A, i, j))$$

and the minor (i, j) is the determinant of the matrix obtained excluding the row i and the column j.

Inverse of a 2×2 matrix

- Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a 3×3 matrix

- Given a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

its inverse is

$$\frac{1}{\det(A)} \begin{pmatrix} +\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & +\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\ -\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & +\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\ +\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} & +\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \end{pmatrix}$$

Transition matrix

- For the ***Transition matrix*** $\Phi(s) = (sI - A)^{-1}$
    - Each element is a rational function in ***s*** variable:
        - denominator of degree ***n*** given by $\mathbf{det}(\boldsymbol{sI - A}) = \boldsymbol{p_A(s)}$, whose roots are the **eigenvalues** of **A**.
        - numerator of element (i,j) corresponds to the algebraic complement of element (j,i) which by construction is a degree at most ***n***-1

⇨ $$\boldsymbol{\Phi_{ij}(s) = \frac{N(s)}{D(s)}}$$

$\boldsymbol{N(s)}$ of degree at most ***n*-1**

$\boldsymbol{D(s)}$ of degree **n,**

LTI systems in the Laplace domain

- Transforming both the sides of the equation (1b), we have

$$L(y(t)) = L\big(Cx(t) + Du(t)\big) \Leftrightarrow Y(s) = CX(s) + DU(s)$$

- and by substituting the previous equation, $X(s) = \Phi(s)x_0 + \Phi(s)BU(s)$

$$Y(s) = C\Phi(s)x_0 + C\Phi(s)BU(s) + DU(s)$$

$$Y(s) = C\Phi(s)x_0 + (C\Phi(s)B + D)U(s)$$

- The matrix function $G(s) = C\Phi(s)B + D = C(sI - A)^{-1}B$ is called ***transfer function,*** therefore

$$Y(s) = C\Phi(s)x_0 + G(s)U(s)$$

- For ***Single Input Single Output (SISO)*** systems the transfer function $G(s)$ is a scalar function;

- For ***Multiple Input Multiple Output (MIMO)*** systems the transfer function $G(s)$ is a matrix whose element $G(s)_{ij}$ will connect the output i with the input j.

Transfer function

- For the ***Transfer function*** $\boldsymbol{G}(\boldsymbol{s}) = \boldsymbol{C}\Phi(s)B + D = C(sI - A)^{-1}B + D$

$$\boldsymbol{\Phi_{ij}}(\boldsymbol{s}) = \frac{\boldsymbol{N}(\boldsymbol{s})}{\boldsymbol{D}(\boldsymbol{s})}$$

$\boldsymbol{N}(\boldsymbol{s})$ of degree at most ***n*-1**

$\boldsymbol{D}(\boldsymbol{s})$ of degree **n,**

$\boldsymbol{G}(\boldsymbol{s})$ is a rational function in ***s*** variable*:*

$$G(s) = \frac{a_m\, s^m + a_{m-1}\, s^{m-1} + \cdots + a_1\, s + a_0}{b_n\, s^n + a_{n-1}\, s^{n-1} + \cdots + b_1\, s + b_0}$$

- Since the multiplication on the left of $\boldsymbol{\Phi}(\boldsymbol{s})$ by $\boldsymbol{C}$ and the one on the right by $\boldsymbol{B}$ correspond to a linear combination of $\boldsymbol{\Phi}(\boldsymbol{s})$ elements, all with the same denominator, i.e. $\det(sI - A)$, then all the elements of $\boldsymbol{C}\Phi(s)B$ are rational functions in ***s*** with a denominator polynomial of degree ***n*** and a numerator of degree $\boldsymbol{m} \leq n - 1$:
- **If *D*=0, *m* < *n*,** ***the system is said strictly proper.***
- **If *D*≠ 0, *m* = *n*,** ***the system is said proper.***

Transfer function calculation: examples

$$\dot{x} = \begin{pmatrix} -2 & -1.5 \\ 2 & 0 \end{pmatrix} x + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} x$$

$$\Downarrow$$

$$G(s) = \frac{s+2}{s^2+2s+3}$$

$$\dot{x} = \begin{pmatrix} -4 & -2.5 \\ 2 & 0 \end{pmatrix} x + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} -1.5 & -1.25 \end{pmatrix} x + u$$

$$\Downarrow$$

$$G(s) = \frac{s^2+s}{s^2+4s+5}$$

Transfer function

- Given a ***transfer function***

$$G(s)=\frac{N(s)}{D(s)}=\frac{a_m\, s^m+a_{m-1}\, s^{m-1}+\cdots+a_1\, s+a_0}{b_n\, s^n+a_{n-1}\, s^{n-1}+\cdots+b_1\, s+b_0}$$

- The roots of the N(s) are said ***zeros.***
- The roots of the D(s) are said ***poles.***
- The polynomial D(s) is defined as $D(s)=\det(sI-A)$, hence
  - ***D(s) coincides with the characteristic polynomial of the system***
  - ***the poles coincide with the eigenvalues of the system*** except for possible pole-zero cancellation

LTI systems in the Laplace domain

Then, for a LTI system, by Laplace transform the state equation:

$$L(\dot{x}(t)) = L\big(Ax(t) + Bu(t)\big), \quad x(t_0) = x_0 \Rightarrow$$

$$\Rightarrow sX(s) - x_0 = AX(s) + BU(s) \Rightarrow (sI - A)X(s) = x_0 + BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1} x_0 + (sI - A)^{-1} BU(s)$$

$$X(s) = \boxed{\Phi(s)} x_0 + \Phi(s)BU(s)$$

Transition matrix

By Laplace transform the output equation: $L(y(t)) = L\big(Cx(t) + Du(t)\big)$

$$Y(s) = CX(s) + DU(s)$$

$$\Rightarrow = C\Phi(s)x_0 + C\Phi(s)BU(s) + DU(s)$$

$$= C\Phi(s)x_0 + \boxed{\big(C\Phi(s)B + D\big)}U(s)$$

G(s): transfer function

$$Y(s) = \overbrace{C\Phi(s)x_0}^{Y_{free}(s)} + \underbrace{G(s)U(s)}_{Y_{forced}(s)}$$

Laplace antitransform

- For SISO systems ***the free evolution in the Laplace domain is given by the ratio of polynomial functions***

$$Y_{free}(s) = C\Phi(s)x_0$$

- ***This is also true for the forced evolution in case we restrict our attention to the case of polynomial and sinusoidal inputs***

$$Y_{forced}(s) = G(s)U(s)$$

- It is convenient to antitransform $Y(s)$ by reducing the ratio of high degree polynomial functions to the sum of selected signals transform such as

$$L\left(e^{\alpha t}\cos(\omega t)\cdot 1(t)\right) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2} \qquad L\left(e^{\alpha t} sen(\omega t)\cdot 1(t)\right) = \frac{\omega}{(s-\alpha)^2+\omega^2}$$

$$L(e^{\alpha t}1(t)) = \frac{1}{s-\alpha}$$

LTI system, first order, strictly proper (d=0)

Consider a first order differential equation, linear with constant coefficients,

$$\dot{y}(t) + a_0 y(t) = b_0 u(t), \quad y(t_0) = y_0$$

that can be described by LTI system as

$$\dot{x}(t) = ax(t) + bu(t), \quad x(t_0) = x_0$$

$$y(t) = x(t)$$

Note that $b = b_0, a = -a_0, x_0 \ = y_0$

By Laplace transform

$$Y(s) = X(s) = \Phi(s)x_0 + G(s)U(s),$$

Where

$$\Phi(s) = \frac{1}{s-a}, G(s) = \Phi(s)b = \frac{b}{s-a}$$

Then,

$$Y(s) = \overbrace{\frac{y_0}{s-a}}^{\boldsymbol{Y(s)_{free}}} + \underbrace{\frac{b}{s-a}U(s)}_{\boldsymbol{Y(s)_{forced}}}$$

LTI system, first order, strictly proper: free and force responses

$$\tau = -\frac{1}{a}$$

$$Y_{free}(s) = \frac{y_0}{s-a} \xrightarrow{\mathcal{L}^{-1}} y_{free}(t) = e^{\boldsymbol{at}} y_0 1(t) = e^{-\frac{t}{\tau}} y_0 1(t)$$

$$u(t) = U_0 1(t)$$

$$Y_{forced}(s) = \frac{bU_0}{s(s-a)} = \frac{A}{s} + \frac{B}{s-a}$$

Compute ***A*** and ***B*** by substitution:

$$Y_{\boldsymbol{forced}}(s) = \frac{A(s-a)+Bs}{s(s-a)}$$

$$= \frac{(A+B)s - Aa}{s(s-a)}$$

$$\begin{cases} A + B = 0 \\ -Aa = bU_0 \end{cases} \Rightarrow \quad \boldsymbol{A} = \frac{bU_0}{-a} \quad \boldsymbol{B} = \frac{bU_0}{a}$$

Or by residual method:

$$A = (s-0)Y_{forced}(s)|_{s=0}$$

$$= \frac{bU_0}{s-a}|_{s=0} = \frac{bU_0}{-a}.$$

$$B = (s-a)Y_f(s)|_{s=a}$$

$$= \frac{bU_0}{s}|_{s=a} = \frac{bU_0}{a}.$$

LTI system, first order, strictly proper: free and force responses

$$Y_{forced}(s) = \frac{bU_0}{s(s-a)} = \frac{A}{s} + \frac{B}{s-a} = \frac{\frac{bU_0}{-a}}{s} + \frac{\frac{bU_0}{a}}{s-a}$$

By denoting with $G_0 = \frac{b}{-a}, \tau = -\frac{1}{a}$

$$Y_{forced}(s) = \frac{G_0U_0}{s} - \frac{G_0U_0}{s-a}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y_{forced}(t) = G_0U_0\left(1 - e^{-\frac{t}{\tau}}\right)1(t),$$

Then,

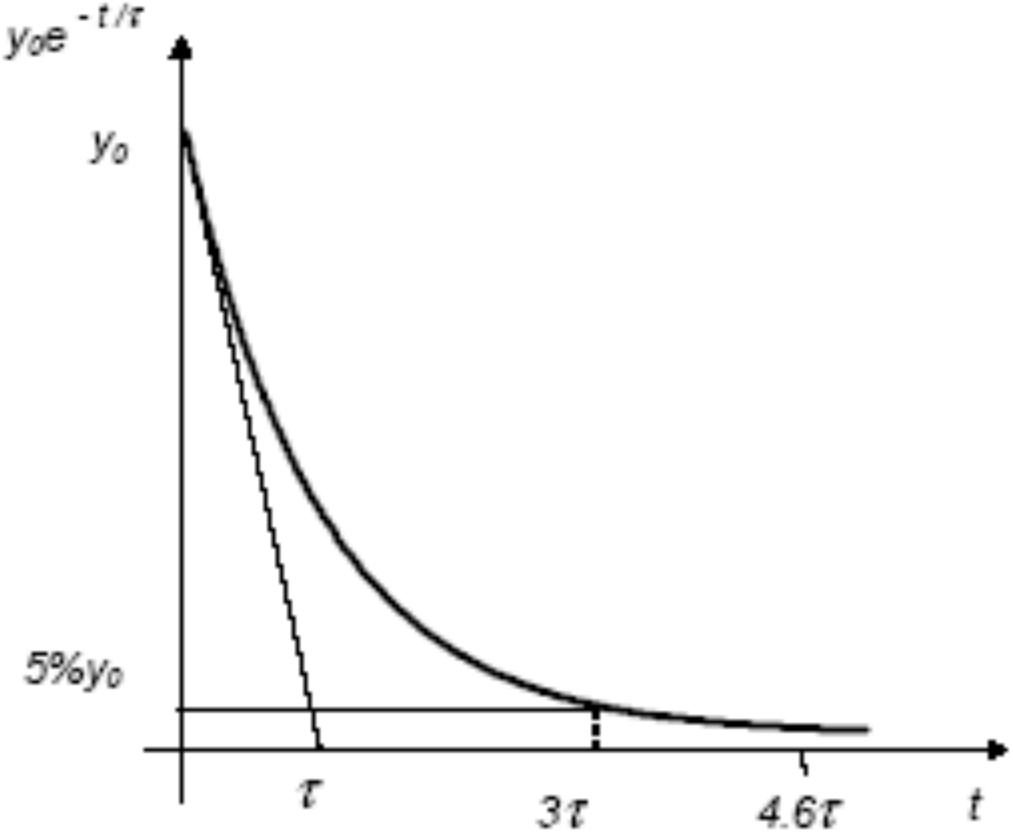
$$y(t) = y_{free}(t) + y_{forced}(t) = \left(e^{-\frac{t}{\tau}}y_0 + G_0U_0\left(1 - e^{-\frac{t}{\tau}}\right)\right)1(t)$$

LTI system, first order, strictly proper: free response

$$\dot{x}(t) = ax(t) + bu(t)$$
$$y(t) = x(t)$$
$$x(t_0) = \boldsymbol{x_0}$$

$$u(t) = \mathbf{0}$$

$$y_{free}(t) = e^{-\frac{t}{\tau}} y_0$$



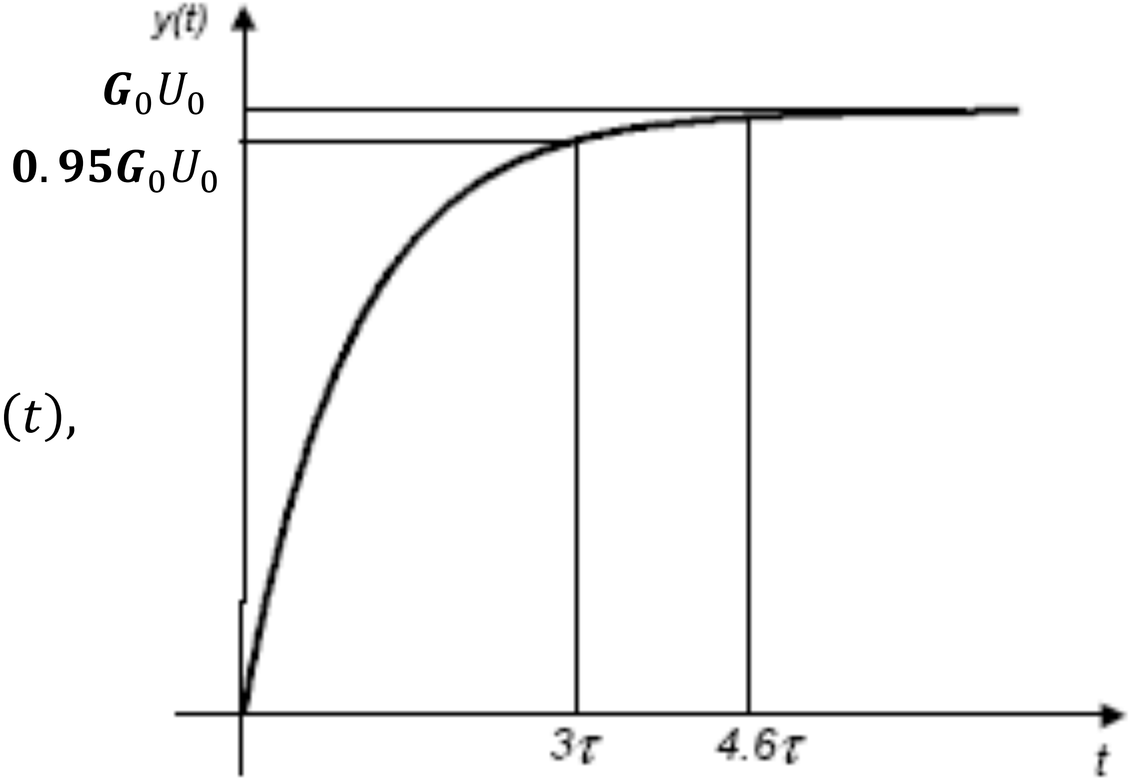
LTI system, first order, strictly proper: step response

$$\dot{x}(t) = ax(t) + bu(t)$$
$$y(t) = x(t)$$
$$x(t_0) = \mathbf{0}$$

$$u(t) = U_0 1(t)$$

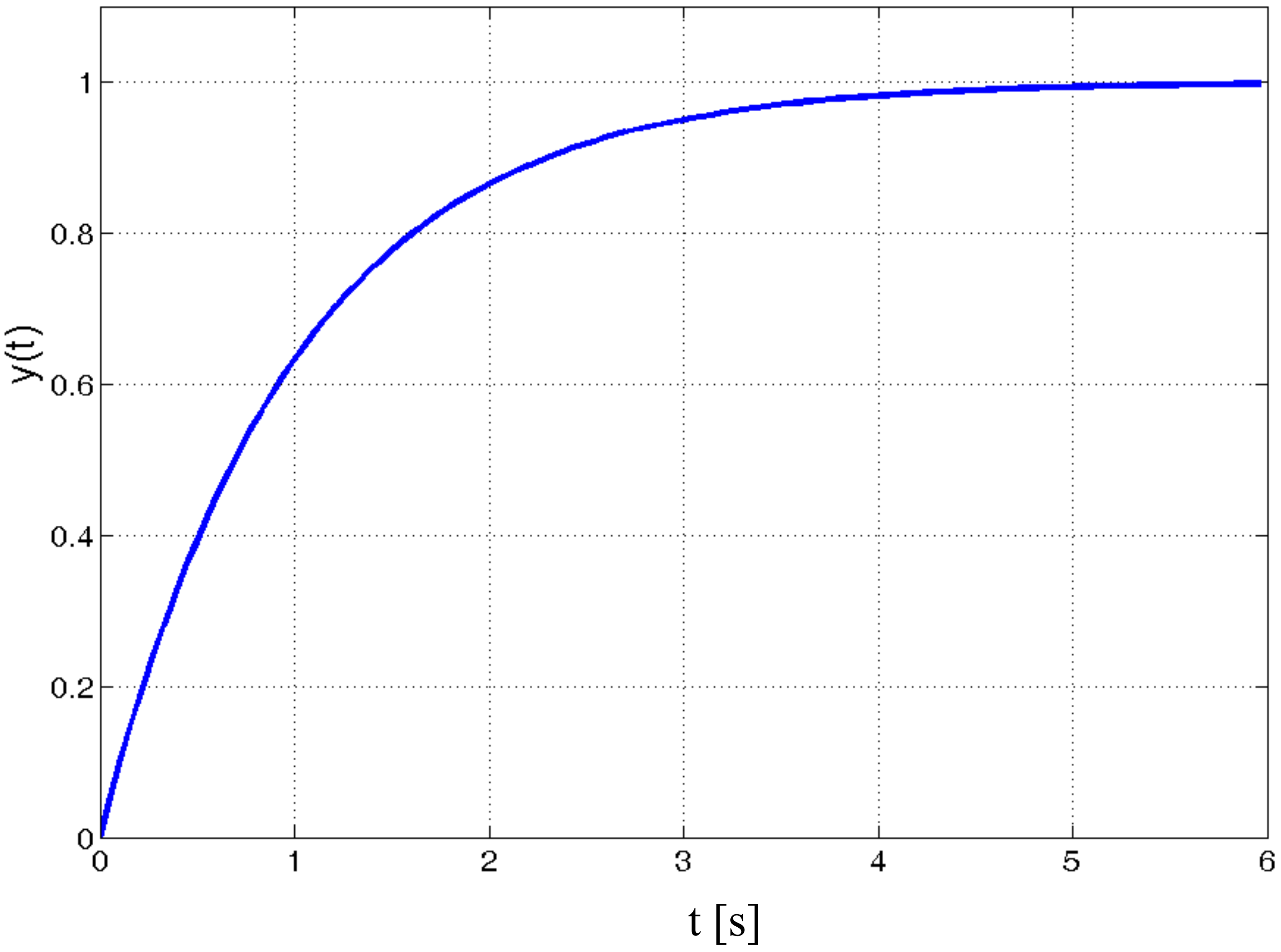
$$y_{step}(t) = \boldsymbol{G}_0 U_0 \left(1 - e^{-\frac{t}{\tau}}\right) 1(t),$$

$$\boldsymbol{G}_0 = \frac{b}{-a}, \tau = -\frac{1}{a}$$



LTI system, first order, strictly proper: step response

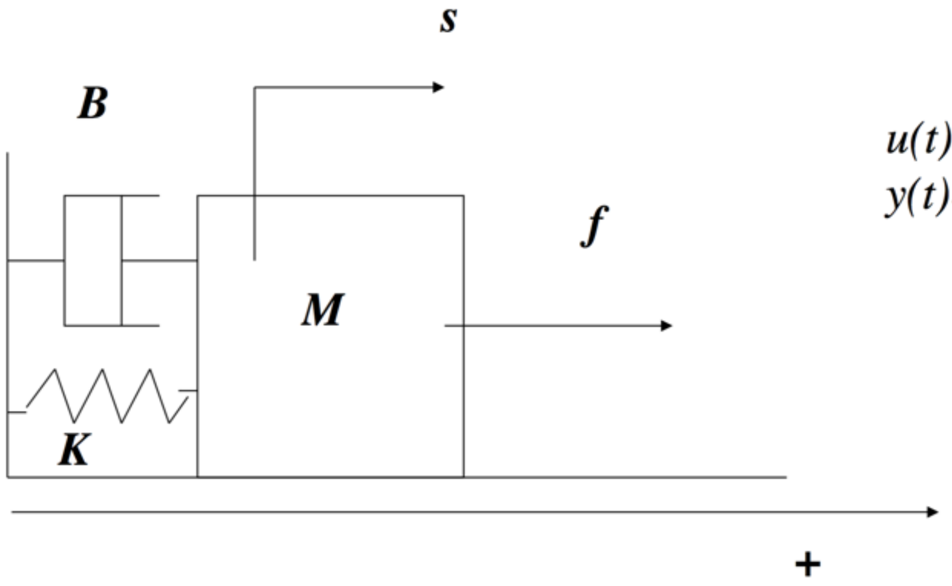
Response to a step input for a first order, strictly proper system ($\boldsymbol{G_0} = 1$, $\tau = 1$)



LTI system, first order, strictly proper: parameters for the qualitative step response

- ***Initial value*** $y(0) = 0$
- ***Final value*** $lim_{t\to\infty} y(t) = G_0 U_0$
- ***Settling time***
  - $t_{s\ 5\%} = 3\tau$
  - $t_{s\ 1\%} = 4.6\tau$
- ***Rise time*** $t_r \cong 2.2\tau$

Example: mass-spring-damper system



$u(t)=f(t)$
$y(t)=s(t)$

- **Input output representation**

$$\boldsymbol{M\ddot{y}(t) + B\dot{y}(t) + Ky(t)=u(t)}$$

- **State space representation**

$$x_1=s \text{ e } x_2=v=ds/dt$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ \dot{s} \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/M \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

In general a second order system…as a mass-spring-damper system

- **Input output representation**

$$\ddot{y}(t)+a_1\dot{y}(t)+a_0y(t)=bu(t)$$

- State space representation

  $x_1=y$ e $x_2=\dot{y}=dy/dt$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}y\\\dot{y}\end{pmatrix}$$

$$\dot{x}=\begin{pmatrix}\dot{x}_1\\\dot{x}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\-a_0 & -a_1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0\\b\end{pmatrix}u,$$

$$y=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

- Transition matrix

$$\Phi(s)=\frac{\begin{pmatrix}s+a_1 & 1\\-a_0 & s\end{pmatrix}}{s^2+a_1s+a_0}$$

- Transfer function

$$G(s)=\frac{b}{s^2+a_1s+a_0}$$

$$Y(s)=C\Phi(s)x_0+\ G(s)U(s)$$

In general a second order system… as a mass-spring-damper system

$$Y(s) = \underbrace{C\Phi(s)x_0}_{Y(s)_{free}} + \underbrace{G(s)U(s)}_{Y(s)_{forced}}$$

$$x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

➡

$$Y(s)_{free} = \frac{(s + a_1)x_{10} + x_{20}}{s^2 + a_1 s + a_0}$$

$$U(s) = \frac{U_0}{s}, \quad Y(s)_{step} = \frac{b}{s^2 + a_1 s + a_0} \cdot \frac{U_0}{s}$$

Characteristic equation …

- The characteristic equation, $\boldsymbol{s^2 + a_1 s + a_0 = 0}$, determines the evolution modes

Three cases:

- ***real and distinct poles***
- ***real multiple poles***
- ***complex conjugates poles***

Laplace Transform and Transfer function

- ***Real and distinct poles,*** **terms as**

$$\frac{1}{s-a}$$

corresponding to a real pole/real eigenvalue of the dynamic matrix

- ***Real multiple poles, a term as***

$$\frac{1}{(s-a)^2}$$

corresponding to a real multiple pole/eigenvalue of the dynamic matrix

- ***Complex conjugates poles,*** **terms as**

$$\frac{\omega}{(s-\alpha)^2+\omega^2} \quad \textbf{or} \quad \frac{s-\alpha}{(s-\alpha)^2+\omega^2}$$

corresponding to a pair of complex conjugate poles/eigenvalues $a \pm j\omega$ of the dynamic matrix

Laplace antitransform: example 1

CASE 1: real and distinct eigenvalues/poles

$$Y_{free}(s) = C\Phi(s)x_0 = \frac{s-10}{s^2+7s+10} = \frac{s-10}{(s+2)(s+5)}$$

- Appling the residual method we have

$$Y_{free}(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+5)}$$

with

$$A_1 = \lim_{s\to -2}(s+2)Y_{free}(s) = \lim_{s\to -2}\frac{s-10}{s+5} = -4$$

$$A_2 = \lim_{s\to -5}(s+5)Y_{free}(s) = \lim_{s\to -5}\frac{s-10}{s+2} = 5$$

Hence

$$y_{free}(t) = (-4e^{-2t} + 5e^{-5t}) \cdot 1(t)$$

Laplace antitransform: example 2

CASE 2: real multiple eigenvalues/poles

$$Y_{forced}\,(s) = G(s)U(s) = \frac{18}{s^2+6s+9}U(s) \qquad \text{with } u(t) = 1(t)$$

- This function can be written as the sum of three terms

$$Y_{forced}\,(s) = \frac{18}{s(s+3)^2} = \frac{A_1}{s} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+3)^2}$$

- The residual method can be applied to evaluate A_1 and A_3, while A_2 can be evaluated by substitution

$$A_1 = \lim_{s\to 0} sY_{forced}(s) = 2 \qquad A_3 = \lim_{s\to -3} (s+3)^2\, Y_{forced}(s) = -6$$

while $A_2 = -2$.

Hence, $$y_{forced}(t) = (2 - 2e^{-3t} - 6te^{-3t}) \cdot 1(t)$$

Laplace antitransform: example 3

CASE 3: complex conjugate eigenvalues/poles

$$Y_{free}\,(s) = C\Phi(s)x_0 = \frac{s+3}{(s^2+4s+13)}$$

- This function can be written as the sum of two terms

$$Y_{free}\,(s) = \frac{s+3}{(s^2+4s+4+9)} = \frac{s+2-2+3}{((s+2)^2+3^2)} =$$

- Hence, $Y_{free}\,(s) = \frac{s+2}{(s+2)^2+3^2} + \frac{1}{3}\frac{3\cdot 1}{(s+2)^2+3^2}$

$$y_{free}(t) = \left(e^{-2t}\cos(3t) + \frac{1}{3}e^{-2t}\sin(3t)\right)\cdot 1(t)$$

$$y_{free}(t) = e^{-2t}\left(\cos(3t) + \frac{1}{3}\sin(3t)\right)\cdot 1(t)$$

Stability

- A linear system is said ***stable*** if no evolution mode is divergent (only convergent and constant evolution modes).

- It happens if all the eigenvalues of the matrix A (pole of $G(s)$) have a negative or null real part and the eigenvalues with null real part have multiplicity 1.

- In a stable system
  - ***the free evolution doesn't tend to infinity***
  - ***the free evolution doesn't converge to zero*** if the constant evolution mode is excited

Asymptotic stability

- A linear system is said ***asymptotically stable*** if all evolution modes are convergent.
- It happens if all the eigenvalues of the matrix A (pole of $G(s)$ have negative real part
- In an ***asymptotically stable*** system
  - ***the free evolution converges to zero***

Unstability

- A linear system is said ***unstable*** if there is a divergent evolution mode.

- It happens if an eigenvalues of the matrix A (pole of $G(s)$ have a real part positive or an eigenvalue (pole) with null real part with multiplicity >1.

- In an ***unstable*** system
  - ***the free evolution tends to infinity***

Example 1

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In order to evaluate the eigenvalues of the matrix A, we can calculate the roots of the characteristic polynomial
- In Matlab, it is possible to use the command ***eig(A)***
- In this example we have $p_1 = p_2 = -1$.
- This system is ***asymptotically stable*** because it has all eigenvalues with negative real part

Example 2

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In this example we have $p_1 = p_2 = 1$.

- The system is ***unstable*** because it has two eigenvalues with positive real part

Example 3

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In this example we have $p_1 = 0$, $p_2 = -1$.

- The system is ***stable*** because it has
  - a null eigenvalue
  - an eigenvalue with negative real part

Example 4

- Let us consider the transfer function of an LTI system

$$G(s) = \frac{s+1}{s^2(s+5)}$$

- This system is ***unstable*** because it has two null poles.