Course of “Automatic Control Systems” 2023/24

PID controller

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

Università degli Studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code: **mfs9zfr**

PID controller

- The PID are controllers for feedback control systems widely used in the industrial control systems.

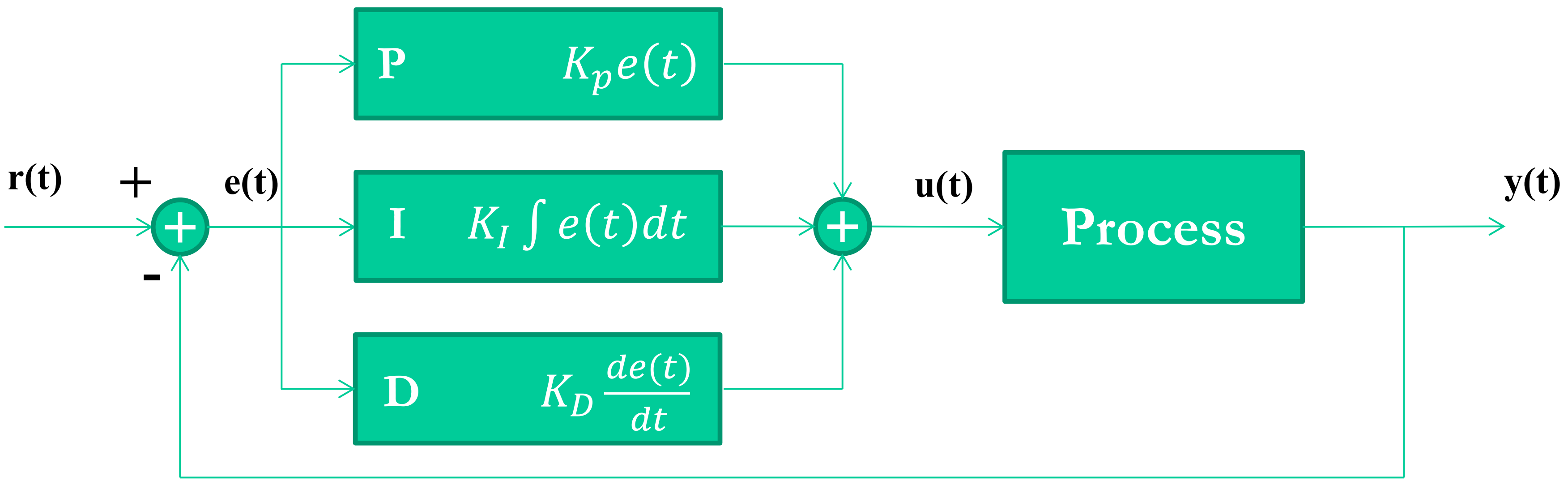
- A PID controller consists of the sum of three control actions, namely a Proportional action, an Integral action, a Derivative action. All these terms act on the tracking error signal $e(t) = r(t) - y(t)$.

- A PID can be written in the time domain as

$$u(t) = \underbrace{K_p e(t)}_{\text{Proportional term}} + \underbrace{K_I \int e(t)dt}_{\text{Integral term}} + \underbrace{K_D \frac{de(t)}{dt}}_{\text{Derivative term}}$$

PID controller in the time domain

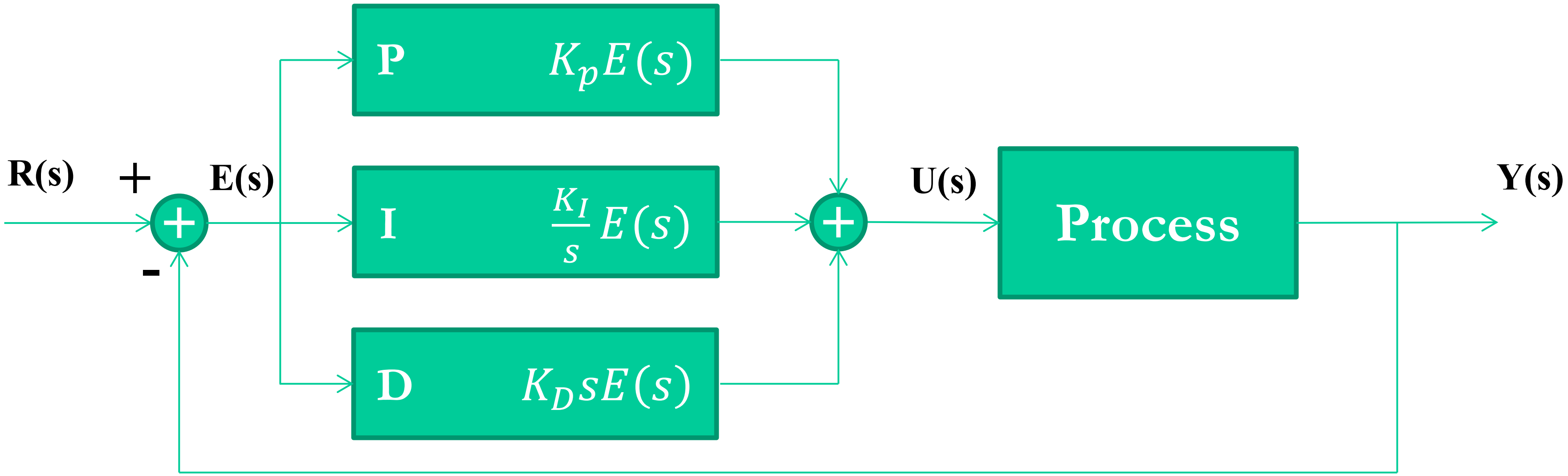
- A PID controller can be represented as



PID controller in the Laplace domain

- A PID controller in the Laplace domain is

$$U(s) = K_p E(s) + \frac{K_I}{s} E(s) + K_D s E(s)$$



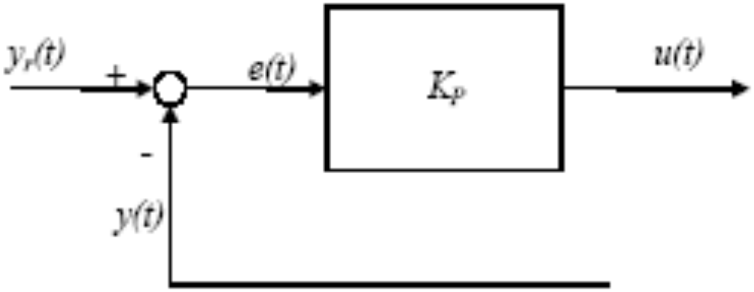
- In the following, in order to evaluate pros and cons of each action, we will consider elementary PID controllers made up of just one action, namely Proportional (P controller), Integral (I controller), Derivative (D controller).

Proportional (P) controller

- The P controller can be written as

$$u(t) = K_p e(t) \quad \rightarrow \quad U(s) = K_p E(s)$$

- The control signal is proportional to the tracking error



- The coefficient K_p is said proportional gain

P controller: steady-state effect

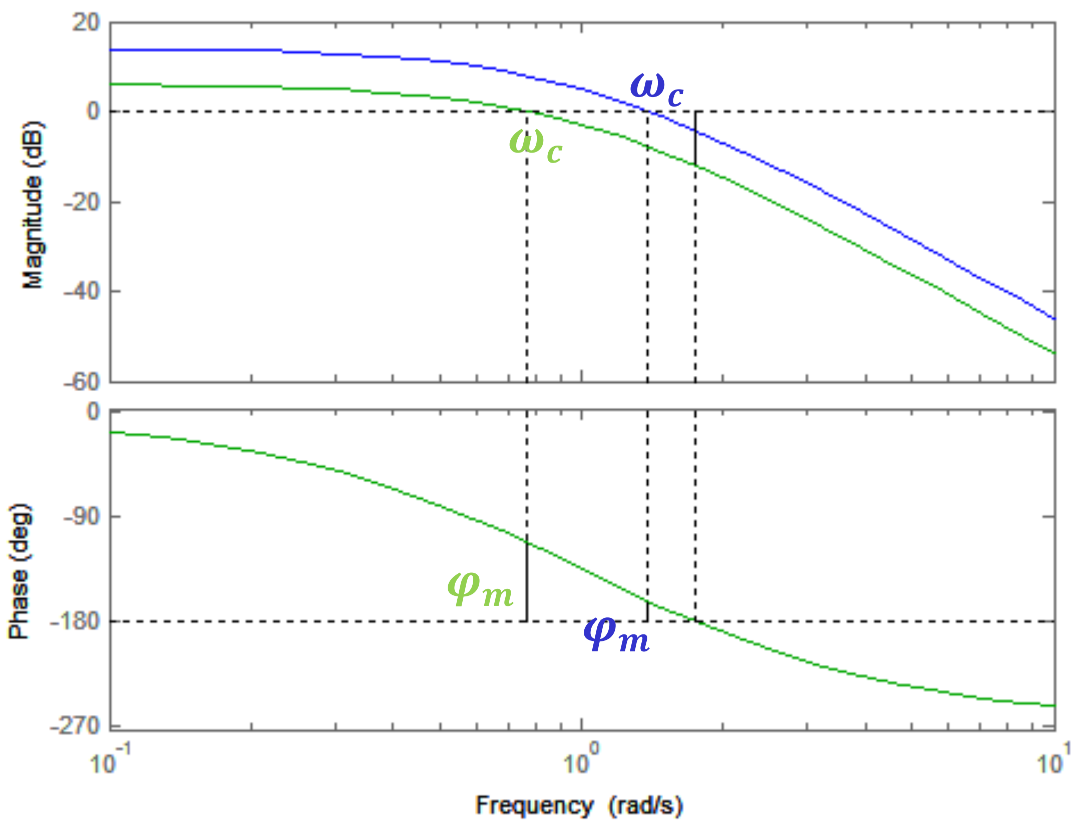
- Assuming that the steady-state error to polynomial signal is finite, increasing the proportional gain it is possible to reduce the steady-state error with respect to reference and disturb polynomial signals.

Order k / Type n	Step R_0/s	Ramp R_0/s^2	Quadratic R_0/s^3
$n = 0$	$\frac{R_0}{1+F_0}$	∞	∞
$n = 1$	0	$\frac{R_0}{F_0}$	∞
$n = 2$	0	0	$\frac{R_0}{F_0}$

$$F_0 = K_p G_0$$

P controller: transient effect

- Assuming a regular stable system, the increase of the proportional gain will cause:
  - Increase of the bandwidth : faster response to input signals
  - Decrease of the phase margin: overshoot and robust stability loss



$$F(s) = \frac{1}{(1+s)^3}$$

$K_p = 2$ $\quad$ $K_p = 5$

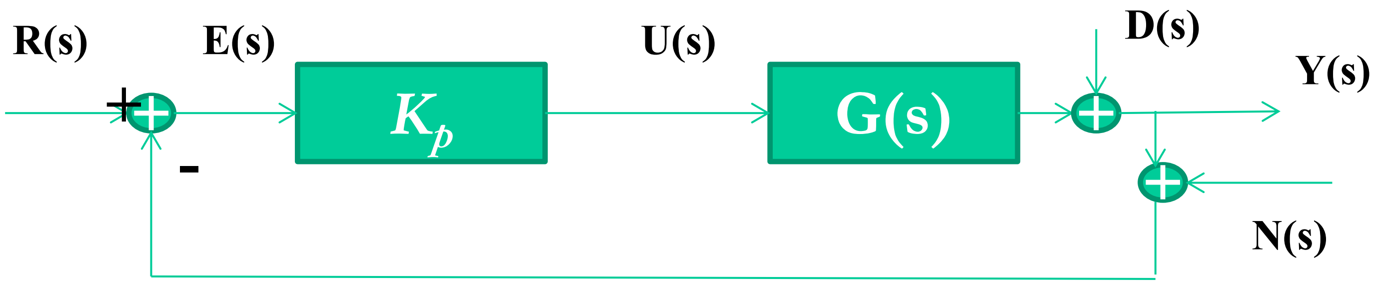
$$\omega_c < \omega_c$$

$$\varphi_m > \varphi_m$$

The increase of the proportional gain can also make the closed loop system unstable

P controller with a plant of first order: analytic results

- Let us consider a closed loop system in the form

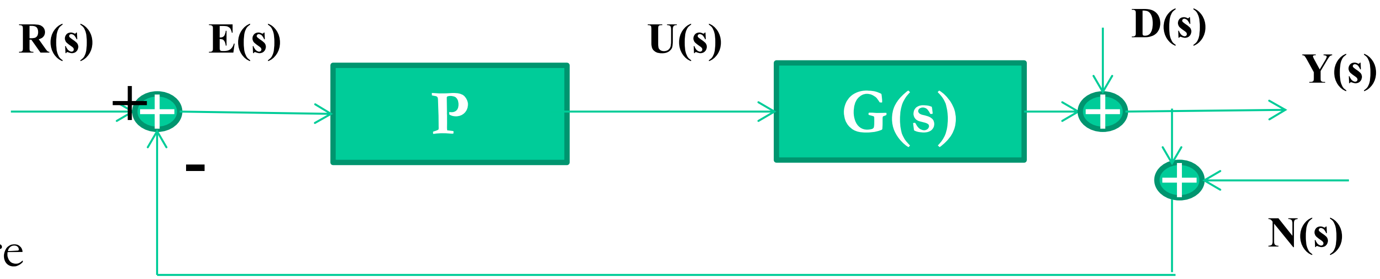


where

$$G(s) = \frac{G_0}{(1 + s\tau)}$$

$$\Rightarrow \quad W(s) = \frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\frac{G_0 K_p}{1 + G_0 K_p}}{(1 + s\,\frac{\tau}{1 + G_0 k_p})}$$

- Let us consider a closed loop system in the form



where

$$G(s) = \frac{G_0}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1}$$

$$\Rightarrow \quad W(s) = \frac{\frac{G_0 K_p}{1 + G_0 K_p}}{\left(\frac{s^2}{(1 + G_0 K_p)\omega_n^2} + \frac{2\zeta}{(1 + G_0 K_p)\omega_n} s + 1\right)}$$

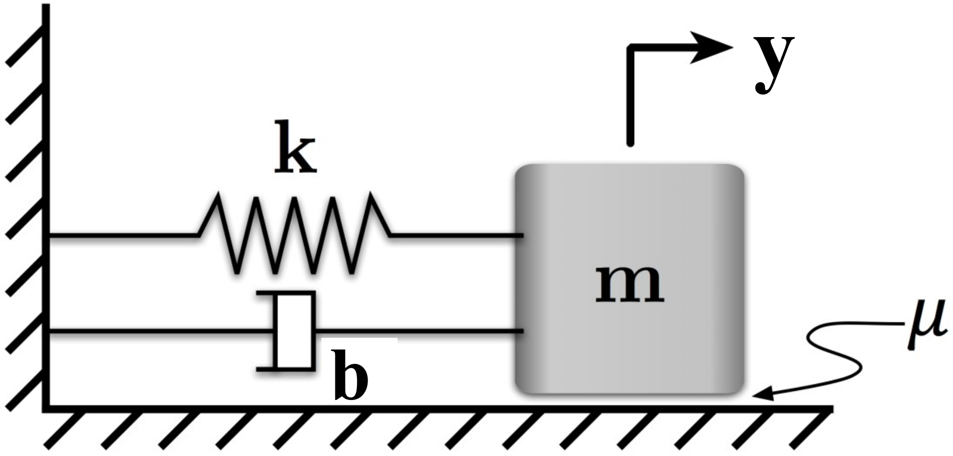
- $\omega_{nc} = \omega_n \sqrt{1 + G_0 K_p}$
- $\zeta_c = {}^{\zeta}/_{\sqrt{1+G_0 K_p}}$

- Let us consider a mass-spring-damper system

- Indicate with $s(t)$ the movement of the mass with respect to a reference position and with $f(t)$ an external control force applied to the mass

- Said $\mathrm{y}(\mathrm{t}) = s(t)$ and $\mathrm{u}(t) = f(t)$, the system can be written as

$$m\ddot{y} + b\dot{y} + ky = u$$



P controller: example

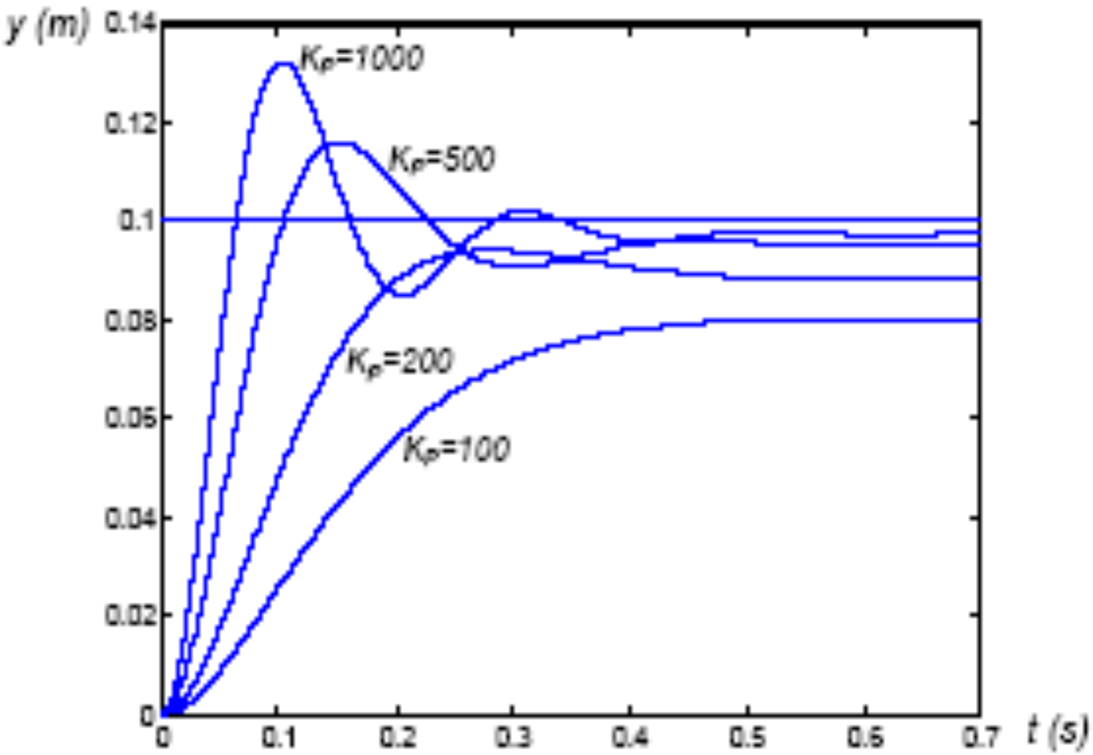
- The aim of the closed loop control is to bring the mass in a new constant position, that is $r(t) = R_0 1(t)$
- Let us assume to use a proportional controller taking care of both the transient behavior and the steady-state performance $u(t) = K_p e(t) = K_p\big(R_0 - y(t)\big)$
- The closed loop equation can be written as

$$m\ddot{y} + b\dot{y} + (k + K_P)y = K_P R_0$$

$$\Rightarrow W(s) = \frac{Y(s)}{R(s)} = \frac{K_P}{ms^2 + bs + k + K_P}$$

P controller: example

- $m = 1 \text{ Kg}$
- $k = 25 \text{ N/m}$
- $b = 20 \text{ Ns/m}$
- $R_0 = 0.1 \text{ m}$

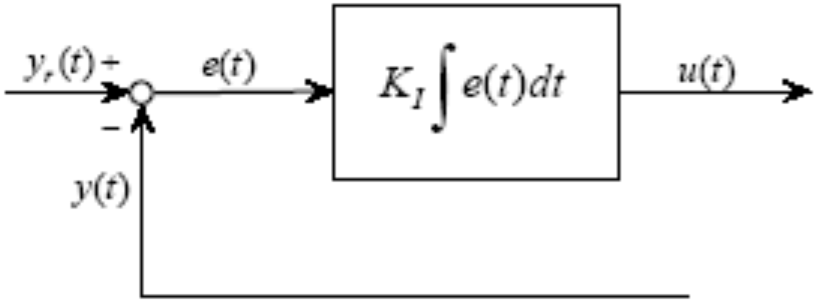


Integral (I) controller

- The integral (I) action can be written as

$$u(t) = K_I \int e(t)dt \quad \rightarrow \quad U(s) = \frac{K_I E(s)}{s}$$

- The control signal is proportional to the integral of the tracking error



- The coefficient K_I is said integral gain

I controller: steady-state effect

- Assuming that the steady-state error to polynomial signal is finite

$$\textit{order of the polynomial signal} = \textit{type of } F(s)$$

adding an integral action it is possible to eliminate the steady-state error with respect to reference and disturb polynomial signals.

Order k / Type n	Step R_0/s	Ramp R_0/s^2	Quadratic R_0/s^3
$n = 0$	$\frac{R_0}{1+F_0}$	∞	∞
$n = 1$	**0**	$\frac{R_0}{F_0}$	∞
$n = 2$	**0**	**0**	$\frac{R_0}{F_0}$

I controller: transient effect

- Assuming a regular stable system, the addition of an integral action will cause:
  - Decrease of the bandwidth : slower response to input signals

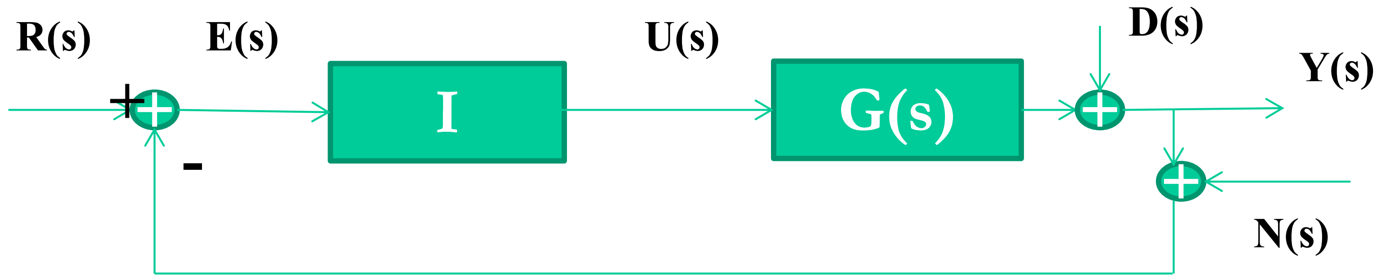
  To maintain the initial bandwidth, it is possible to increase the integral gain K_I

  - 90° phase lag at all frequencies:

  It causes a decrease of the phase margin and therefore a possible increase of the overshoot or the closed loop instability

I controller with a plant of first order

- Let us consider a closed loop system in the form



where

$$G(s) = \frac{G_0}{(1 + s\tau)}$$

$$\Rightarrow \quad W(s) = \frac{\mathbf{1}}{\left(\frac{\tau}{G_0 K_I} s^2 + \frac{\mathbf{1}}{G_0 K_I} s + 1\right)}$$

- $\omega_{nc} = \sqrt{G_0 K_I / \tau}$
- $\zeta_c = \frac{\mathbf{1}}{\mathbf{2}\sqrt{G_0 K_I \tau}}$

➢ Let us consider again the tracking problem of the mass-spring-damper system

$$m\ddot{y} + b\dot{y} + ky = u$$

➢ Let us assume to use an integral controller taking care of both the transient behavior and the steady-state performance

$$u(t) = K_I \int e(t)dt = K_I \int \big(R_0 - y(t)\big)dt$$

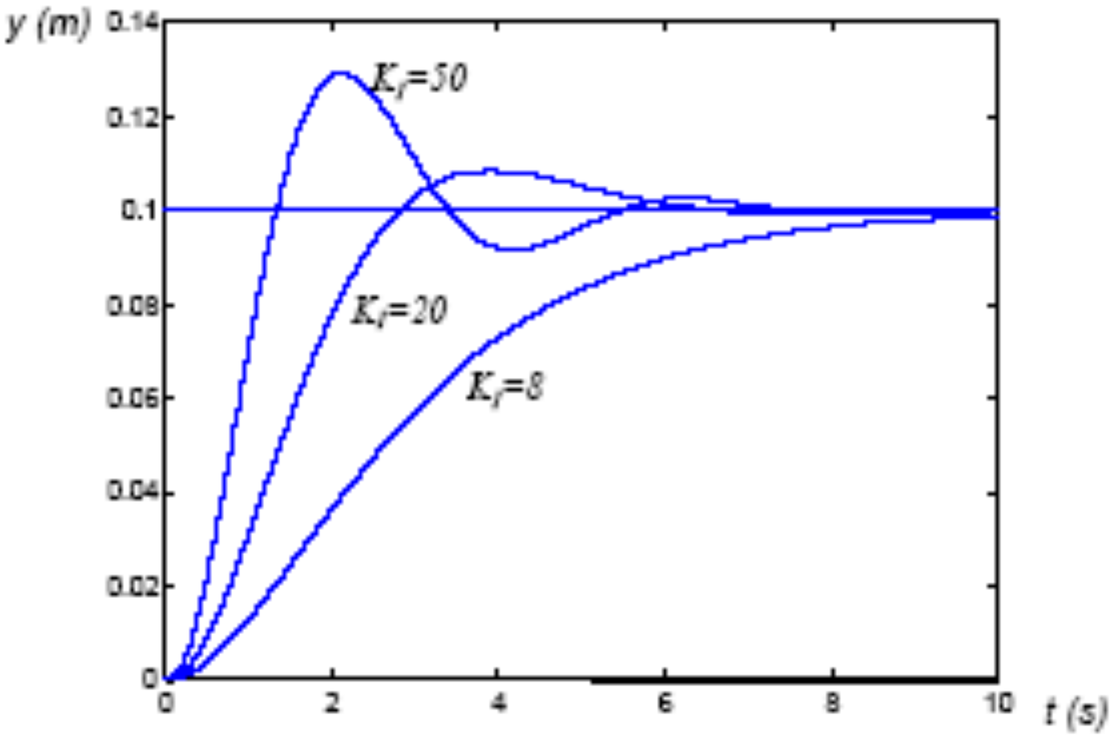
➢ By substituting the equation of $u(t)$ and assuming u as a step signal

$$m\dddot{e} + b\ddot{e} + k\dot{e} + K_I e = 0$$

➢ If the choice of K_I is made in such a way as to guarantee that the system is asymptotically stable, then the error tends to zero and therefore the controlled output tracks at infinity the value of the reference signal.

I controller: example

- $m = 1$ Kg
- $k = 25$ N/m
- $b = 20$ Ns/m
- $R_0 = 0.1$ m



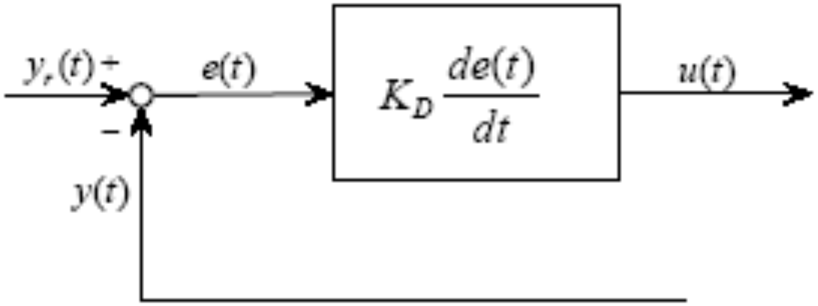
Derivative (D) controller

- The PID integral action can be written as

$$u(t) = K_D \frac{de(t)}{dt} \rightarrow \qquad U(s) = K_D s E(s)$$

- It should be noted that a pure Derivative controller is not Physically Realizable. Indeed, it is realized by adding a pole at a very high frequency, so that to not much modify the performance of the ideal D controller.

- The control signal is proportional to the derivative of the tracking error



- The coefficient K_D is said derivative gain

D controller: steady-state effect

- Assuming that the steady-state error to polynomial signal is finite

$$order\ of\ the\ polynomial\ signal = type\ of\ F(s)$$

adding a derivative action we have a null steady-state output.

- Therefore, the derivative action is never used alone.

D controller: transient effect

- Assuming a regularly stable system, the addition of a derivative action will cause:
  - Increase of the bandwidth : faster response to input signals

  To maintain the initial bandwidth, it is possible to decrease the derivative gain K_D

  - 90° phase lead at all frequencies:

  It has a stabilizing effect because it causes an increment of the phase margin and therefore a possible decrease of the overshoot

D controller: example

- Let us consider again the tracking problem of the mass-spring-damper system

$$m\ddot{y} + b\dot{y} + ky = u$$

- Let us assume to use a derivative controller taking care of both the transient behavior and the steady-state performance

$$u(t) = K_D \frac{de(t)}{dt} = K_D \frac{d\big(R_0 - y(t)\big)}{dt}$$

- The closed loop equation can be written as

$$m\ddot{y} + (b + K_D)\dot{y} + ky = 0$$

- The effect of the derivative action is
  - Null steady-state output
  - An increase of the damping coefficient $\quad \zeta = \dfrac{b + K_D}{2\sqrt{km}}$