Course of
"Industrial Control System Security"
2024/25

Analysis of LTI systems in the time domain

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LTI systems in the time domain

- ***Linear time invariant (LTI) systems*** in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}, \quad x(t_0) = x_0$$

with $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$, where $x(t)$ is the state vector, $u(t)$ is the input vector and $y(t)$ is the output vector of the system.

Lagrange Formula

- Let us consider a ***Linear Time Invariant*** (***LTI***) system in the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) = Cx(t) + Du(t) \tag{1}$$

The solution of the linear differential equation (1) defines the ***time evolution of the state variables*** and it is given by the Lagrange Formula

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B\, u(\tau)\, d\tau, \quad t \geq t_0 \tag{2}$$

- The ***time evolution of the outputs*** turns out to be

$$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(t-\tau)}B\, u(\tau)\, d\tau + D\, u(t), \quad t \geq t_0 \tag{3}$$

Lagrange Formula

- Taking into account that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)\,d\tau = f\big(t,b(t)\big)\frac{db(t)}{dt} - f\big(t,a(t)\big)\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{d}{dt} f(t,\tau)\,d\tau$$

- Lagrange formula (2) can be easily verified by derivation (assuming $t_0 = 0$)

$$\dot{x}(t) = \frac{d}{dt}(e^{At}x_0) + e^{A(t-t)}Bu(t) + \int_0^t \frac{d}{dt}\left[e^{A(t-\tau)}B\,u(\tau)\right]d\tau$$

$$= Ae^{At}x_0 + Bu(t) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau$$

$$= A\left[e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right] + Bu(t) = Ax(t) + Bu(t)$$

Free and forced evolution of LTI systems

- The ***time evolution of the state and output variables*** can be conceptually divided in two parts,

$$x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{Free evolution, } x_l(t)} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}B\,u(\tau)\,d\tau}_{\text{Forced evolution, } x_f(t)}, \quad t \geq t_0$$

Free evolution, $x_l(t)$ ***Forced evolution,*** $x_f(t)$

$$y(t) = \overbrace{Ce^{A(t-t_0)}x_0}^{\text{Free evolution}} + \overbrace{C\int_{t_0}^{t} e^{A(t-\tau)}B\,u(\tau)\,d\tau + D\,u(t)}^{\text{Forced evolution}}, \quad t \geq t_0$$

- The ***free evolution*** indicate the evolution of state and output vectors that would be obtained in the absence of input ($u(t) = 0$).

- The ***forced evolution*** indicate the evolution of state and output vectors that would be obtained in the presence of input and null initial conditions ($x_0 = 0$)

Free evolution: matrix $\boldsymbol{A}$ diagonalizable

- The free evolution of an LTI system in the time domain is defined by the matrix exponential $\boldsymbol{e^{At}}$. Generalizing the Taylor expansion of an exponential to the matrix case, we have

$$e^M = \sum_{i=0}^{\infty} \frac{1}{i!} M^i = I_n + M + \frac{M^2}{2!} + \cdots$$

- In case the matrix A has real and distinct eigenvalues, λ_i, i=1,…,n, it is diagonalizable: it is possible to find a matrix $\boldsymbol{T_D}$, such that $\widehat{\boldsymbol{A}}=\boldsymbol{T_D A\ T_D}^{-1}= \text{diag}\{\lambda_1, \dots, \lambda_n\}$. Then $\boldsymbol{A}=\boldsymbol{T_D}^{-1}\widehat{\boldsymbol{A}}\ \boldsymbol{T_D}$ and $\boldsymbol{e^{At}}$ turns out to be

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} (A\,t)^i = T_D^{-1} \sum_{i=0}^{\infty} \frac{1}{i!} (\widehat{A}\,t)^i\, T_D$$

$$= T_D^{-1} \text{diag}\left\{ \sum_{i=0}^{\infty} \frac{(\lambda_1 t)^i}{i!}, \sum_{i=0}^{\infty} \frac{(\lambda_2 t)^i}{i!}, \cdots, \sum_{i=0}^{\infty} \frac{(\lambda_n t)^i}{i!} \right\}$$

$$= \boldsymbol{T_D^{-1} \textbf{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t} \dots, e^{\lambda_n t}\} T_D}.$$

Remark: ***$\lambda_1, \lambda_2 \dots \lambda_n$ are the eigenvalues of the A matrix, T_D is the transformation (i.e., $\hat{x} = T_D x$) that allows achieving a diagonal $\widehat{A}$ matrix.***

Free evolution: aperiodic and pseudo-periodic modes

- The exponential terms,

$$\boldsymbol{e^{\lambda_i t}}$$

are the modes of the system, named ***aperiodic modes***

- In case of ***complex conjugate eigenvalues*** $\boldsymbol{\lambda_i = \alpha_i + j\omega_i}$ e $\boldsymbol{\bar{\lambda}_i = \alpha_i - j\omega_i}$, the corresponding complex exponential function determines a term as follows:

$$\boldsymbol{e^{\alpha_i t}} \sin(\boldsymbol{\omega_i t + \varphi_i})$$

These latter modes are named ***pseudo-periodic modes.***

Free evolution: A is not diagonalizable

- In case of a non-diagonalizable $\boldsymbol{A}$ matrix (eigenvalues with multiplicity ($\boldsymbol{\eta}$) greater than one) we must resort to the Jordan form (see text for further information): the matrix $\widehat{\boldsymbol{A}}$ has an almost diagonal structure, with the elements on the diagonal corresponding to the eigenvalues, with the addition of superdiagonal elements, Jordan miniblocks, which determine terms of the type

$$\boldsymbol{t}^{\boldsymbol{\eta}-\mathbf{1}}\boldsymbol{e}^{\boldsymbol{\lambda}_{\boldsymbol{i}}\boldsymbol{t}}, \text{ if } \lambda_i \in \mathbb{R},$$

or

$$\boldsymbol{t}^{\boldsymbol{\eta}-\mathbf{1}}\boldsymbol{e}^{\boldsymbol{\alpha}_{\boldsymbol{i}}\boldsymbol{t}}\sin(\boldsymbol{\omega}_{\boldsymbol{i}}\boldsymbol{t}+\boldsymbol{\varphi}_{\boldsymbol{i}}) \text{ if } \lambda_i \in \mathbb{C}$$

where $\boldsymbol{\eta}$ is an integer between 1 and the maximum size of the Jordan miniblocks associated with λ_i.

Summary: free evolution of LTI system

- **Analysis in the time domain**

$$x_l(t) = e^{At}x_0$$
$$y_l(t) = Ce^{At}x_0$$

- **$x_1(t)$ (and then $y_1(t)$) is given by a combination of terms as**
  - $\boldsymbol{e^{\lambda_i t}}$**,** in the case of real and distinct eigenvalues**;**
  - $\boldsymbol{e^{\alpha_i t}}\sin(\boldsymbol{\omega_i t + \varphi_i})$, in the case of complex conjugate eigenvalues of multiplicity one**;**
  - $\boldsymbol{t^{\eta-1}e^{\lambda_i t}}$**,** in the case of real eigenvalues with multiplicity $\boldsymbol{\eta}$ **>1;**
  - $\boldsymbol{t^{\eta-1}e^{\alpha_i t}}\sin(\boldsymbol{\omega_i t + \varphi_i})$ in the case of complex conjugate eigenvalues of multiplicity $\boldsymbol{\eta}$ **>1.**

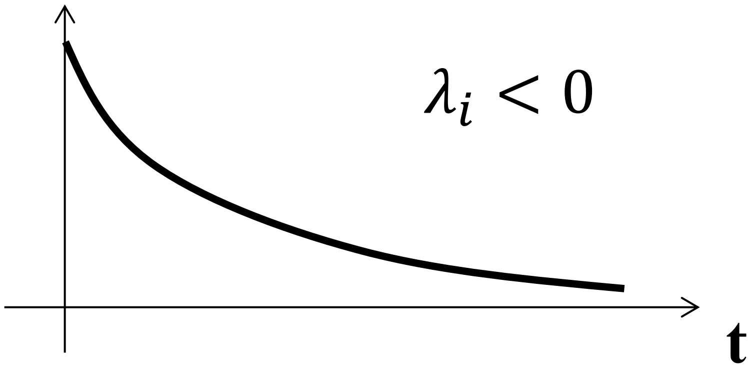
Aperiodic evolution modes (1/3)

- ***An aperiodic mode*** is an evolution mode of a linear system related to a real eigenvalue of the matrix A of multiplicity 1. It can be written in the form

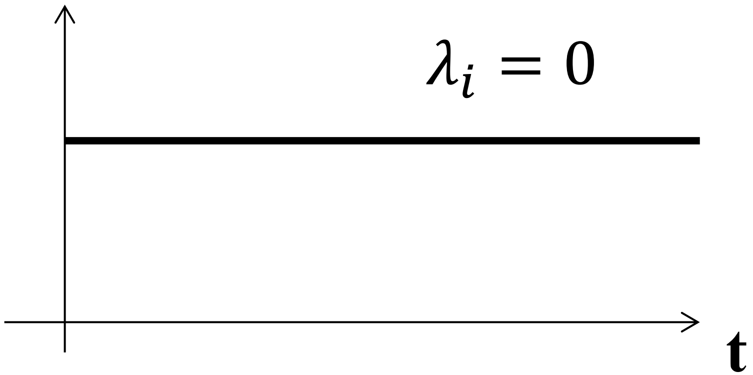
$$e^{\lambda_i t}$$

- Depending on the sign of the eigenvalue λ_i, an aperiod evolution mode can be
  - convergent ($\lambda_i < 0$)
  - constant ($\lambda_i = 0$)
  - divergent ($\lambda_i > 0$)

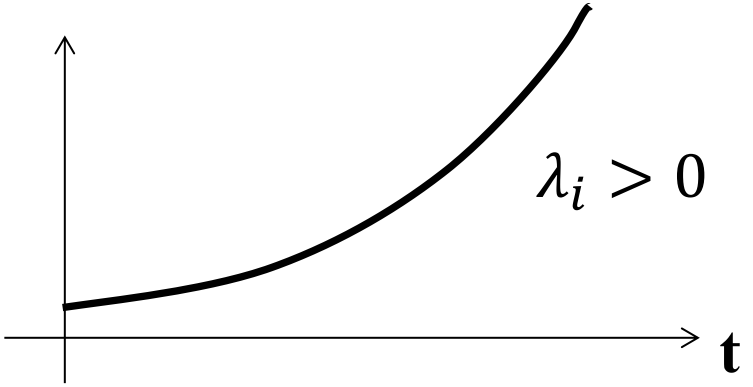
- *Convergent aperiodic mode*



- *Constant aperiodic mode*



- *Divergent aperiodic mode*



Aperiodic evolution modes (3/3)

- When the evolution mode is convergent it is possible to introduce a new parameter said ***time constant of the mode*** defined as

$$\tau_i = -\frac{1}{\lambda_i}$$

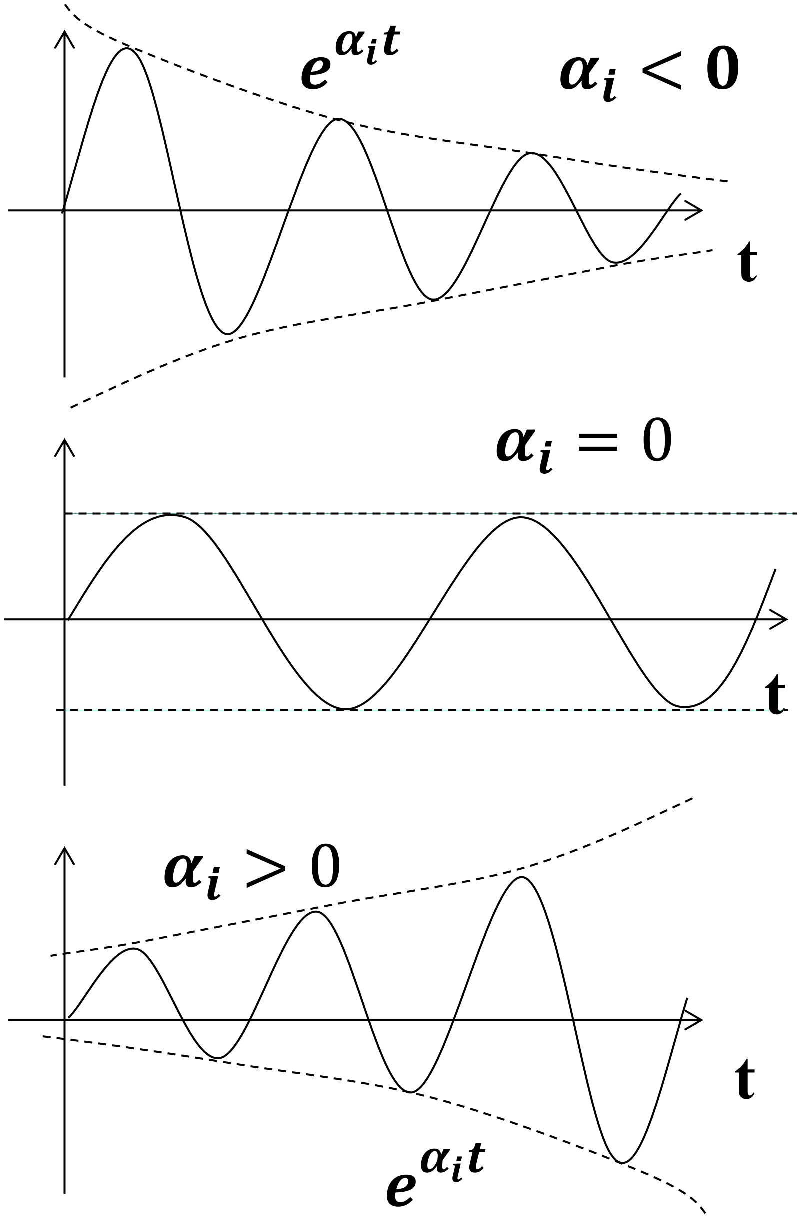
- The time constant gives us an information about the time needed before the convergent mode will be extinguished.

- It is straightforward to verify that
  - ***After a time*** $\bar{t} = 3\tau$ the magnitude of the mode will be reduced to the 5% of the initial value
  - ***After a time*** $\bar{t} = 4.6\tau$ the magnitude of the mode will be reduced to the 1% of the initial value

Pseudo-periodic evolution modes (1/5)

$$e^{\alpha_i t}\sin(\omega_i t+\varphi_i)$$

- *Convergent pseudo-periodic mode*
- *Constant pseudo-periodic mode*
- *Divergent pseudo-periodic mode*



Pseudo-periodic evolution modes (2/5)

- The pseudo-periodic mode is completely characterized by the pair of parameters (α_i, ω_i) that represent the real part and the imaginary part of the complex conjugate eigenvalues.

- These parameters give direct information both on the exponential law that envelops the oscillation peaks (parameter α_i) and on the angular frequency of the oscillations (parameter ω_i).

- Frequently, instead of using these parameters, other pairs of parameters related to (α_i, ω_i) through simple relations are used, and that provide information more directly related to other characteristics of the response, especially in the case of convergent oscillatory motion.

Pseudo-periodic evolution modes (3/5)

- For convergent pseudo-periodic mode, the ***time constant*** is defined as

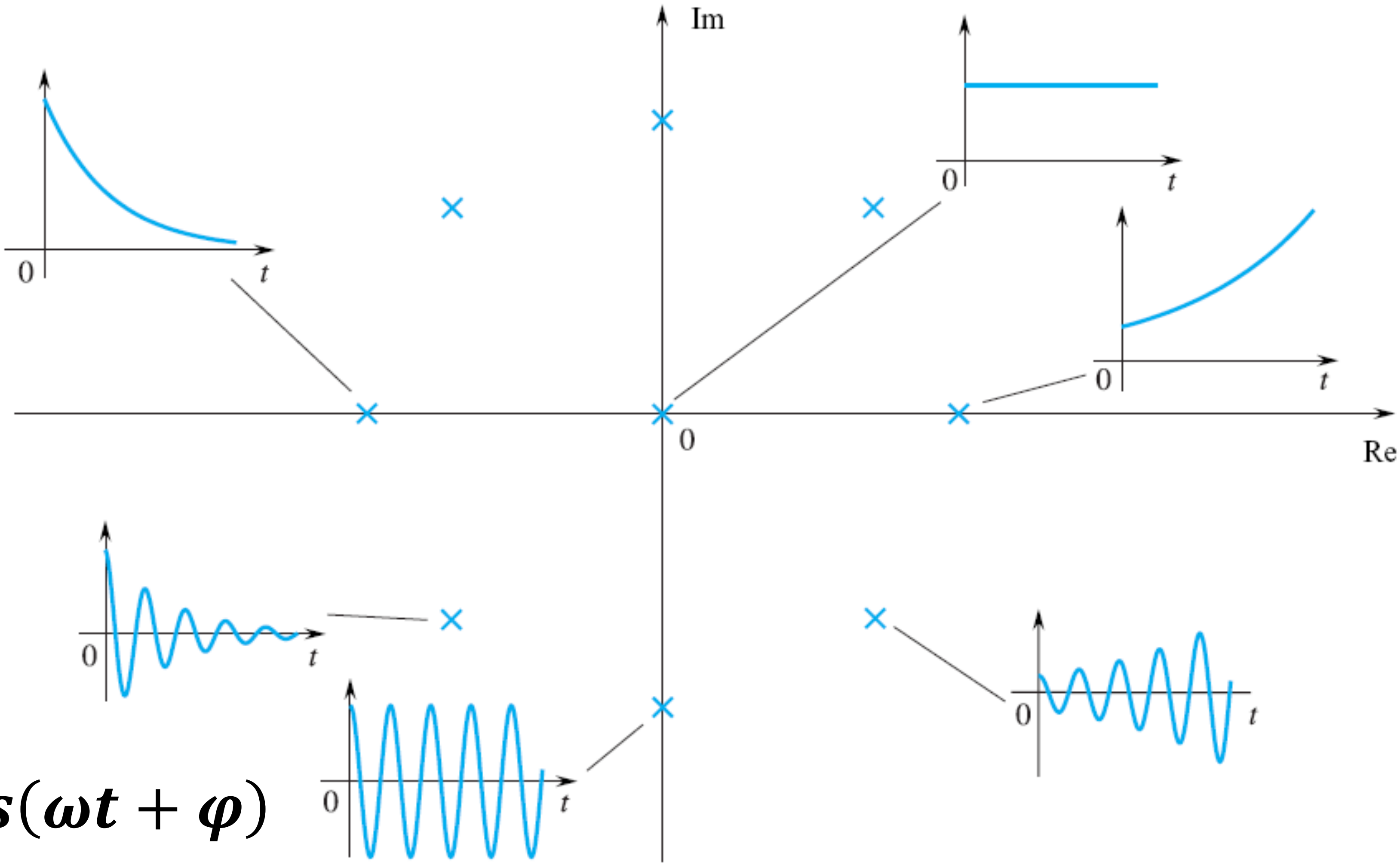
$$\tau_i = -\frac{1}{\alpha_i}$$

- The parameter $\boldsymbol{\omega_i}$ is called the angular frequency of the system, while $\boldsymbol{T}$, related to $\boldsymbol{\omega_i}$ by the relation $T=2\pi/\boldsymbol{\omega_i}$, is called the oscillation period of the system.

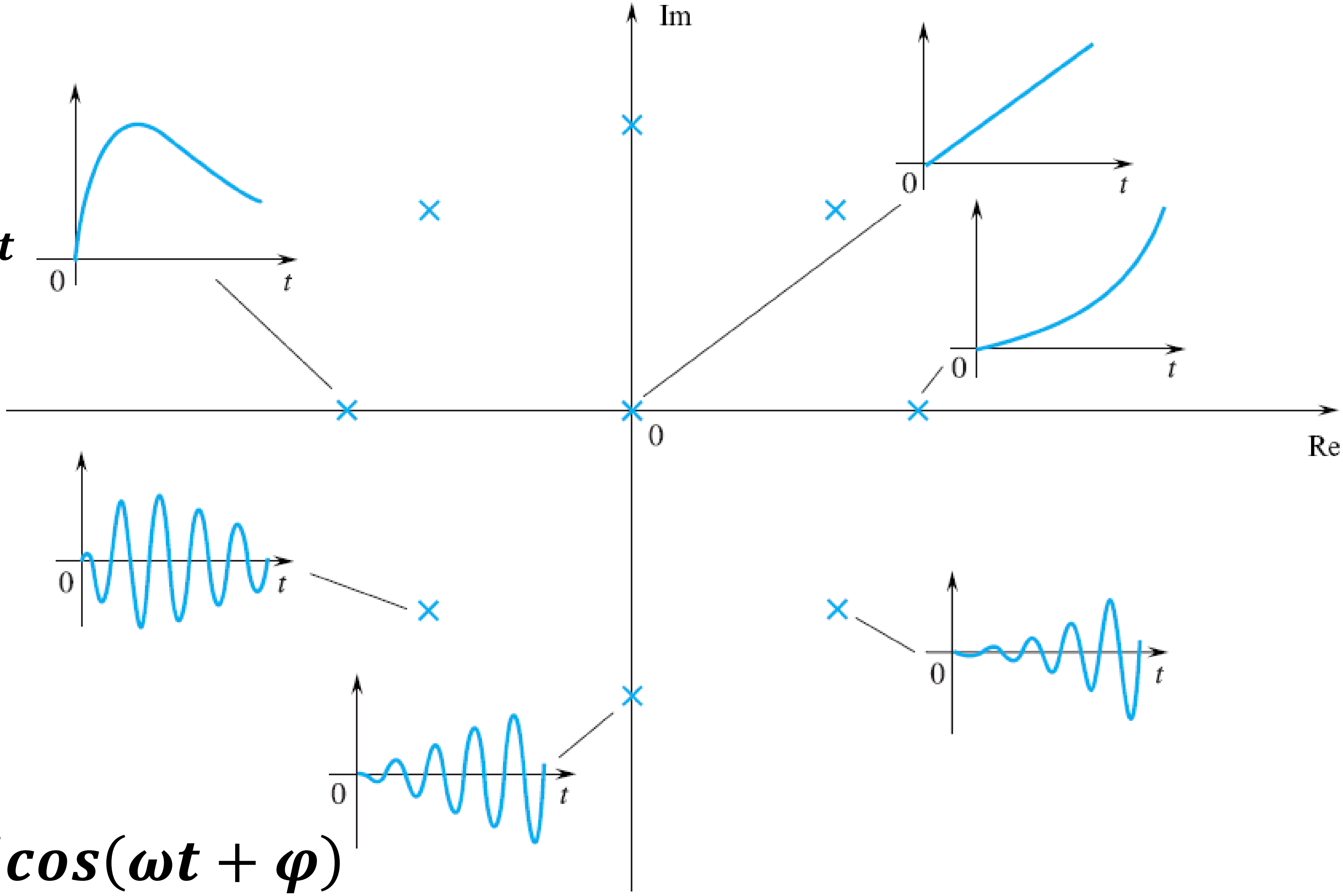
- Sometimes, instead of the period T, the frequency $\boldsymbol{f}=1/\boldsymbol{T}$ is specified.

$e^{\lambda t}$

$e^{\alpha t} cos(\omega t + \varphi)$



Evolution modes with multiple eigenvalues (η=2)



$$te^{\lambda t}$$

$$te^{\alpha t}cos(\omega t+\varphi)$$

Other important parameters for pseudo-periodic mode are the natural frequency ω_n and the damping coefficient ζ.

The natural frequency is defined by

$$\omega_n{}^2 = \alpha^2 + \omega^2$$

and the damping factor

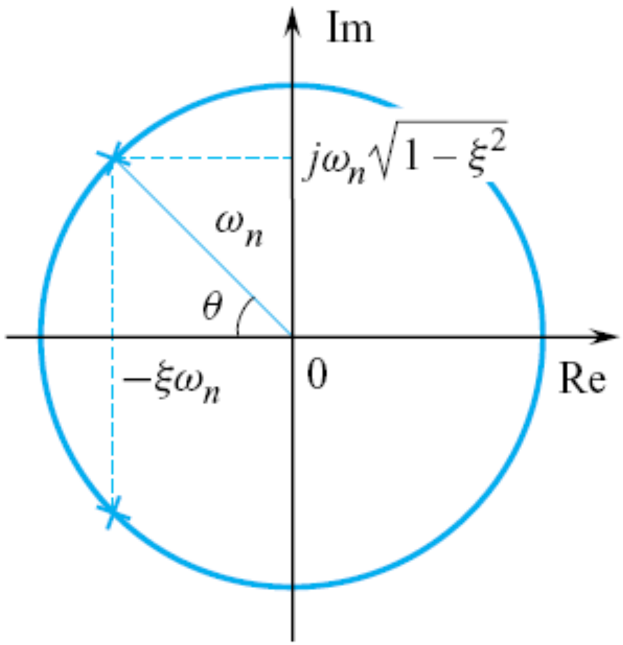
$$\zeta = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

$$\alpha = -\zeta\omega_n,$$

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

$$\omega_n \cos\theta = \zeta\omega_n$$

$$\zeta = \cos\theta$$



- The ***natural frequency*** $\boldsymbol{\omega_n}$ is the oscillation frequency of the pseudo-periodic mode when $\alpha = 0$.

- ***For convergent*** pseudo-periodic modes the ***damping coefficient ζ∈(0,1]*** while ***for divergent pseudo-periodic modes ζ∈[-1,0)***

- ***For convergent*** pseudo-periodic modes, the **damping coefficient** $\boldsymbol{\zeta}$ relates the oscillations of the pseudo-periodic mode to the time before the evolution will extinguish. For $\zeta \ll 1$

$$\zeta = -\frac{\alpha}{\omega_n} \cong -\frac{\alpha}{\omega} = \frac{T}{2\pi\tau} \ll 1$$

where T is the oscillation period. Indeed, the number of the oscillation before the mode will extinguish increases when $\boldsymbol{\zeta}$ becomes small.

$$\zeta = \frac{T}{2\pi\tau} \cong \frac{T}{6\tau} \Longrightarrow \frac{1}{2\zeta} \cong \frac{3\tau}{T}$$

# of oscillations before the mode will extinguish

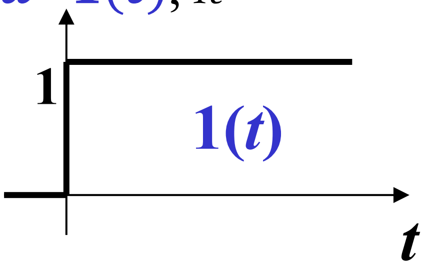
Forced response in the time domain

- Let us consider the forced response of an LTI system in the output $(x_0 = 0)$

$$y_f(t) = C\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau + D\,u(t), \quad t \geq t_0$$

- The evaluation of the forced response in the time domain is demanding due to the presence of the convolution product.

- Only in some particular case, such as the ***step response*** $\boldsymbol{u(t) = \bar{u}\cdot 1(t)}$, it becomes straightforward



$$y_f\,(t) = C\int_0^t e^{A(t-\tau)}B\,\bar{u}\,d\tau + D\,\bar{u}$$

$$= \left[-CA^{-1}e^{A(t-\tau)}B\bar{u}\right]_0^t + D\,\bar{u}$$

$$= CA^{-1}e^{At}B\bar{u} + \left[-CA^{-1}B + D\right]\bar{u}$$

- In the other cases the forced response is evaluated in the ***Laplace domain***

First-order LTI system

- For **a first-oder LTI system** in the form

$$\dot{x}(t) = ax(t) + bu(t)$$
$$y(t) = cx(t) + du(t)', \qquad x(t_0 = 0) = x_0$$

the ***time evolution of the state variable*** in case of $\boldsymbol{u(t) = \bar{u} \cdot 1(t)}$ is given

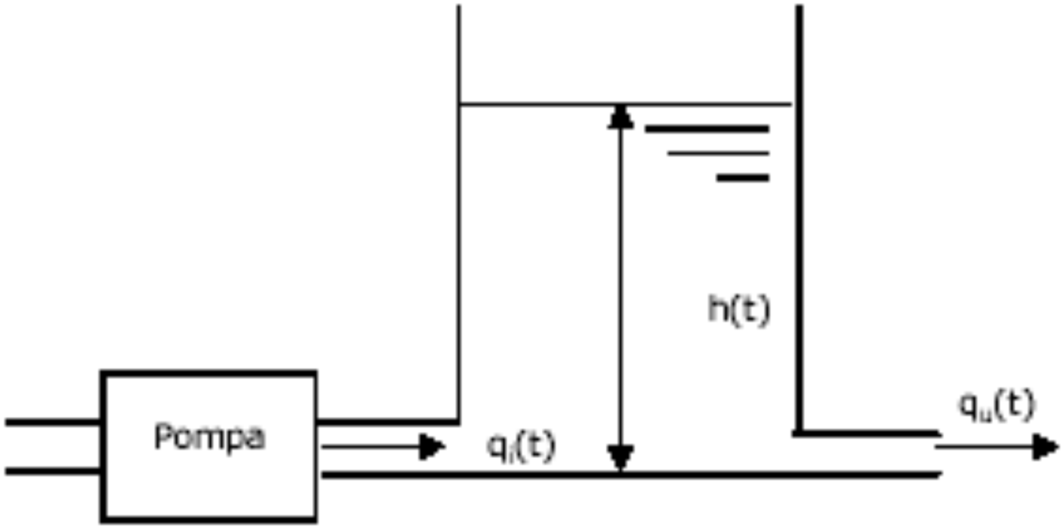
$$x(t) = \underbrace{e^{at}x_0}_{x_l} + \underbrace{\frac{1}{a}e^{at}b\bar{u} - \frac{1}{a}b\bar{u}}_{x_f}, \quad t \geq 0$$

- The ***time evolution of the output*** turns out to be

$$y(t) = ce^{at}x_0 + c\frac{1}{a}e^{at}b\bar{u} - c\frac{1}{a}b\bar{u} + d\bar{u},$$

$$= \underbrace{ce^{at}x_0}_{y_l} \underbrace{- c\frac{b}{a}\bar{u}(1 - e^{at}) + d\bar{u}}_{y_f}$$

Example of first order LTI system:



input: $u(t) = q_i(t)$

output: $y(t) = h(t)$

state: $x(t) = h(t)$

SS representation:

$$\dot{x}(t) = -\frac{k}{S}x(t) + \frac{1}{S}u(t)$$

$$y(t) = x(t)$$

- For a first orer LTI, the *time evolution of the output for* $u(t) = \bar{u} \cdot 1(t)$ *is given by*

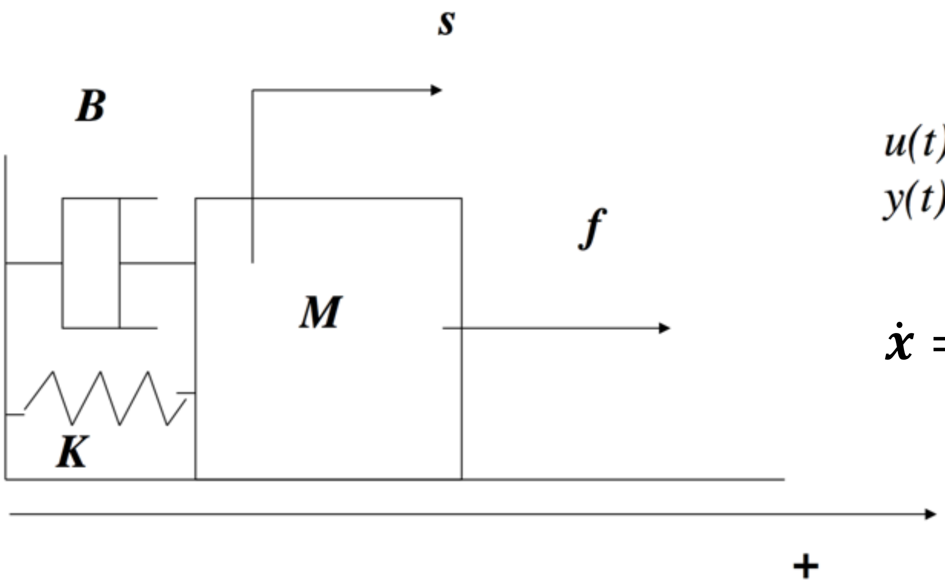
$$y(t) = \underbrace{ce^{at}x_0}_{y_l} - \underbrace{c\frac{b}{a}\bar{u}(1 - e^{at}) + d\bar{u}}_{y_f}$$

- In this case, $a = -\frac{k}{S}, b = \frac{1}{S}, c = 1, d = 0$,

$$y_l(t) = e^{-\frac{k}{S}t}x_0 = e^{-\frac{t}{\tau}}x_0, \text{ with } \tau = -\frac{1}{a} = \frac{S}{k} \qquad y_f(t) = \frac{1}{k}\bar{u}\left(1 - e^{-\frac{k}{S}t}\right)$$

time constant ($\frac{S}{k}$); static gain ($\frac{1}{k}$)

Example of second-order LTI system: mass-spring-damper system



$u(t)=f(t)$
$y(t)=s(t)$

- **State space representation**

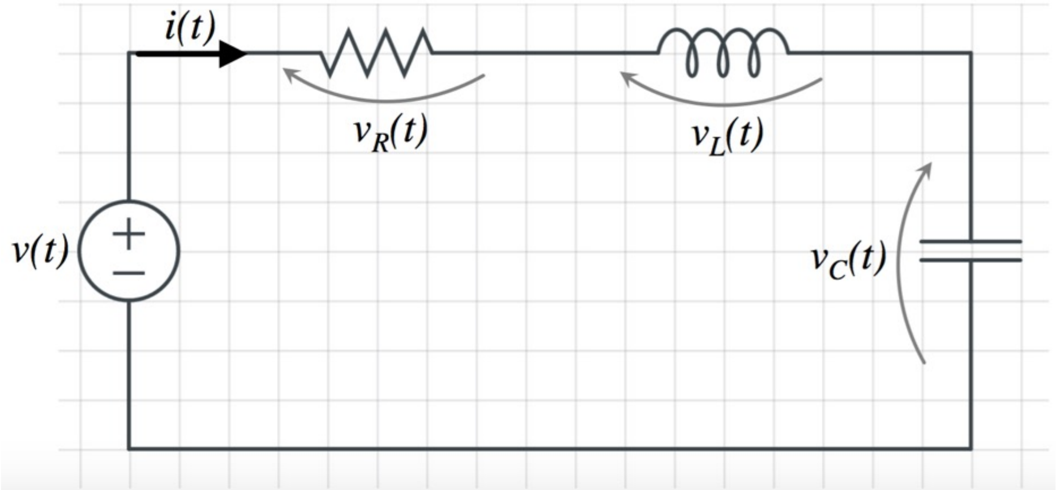
$$x_1 = s,\ x_2 = \frac{ds}{dt} = \dot{s} = \dot{x}_1$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/M \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Problem:

For the mass-spring-damper system reported in figure, assuming *M*=1Kg, *K*=16 N/m, evaluate the evolution modes by varying *B* values [Ns/m] in the interval [0 20].



$$u(t) = v(t), \quad y(t) = v_c(t)$$

$$x_1(t) = v_c(t) \quad x_2(t) = i_L(t)$$

Problem:

For the RLC circuit in series configuration, **compute the input-output and state space representations.**

Assuming the capacity value C=1e-6 μF, and the inductance value L=1e-3 mH, compute the values of ***R*** for which the system exhibits aperiodic and pseudo-periodic modes.

INVERSE OF A MATRIX $N \times N$

Inverse of a matrix

- Given a quadratic and invertible matrix

$$A = \begin{pmatrix} x_{1,1} & \dots & x_{1,j} \\ \vdots & \ddots & \vdots \\ x_{i,1} & \dots & x_{i,j} \end{pmatrix}$$

its inverse is defined as

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \mathrm{cof}(A, x_{1,1}) & \dots & \mathrm{cof}(A, x_{1,j}) \\ \vdots & \ddots & \vdots \\ \mathrm{cof}(A, x_{i,1}) & \dots & \mathrm{cof}(A, x_{i,j}) \end{pmatrix}^T$$

where the cofactor is

$$\mathrm{cof}(A, i, j) = (-1)^{i+j} \det(\mathrm{minor}(A, i, j))$$

and the minor (i, j) is the determinant of the matrix obtained excluding the row i and the column j.

Inverse of a 2×2 matrix

- Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a 3×3 matrix

- Given a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

its inverse is

$$\frac{1}{\det(A)} \begin{pmatrix} +\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & +\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\ -\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & +\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\ +\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} & +\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \end{pmatrix}$$

EIGENVALUES AND EIGENVECTORS

Eigenvalues and eigenvectors

- Given a matrix $A \in R^{n\times n}$, a scalar $\boldsymbol{\lambda \in C}$ is said ***eigenvalue*** of the matrix A if there exists a vector $\boldsymbol{v \in C^n}$, said ***eigenvector***, such that

$$\boldsymbol{Av = \lambda v}$$

- Taking into account account that eigenvalues and eigenvectors of a matrix verify the equation

$$(A - \lambda I)v = 0.$$

The eigenvalues can be found evaluating the roots of the ***characteristic polynomial*** $\boldsymbol{p(\lambda)}$ defined as

$$p(\lambda) = det(A - \lambda I).$$

Examples

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

$$\boldsymbol{p(\lambda) = \lambda^2 + 3\lambda - 10}$$

Eigenvalues

$$\boldsymbol{\lambda_1 = 2,\ \lambda_2 = -5}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

$$\boldsymbol{p(\lambda) = \lambda^2 + 2\lambda + 1}$$

Eigenvalues

$$\boldsymbol{\lambda_1 = \lambda_2 = -1}$$