Course of
"Industrial Control System Security"
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Modelling: Input-output and state-space representations

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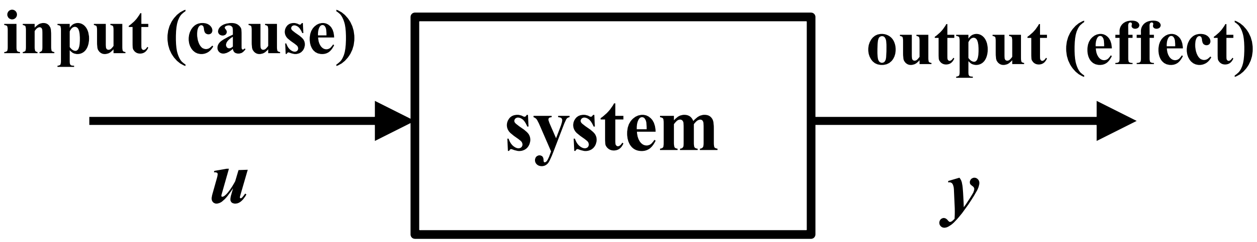
Modelling

- From engineering point of view, it is essential to study the quantitative aspects of systems.
- For quantitative analysis, it is needed to find a **mathematical model** that provides a mathematical description of the behavior of the real system.
- For modelling a system, it needs to identify the quantities of interest that describe the system behavior:
  - quantities that can vary over time independently of the others, called input quantities or simply inputs;
  - quantities whose evolution over time is required to be characterize and is dependent on the input; for these quantities it is possible to find a cause-effect relationship; the inputs are the cause, while these quantities are the effects and appear to be dependent on the inputs; they are called output quantities or simply outputs.

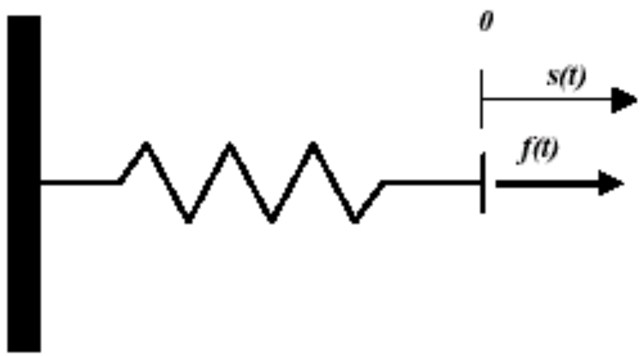
Modelling: input-output representation

Systems Theory studies all those phenomena in which it is possible to find a cause-effect relationship.



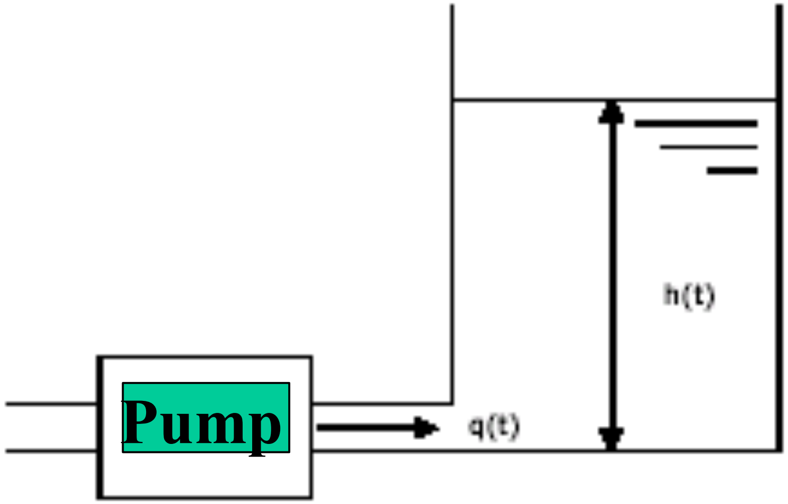
To define the model, it is necessary to identify the mathematical functional relationships between the inputs and outputs.

Example: spring system



input: $\boldsymbol{u(t) = f(t)}$

output: $\boldsymbol{y(t) = s(t)}$



input: $u(t) = q(t)$

output: $y(t) = h(t)$

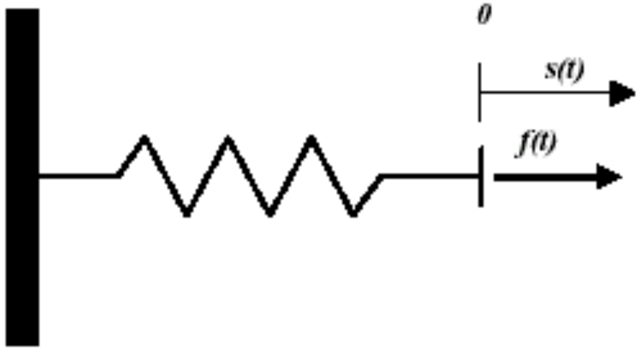
Input – output relationship

- Once the input and output quantities have been identified, to define the model, it is necessary to find the mathematical relationships between the inputs and the outputs.

- These functional relations will be identified on the basis of laws (constitutive laws) that describe the phenomenon under consideration.

- When the model is able to well reproduce the behavior of the output variables as in the real system, then the model (which is an abstract object) and the system are "confused", and these two terms can be used in the same way.

Example: spring system

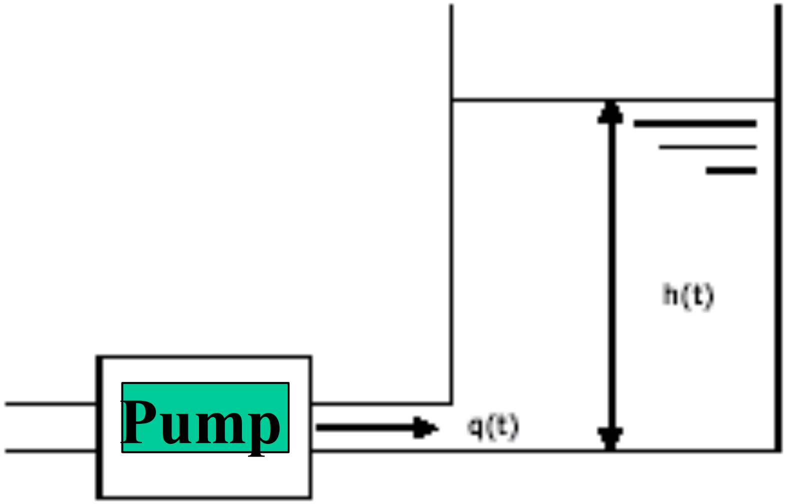


input: $u(t) = f(t)$

output: $y(t) = s(t)$

$$F + F_e = 0 \Rightarrow u(t) - k\,y(t) = 0$$

$$\Rightarrow y(t) = \frac{u(t)}{k}$$



input: $u(t) = q(t)$

output: $y(t) = h(t)$

$$\frac{dV}{dt} = q(t) \Rightarrow A\frac{dh}{dt} = q(t)$$

$$\Rightarrow A\frac{dy}{dt} = u(t) \Rightarrow \dot{y}(t) = \frac{u(t)}{A}$$

Static and dynamic systems

- The comparison between these models highlights a first classification, i.e. between static and dynamic systems.

- In the case of the spring system, the relation is a purely algebraic.

- The output at a generic instant *t* depends only on the value of the input at the same instant, and the system has no memory of the values of the input at previous instants.

- Systems of this type are called **static** or **algebraic systems** or even memoryless systems; in some cases, they are also called **combinatorial systems.**

Static and dynamic systems

- In the case of the tank, the functional relationship is a differential equation.

- In this case, the output at a generic instant *t* also depends on the past values of the input.

- Indeed, by solving the differential equation, in the time interval **[t_0, t],** we get

$$y(t) = y(t_0) + \frac{1}{A}\int_{t_0}^{t} u(\tau)d\tau$$

- Systems of this type are called **dynamic systems.**

The state of a system

- Then, for a dynamic system, the output $\boldsymbol{y(t)}$ at a generic instant $\boldsymbol{t}$ also depends on the past values of the input.
- It is therefore evident that if we are interested in calculating the output in a time interval **[t_0, t],** it is not sufficient to know the values of the input $\boldsymbol{u(t)}$ in that interval.
- What is the missing information to solve the problem?
- For the tank problem, it is clear that, to determine the tank level in a given time interval, it is not enough to assign the input flow rate in that interval.

$$\boldsymbol{y(t) = y(t_0) + \frac{1}{A}\int_{t_0}^{t} u\,(\tau)d\tau}$$

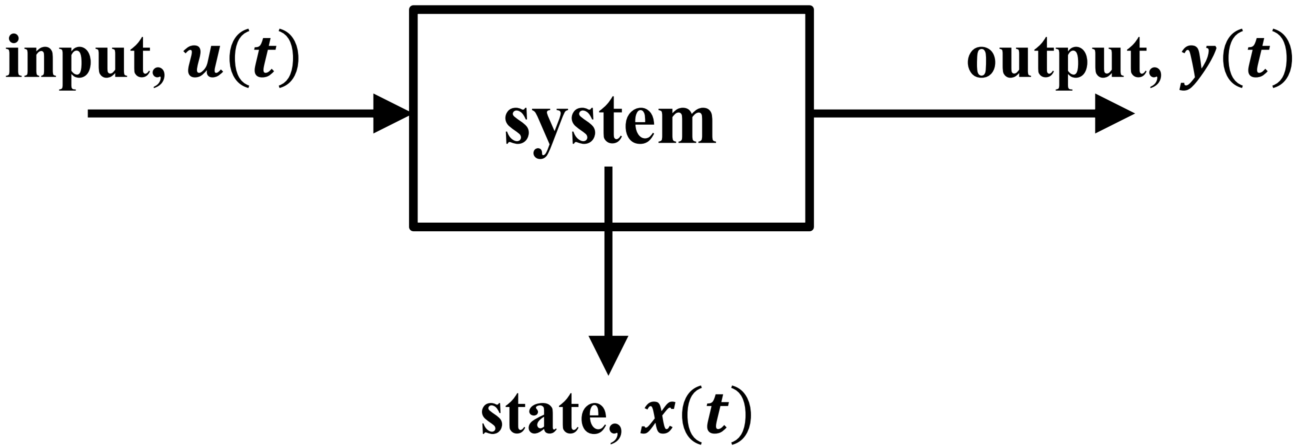
- The missing information is represented by the initial level at time t_0: $\boldsymbol{y(t_0)}$
- From a mathematical point of view, this is confirmed by the fact that, since the model is described by a first order differential equation, to get unique solution we need to assign the initial condition of the output.

State definition

- **Definition**: The state of a system at a given instant t_0 is the information at that instant (i.e., t_0) allowing us to uniquely determine the output $y(t)$, for $t \geq t_0$, once the input $u(t)$ has been assigned $t \geq t_0$.

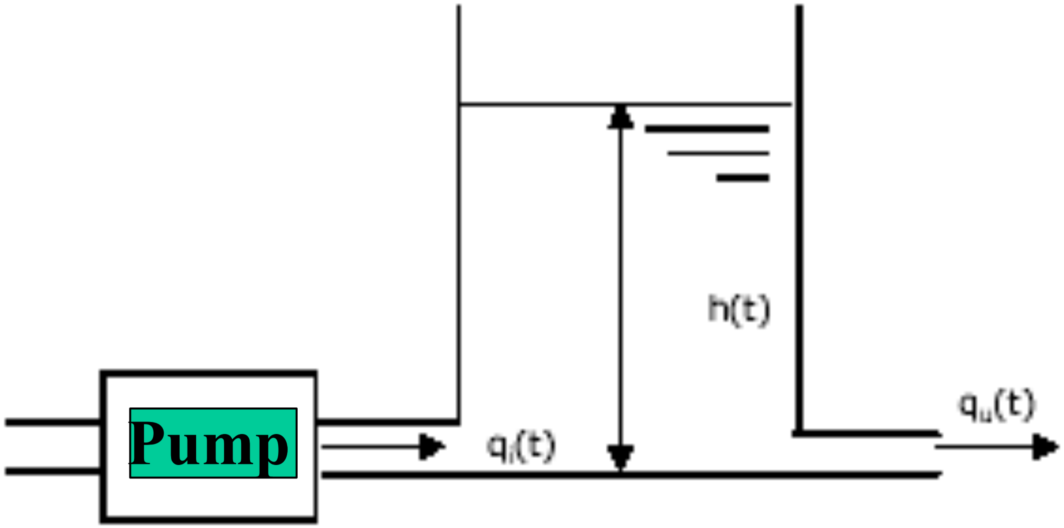
State space representation

- In addition to an input – output representation, another equivalent representation is the state-space representation.
- In this type of representation, in addition to the "external" variables (inputs and outputs), other variables, called state variables, are used.



- Note that state variables are those involved in the derivation operation.
- The number of state variables, n, represents the order of the system.

Example of first-order LTI system: hydraulic system



input: $u(t) = q_i(t)$

output: $y(t) = h(t)$

$$\frac{dV}{dt} = q_i(t) - q_u(t)$$

Input-output representation:

hp. laminar flow ⇒ $S\dot{y}(t) = u(t) - ky(t)$ ⇒ $S\dot{y}(t) + ky(t) = u(t)$

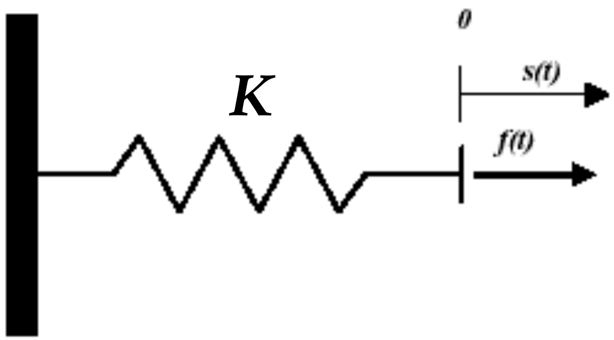
State space representation:

$$\dot{x}(t) = -\frac{k}{S}x(t) + \frac{1}{S}u(t)$$

$$y(t) = x(t)$$

Mechanical systems: example of static and dynamic systems

- **Spring system**



***K*, elastic coeff.**

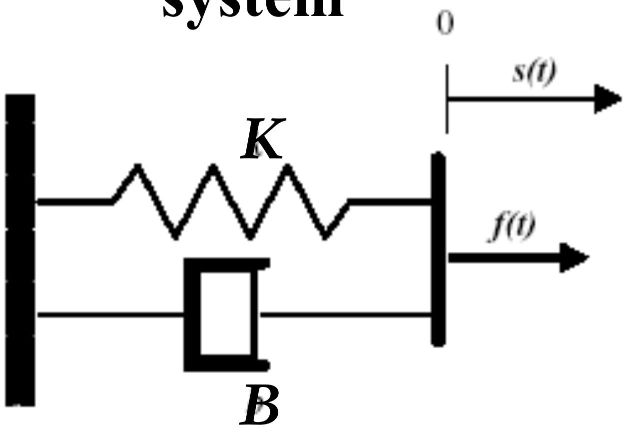
$$u(t)=f(t)$$
$$y(t)=s(t)$$
$$f(t)+f_e(t)=0$$
$$f_e(t)=-K\,s(t)$$

⇒

$$u(t)-K\,y(t)=0$$
$$y(t)=u(t)/K$$

Static system

- **Spring-damper system**



***B*, viscous friction coeff. of the damper**

$$f(t)+f_e(t)+f_d(t)=0$$
$$f_d(t)=-B\dot{s}(t)$$
$$x(t)=s(t)$$

⇒

State – space representation

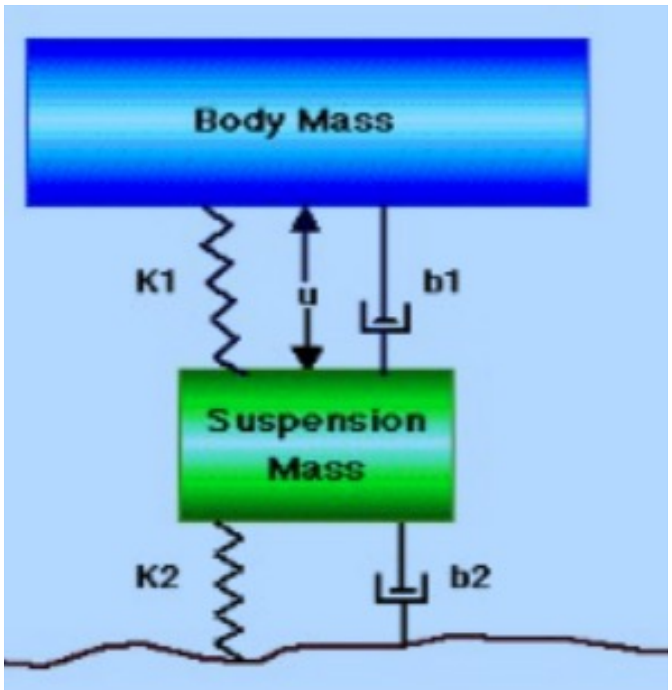
$$u(t)-Kx(t)-B\dot{x}(t)=0$$

⇓

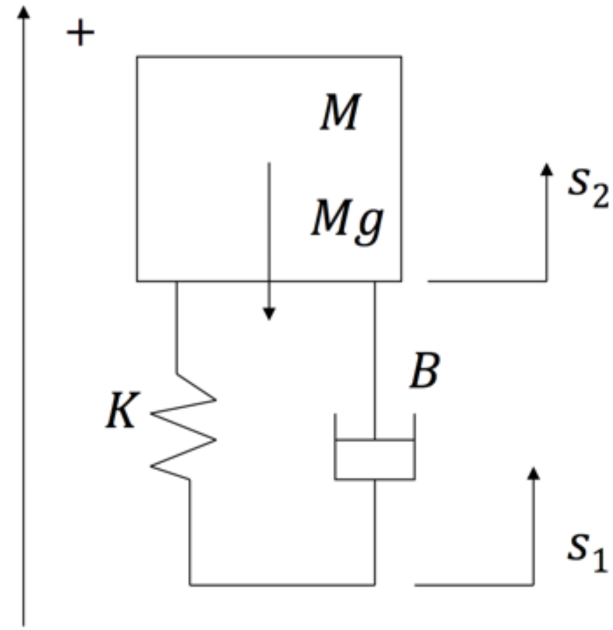
$$\dot{x}(t)=-\frac{k}{b}x(t)+\frac{1}{b}u(t)$$
$$y(t)=x(t)$$

Dynamic system

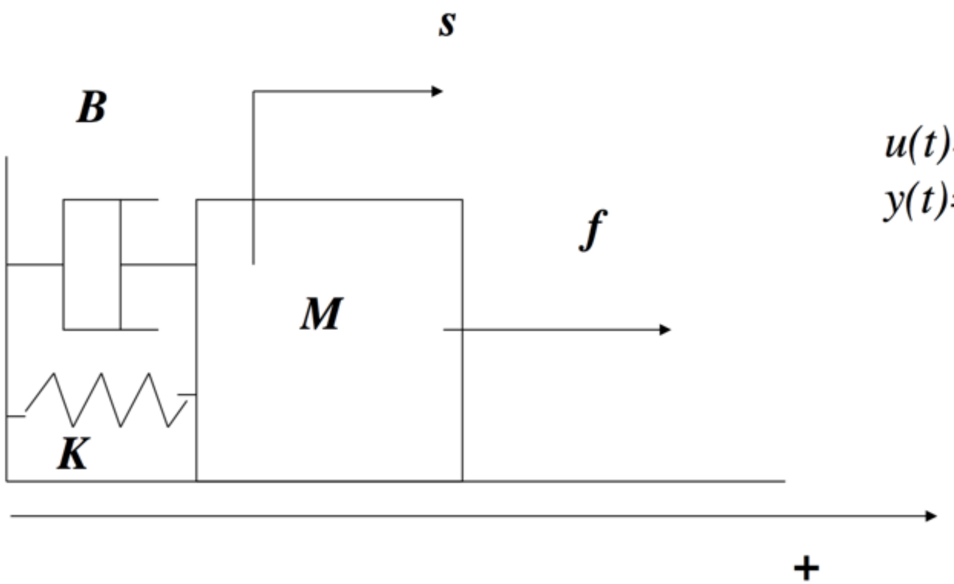
Example: car suspension



Simplified model



Example: mass-spring-damper system



$u(t)=f(t)$
$y(t)=s(t)$

- **Input output representation**

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t)$$

- **State space representation**

$$x_1 = s,\ x_2 = \frac{ds}{dt} = \dot{s} = \dot{x}_1$$

(n.b. omitted time dependence)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ \dot{s} \end{pmatrix}, \quad y = x_1$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{K}{M}x_1 - \dfrac{B}{M}x_2 + \dfrac{1}{M}u \end{cases}$$

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t)$$

$$y = x_1,\quad \dot{y} = x_2,\quad \ddot{y} = \dot{x}_2,$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{B}{M} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1/M \end{pmatrix}}_{B} u,$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

State space model

- In general, the behavior of a dynamical system in the continuous-time domain can be described by the following equations:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}), \boldsymbol{t})$$

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}), \boldsymbol{t})$$

$\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{m}}$, $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$,

$\boldsymbol{t} \in \mathbb{R}$, $\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{p}}$,

- $\boldsymbol{f}$ is the state function;
- $\boldsymbol{g}$ is the output function;
- $\boldsymbol{m}$ is the numper of input signals;
- $\boldsymbol{p}$ is the number of output signal;
- $\boldsymbol{n}$ is the order of the system, corresponding to the number of the state variables.

- It is important to note that, when the state ***x*** is a vector of dimension ***n***, the function ***f*** is also a vector of ***n*** scalar functions.
- Similarly, if the output ***y*** is a vector of dimension ***p***, the function ***g*** is also a vector of ***p*** scalar functions.
- Therefore, assuming that the input ***u*** is also a vector of dimension ***m***, the extended form of the implicit model is as follows:

$$\begin{cases} \dot{x}_1 = f_1(, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \\ \vdots \\ \dot{x}_n = f_n(, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \end{cases}$$

$$x_i(t_0) = x_{i0}, i = 1, \dots, n$$

$$\begin{cases} y_1 = g_1(, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \\ \vdots \\ y_p = g_p(, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \end{cases}$$

SISO and MIMO systems

- Systems with only one input variable and only one output variable ($m = p = 1$) are called Single-Input-Single-Output, **SISO.**

- Systems that are not SISO are called multivariable Multiple-Input-Multiple-Output, **MIMO.**

- In this course we will focus mainly on **SISO** systems.

- In the case of linear functions ***f*** and ***g***, the dynamical system is given by

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}, \quad x(t_0) = x_0$$

with $A(\cdot) \in R^{n\times n}$, $B(\cdot) \in R^{n\times m}$, $C(\cdot) \in R^{p\times n}$, $D(\cdot) \in R^{p\times m}$ (time-dependent matrices), where $x(t)$ is the state vector, $u(t)$ is the input vector and $y(t)$ is the output vector of the system.

When the matrixes are time dependent, the linear system is named ***time variant*** (**LTV**).

Linear systems: input – output representation

- Assuming the input-output representation, a linear system can be put in the form

$$y^{(n)} + A_{n-1}y^{(n-1)} + \cdots A_0 y = B_n u^{(n)} + B_{n-1}u^{(n-1)} + \cdots B_0 u$$

- Note that the matrices A_i and B_i can be functions of time, and they are not necessarily all different from zero.

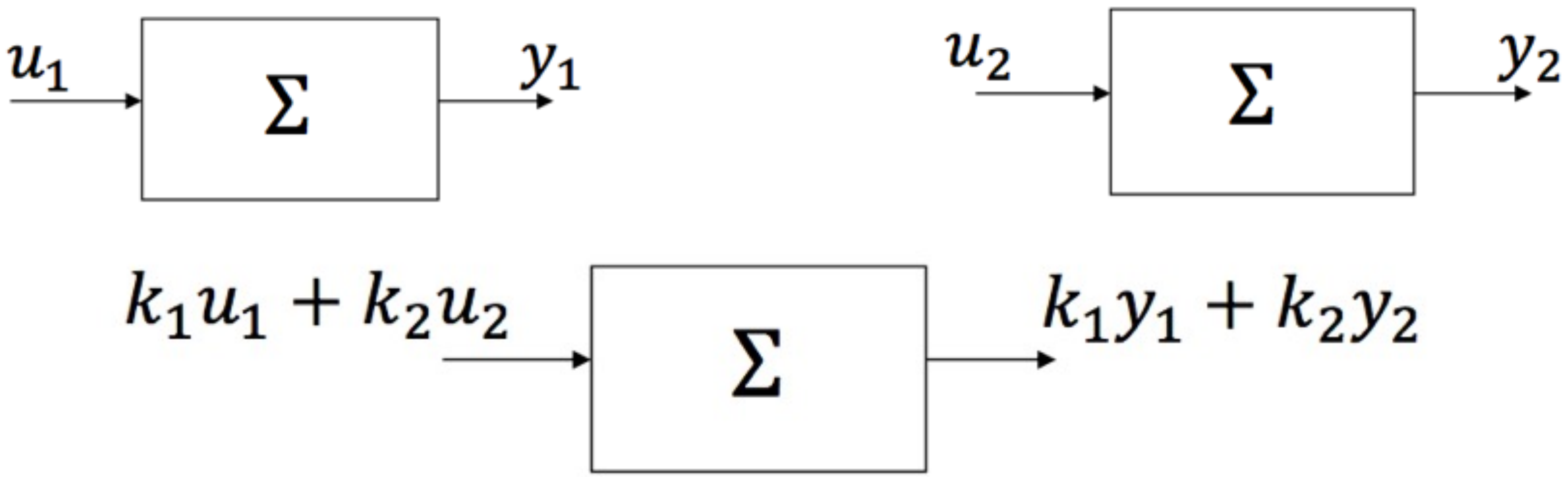
- In general, the input and the output are vectors (MIMO systems):

$$\boldsymbol{u} = \begin{pmatrix} \boldsymbol{u_1} \\ \vdots \\ \boldsymbol{u_m} \end{pmatrix} \in \mathbb{R}^{\boldsymbol{m}} \qquad \boldsymbol{y} = \begin{pmatrix} \boldsymbol{y_1} \\ \vdots \\ \boldsymbol{y_p} \end{pmatrix} \in \mathbb{R}^{\boldsymbol{p}}$$

Linear property

- The linear systems satisfy a particular property, also called superposition principle/property.

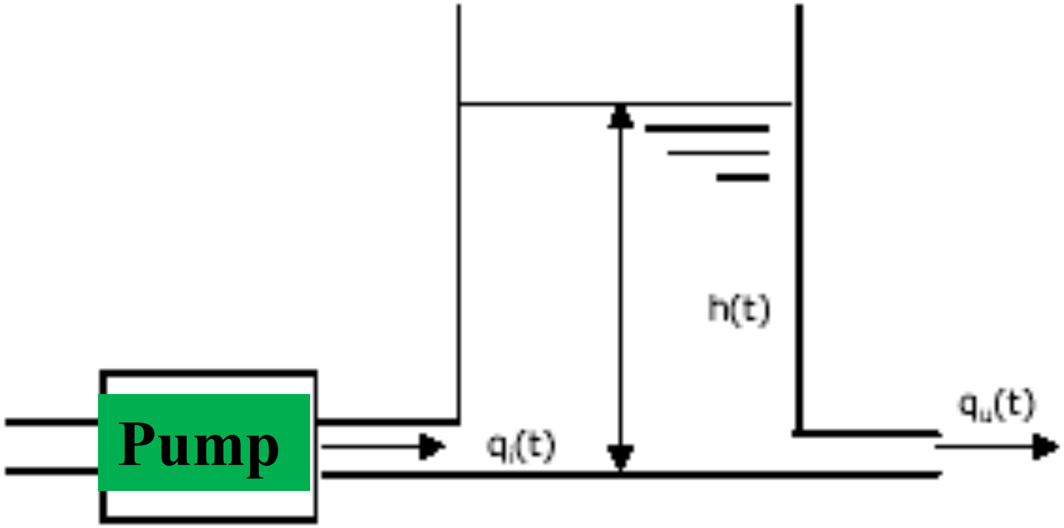
- A linear combination of inputs corresponds to a linear combination of the corresponding outputs.



Nonlinear systems

- A system is nonlinear when it is not linear.

- In this regard, see the next example.



hp. laminar flow:

$$q_u(t) = kh(t)$$

State space representation:

$$\dot{x}(t) = -\frac{k}{S}x(t) + \frac{1}{S}u(t)$$

$$y(t) = x(t)$$

Linear system

hp. turbolent flow:

$$q_u(t) = k\sqrt{h(t)}$$

State space representation:

$$\dot{x}(t) = -\frac{k}{S}\sqrt{x(t)} + \frac{1}{S}u(t)$$

$$y(t) = x(t)$$

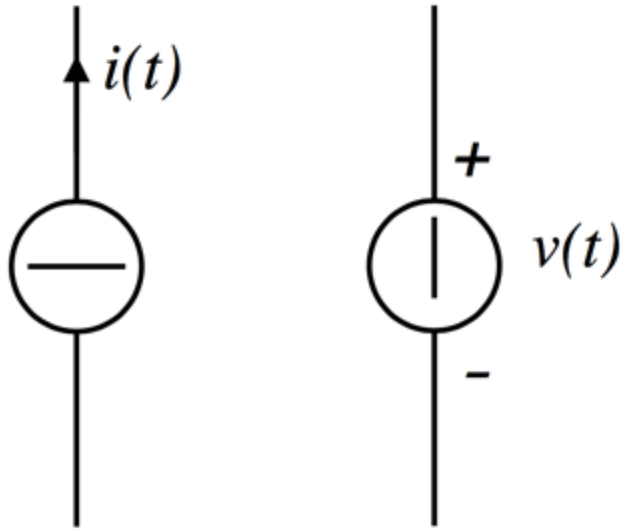
Nonlinear system

- In the case of constant matrices (i.e. ***A***, ***B***, ***C***, ***D*** are not time-dependent), the system is ***linear time invariant (LTI)*** and is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\text{'} \end{aligned} \qquad x(t_0) = x_0$$

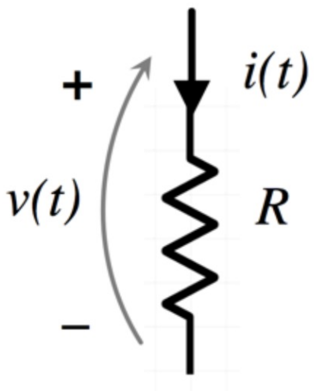
with $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$, where $x(t)$ is the state vector, $u(t)$ is the input vector and $y(t)$ is the output vector of the system.

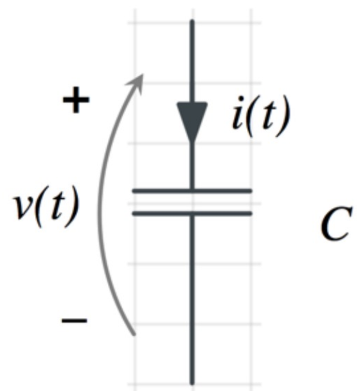
- Ideal current and voltage sources.

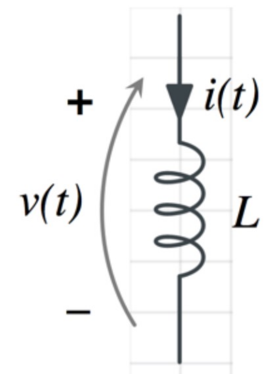


- An ideal current (voltage) source provide a current that passes through it (a voltage at its terminals) that is independent of the network in which it is inserted.

LTI systems – circuit elements



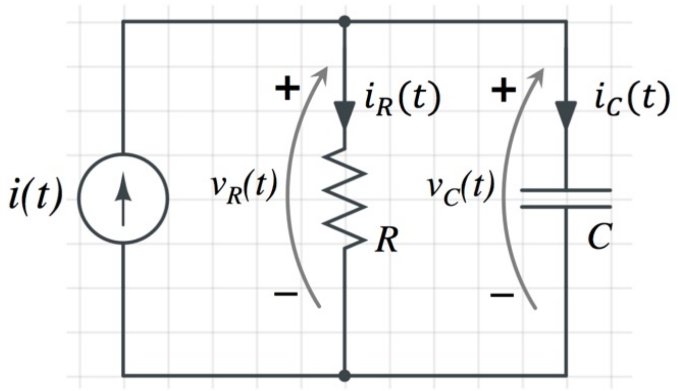




$$\mathbf{v}(\boldsymbol{t}) = \boldsymbol{R}\, \boldsymbol{i}(\boldsymbol{t})$$

$$\boldsymbol{i}(\boldsymbol{t}) = \boldsymbol{C}\frac{\boldsymbol{dv}(\boldsymbol{t})}{\boldsymbol{dt}}$$

$$\boldsymbol{v}(\boldsymbol{t}) = \boldsymbol{L}\frac{\boldsymbol{di}(\boldsymbol{t})}{\boldsymbol{dt}}$$



$$u(t) = i(t), \quad y(t) = v_c(t)$$

R and C in parallel: $v_c(t) = v_R(t)$

$$i(t) = i_R(t) + i_c(t) = \frac{v_c(t)}{R} + C\frac{dv_c(t)}{dt}$$

Input-output representation:

$$\Longrightarrow \quad C\dot{y}(t) + \frac{y(t)}{R} = u(t)$$

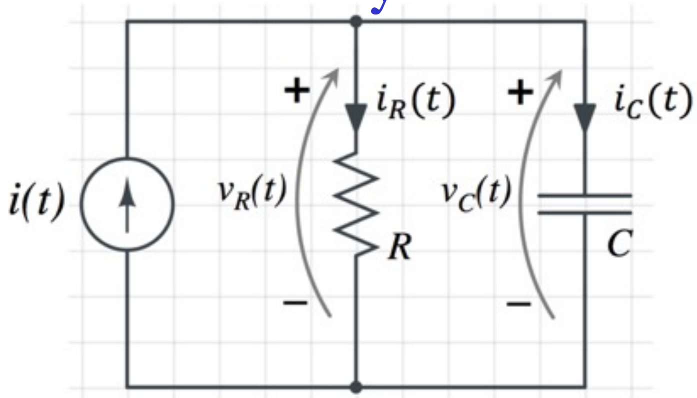
State space representation:

$$\Longrightarrow \quad \dot{x} = -\frac{1}{CR}x + \frac{1}{C}u$$

$$y = x$$

Example: Linear and nonlinear systems

Linear system

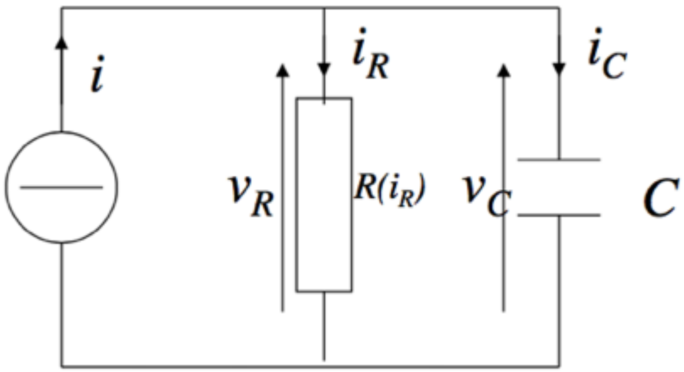


- $u(t) = i(t);\ y(t) = v_c(t)$
- **IU:** $C\dot{y}(t) + \frac{y(t)}{R} = u(t)$
- **SS:** $x(t) = v_c(t)$

$$\dot{x}(t) = -\frac{1}{CR}x(t) + \frac{1}{C}u(t)$$

$$y(t) = x(t)$$

Nonlinear system



$R(i_R) = ki_R^2 \Rightarrow v_R = R(i_R)i_R = ki_R^3$

- **SS:** $v_R(t) = v_c(t)$

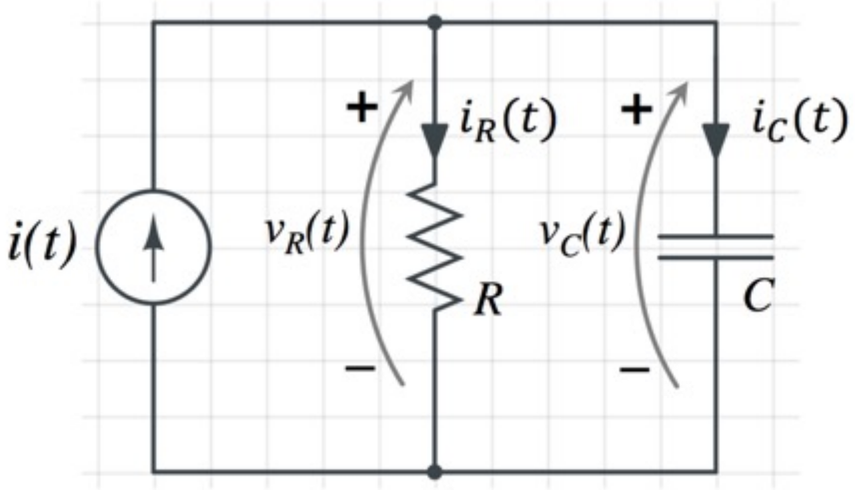
$\Rightarrow i_R(t) = \frac{1}{\sqrt[3]{k}}\sqrt[3]{x(t)}$

$$\dot{x}(t) = -\frac{1}{C\sqrt[3]{k}}\sqrt[3]{x(t)} + \frac{1}{C}u$$

$$y(t) = x(t)$$

Example: Linear Time Invariant (LTI) and Linear Time Variant (LTV) systems

LTI system

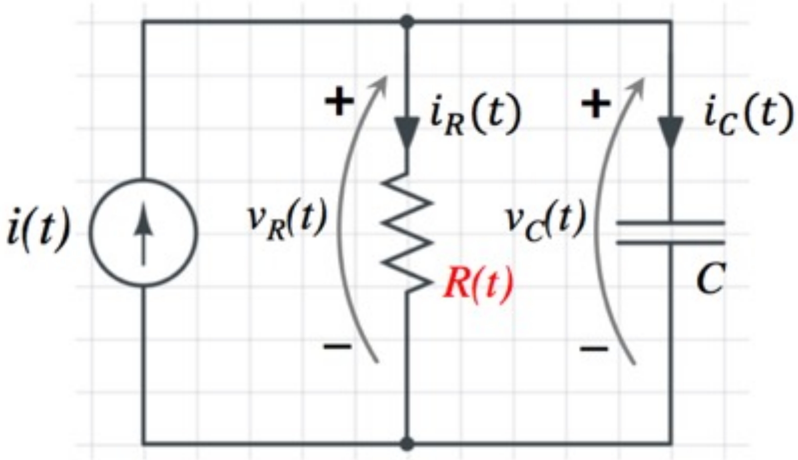


- **SS:**

$$\dot{x}(t) = -\frac{1}{CR}x(t) - \frac{1}{C}u(t)$$

$$y(t) = x(t)$$

LTV system



- **SS:**

$$\dot{x}(t) = -\frac{1}{CR(t)}x(t) - \frac{1}{C}u(t)$$

$$y(t) = x(t)$$

A strictly proper system

- A linear system is said to be ***strictly proper*** if the following situations occur:
  - For state space representations, the matrix $\boldsymbol{D}=0$.
  - For input-output representations, the maximum order of derivation of the input is less than that of the output.
- A system is said to be ***proper*** if it is not strictly proper.

System classification

- In summary, we have classified the systems into the following categories:
  - **Linear** and **nonlinear** systems
  - Stationary and nonstationary systems, i.e., **Linear Time Invariant** (**LTI**) and **Linear Time Variant** (**LTV**)
  - **Proper** and **strictly proper** Systems
  - **Static** and **dynamic** systems
  - In this course we will focus our attention on the class of **LTI** dynamical systems

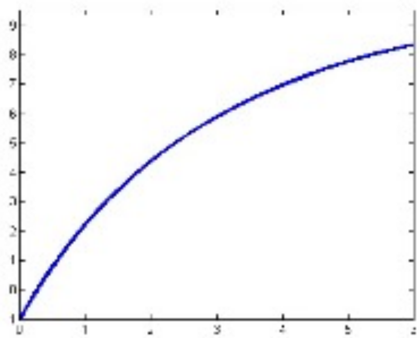
Note

So far, we consider systems in which the variables change continuously over time:

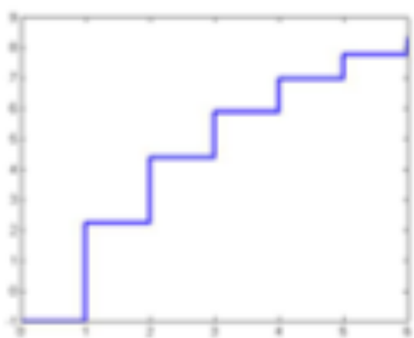
continuous-time systems

Continuous-time signals

The time variable *t* varies continuously in an interval of ℝ.

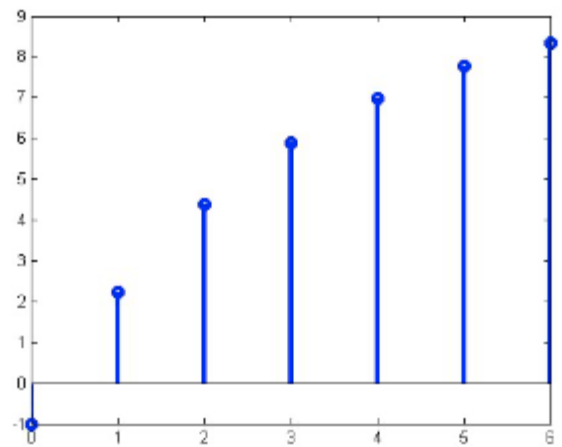


analog signals, if the amplitude can vary continuously in an interval of ℝ

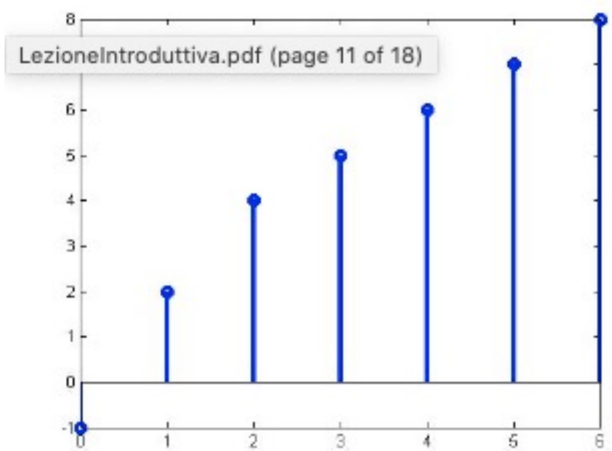


quantized signals, if the amplitude can assume only a finite set of values

The time variable can assume only a set (even infinite) of discrete values.



sampled data signals, if the amplitude can vary continuously in an interval of $\mathbb{R}$



digital signals, if the amplitude is quantized.
Digital signals are represented with a finite number of binary digits.

Discrete-time systems

- Discrete-time systems are characterized by the fact that the time variable is integer rather than real.
- So input and output are sequences of numbers,

$$\{u(k)\}_{k\in N} \qquad \{y(k)\}_{k\in N}$$

- … and are denoted by $u(k)$ and $y(k)$. Then the discrete time system can be denoted by