Course of
"Automatic Control Systems"
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Control requirements: Steady-state performance

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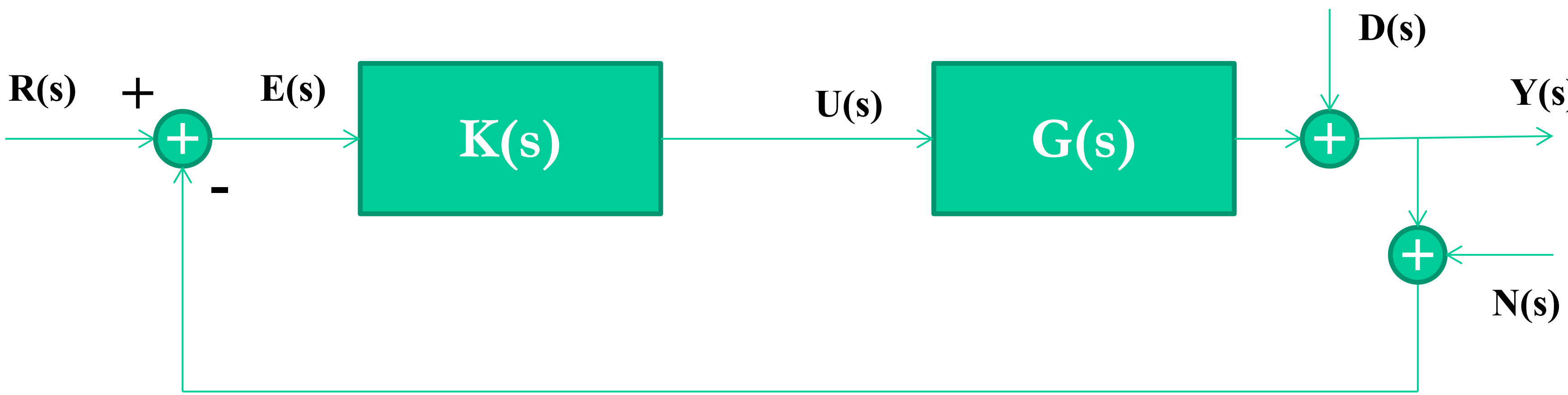
Closed loop transfer function

- A SISO closed loop control system in the Laplace domain can be indicated as



- G(s) plant to be controlled
- K(s) controller
- R(s) reference
- Y(s) controlled output
- U(s) control variable
- E(s) tracking error
- D(s) disturb
- N(s) measurement noise

$$\frac{Y(s)}{R(s)} = T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

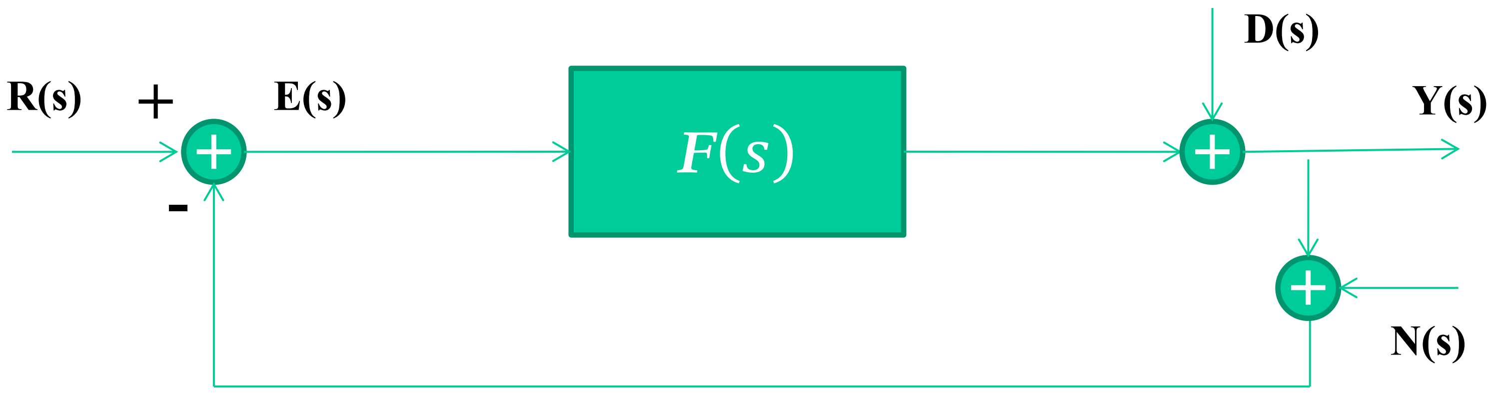
$$\frac{Y(s)}{D(s)} = S(s) = \frac{1}{1 + G(s)K(s)}$$

$$\frac{Y(s)}{N(s)} = -T(s) = \frac{G(k)K(s)}{1 + G(s)K(s)}$$

$$Y(s) = T(s)R(s) + S(s)D(s) - T(s)N(s)$$

Open loop function $F(s)$

- The transfer function given be the series of controller $K(s)$ and plant $G(s)$ is called ***Open Loop (O.L.) function*** $\boldsymbol{F(s) = G(s)K(s)}$



- The O.L. function $F(s)$ assumes a main role in the control theory
- Indeed, it is easier to design a controller $K(s)$ able to modify as desired $F(s)$ instead of closed loop function $T(s)$ (or $W(s)$ using different notation)
- It makes important to convert the closed loop requirements in terms of $F(s)$ constraints

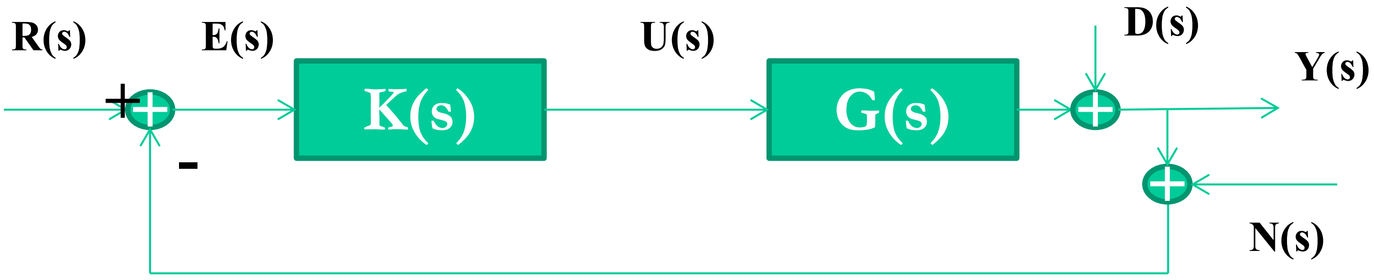
Control requirements

- The closed loop control requirements can be divided in four classes:
  - ***Stability*** ***(DONE)***
  - ***Robust stability*** ***(DONE)***
  - ***Steady-state performances***
  - ***Transient performances***

- ***In the following we will assume that the considered closed loop systems are asymptotically stable***

Closed loop tracking performances

- The performance of the closed loop system are evaluated in terms of
  - ***Tracking of the reference input***
  - ***Rejection of the disturbs***
  - ***Insensibility to the noise***



- When the stability of the C.L. system is guaranteed, the ***response of the system can be divided in a transient and a steady-state parts.***

- The ***steady-state performance*** cares about the steady-state behavior of the closed loop system while the ***transient performance*** cares about the tracking of the reference signal during the transient phase

Steady-state performance

- The steady-state performance depends on the class of input signals $R(s), D(s), N(s)$ and the type of polynomial transfer function $F(s)$

- ***Tracking of the reference input $R(s)$***
  - Null or bounded steady-state error to ***polynomial inputs*** (step, ramp,…)
  - Null or bounded steady-state error to ***sinusoidal inputs*** at fixed frequency

- ***Rejection of the disturbs D(s)***
  - Null or bounded steady-state error to ***polynomial inputs***
  - Bounded error steady-state to ***multi-frequency sinusoidal inputs***

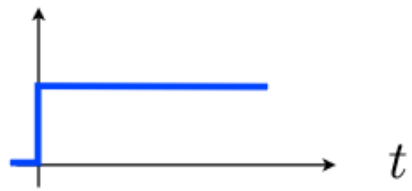
- ***Insensibility to the noise $N(s)$***
  - Bounded steady-state error to ***multi-frequency sinusoidal inputs***

Due to the superposition principle, the three requirements are treated separately.

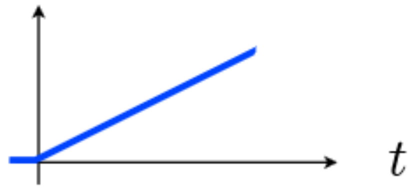
Polynomial function of order k

- ***A polynomial canonic signal of order k*** is defined as $r(t) = \frac{t^k}{k!} 1(t)$.

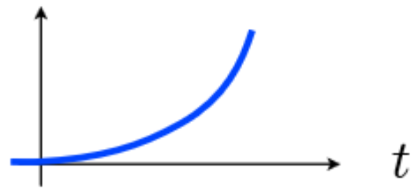
order 0 (step function) $\quad 1(t)$



order 1 (ramp function) $\quad t \cdot 1(t)$



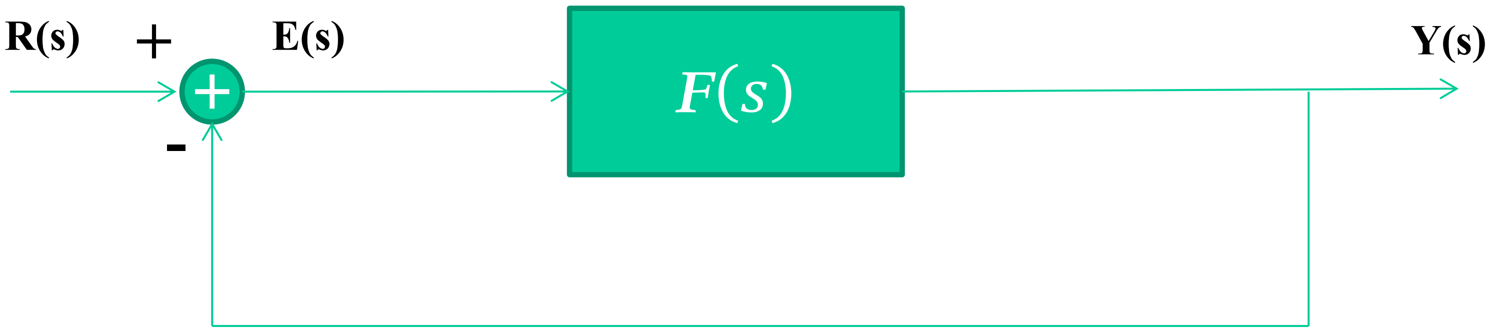
order 2 (quadratic function) $\quad \frac{t^2}{2} \cdot 1(t)$



Tracking of a polynomial reference of order 0

- Let us consider a polynomial reference of order 0 (step function) and amplitude R_0

$$r(t) = R_0 1(t) \quad \rightarrow \quad R(s) = R_0 \frac{1}{s}$$



- The tracking error $E(s)$ is defined as

$$E(s) = S(s)R(s) = \frac{1}{1+F(s)} R(s) = \frac{1}{1+F(s)} \frac{R_0}{s}$$

Tracking of a polynomial reference of order 0

- Making use of the ***Final Value Theorem***

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) =$$

$$= \lim_{s\to 0} \; s \frac{1}{1+F(s)} \frac{R_0}{s} =$$

$$= \frac{R_0}{1+\lim_{s\to 0} F(s)}$$

- Hence, the steady-state error at reference signal of order 0 is null if

$$\lim_{s\to 0} F(s) \to \infty$$

that is $\boldsymbol{F(s)}$ ***has one or more poles in the origin.***

- An ***O.L. transfer function*** $\boldsymbol{F(s)}$ is said to be of ***type*** $\boldsymbol{n}$ if the number of poles in the origin is n.

Tracking of a polynomial reference

- In case of a ***reference signal of order*** $\boldsymbol{k = 0}$, that is $\boldsymbol{r(t) = R_0\, 1(t)}$.

  - For O.L. function $F(s)$ of type $n = 0$, the steady-state tracking error is finite

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{R_0}{1+F_0}$$

  - For O.L. function $F(s)$ of type $n > 0$, the steady-state tracking error is null

Tracking of a polynomial reference

- In case of a ***reference signal of order*** $\boldsymbol{k=1}$, that is $\boldsymbol{r(t)=R_0 t\, 1(t)}$.

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+F(s)}\frac{R_0}{s^2}$$

- For O.L. function $F(s)$ of type $n=0$, the steady-state tracking error is infinite

- For O.L. function $F(s)$ of type $n=1$, the steady-state tracking error is finite

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{1+F_0/s}\frac{R_0}{s} = \frac{R_0}{F_0}$$

- For O.L. function $F(s)$ of type $n>1$, the steady-state tracking error is null

Tracking of a polynomial reference

- The previous results can be summarized as follows

Order k / Type n	Step R_0/s	Ramp R_0/s^2	Quadratic R_0/s^3
$n = 0$	$\frac{R_0}{1+F_0}$	∞	∞
$n = 1$	**0**	$\frac{R_0}{F_0}$	∞
$n = 2$	**0**	**0**	$\frac{R_0}{F_0}$

Internal model principle

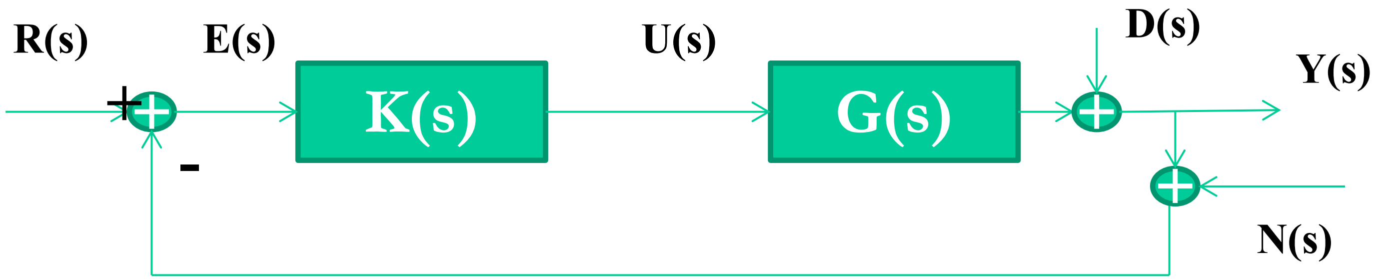
- The previous table allows to derive the so called ***Internal Model Principle:***

In order to make a C.L. system able to track a reference signal of order $\mathbf{n}$ with null steady-state error, it is necessary an O.L. system of type $\mathbf{n+1}$

- Taking into account that $F(s) = K(s)G(s)$, if the plant $G(s)$ doesn't contain enough integrators, they must be supplied by the controller $K(s)$.

Rejection of polynomial disturbs

- Let us consider the initial closed loop scheme:

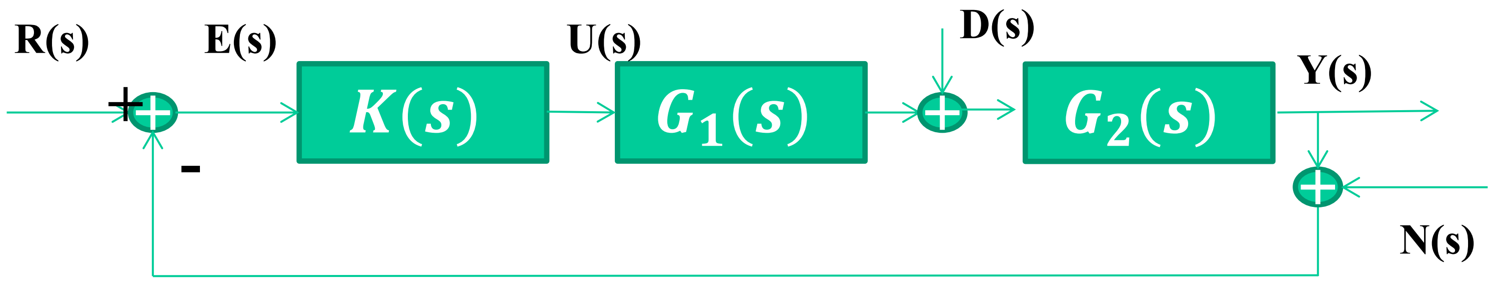


- The transfer function $D(s) \rightarrow Y(s)$ o $(E(s))$ is the same as the transfer function $R(s) \rightarrow E(s)$ (except for the sign for $E(s)$=-$S(s)D(s)$).

$$Y(s) = S(s)D(s) = \frac{1}{1 + F(s)} D(s)$$

- Considering that we are interested in achieving a null contribution, the previous table can be used to quantify the rejection of polynomial disturbances when they are additive to the output

- Let us consider a polynomial disturbance summed to the input of the transfer function $G_2(s)$.



- The transfer function $D(s) \to Y(s)$ is now

$$Y(s) = \frac{G_2(s)}{1 + K(s)G_1(s)G_2(s)} D(s)$$

- Applying the Final Value Theorem, it is possible to verify that only the integrators in $K(s)$ and $G_1(s)$ affects the steady state response to polynomial signals.

Effect of integrators on the closed loop stability

- The previous analysis has shown that, provided the control system remains stable, adding integrators in the feedforward path has beneficial effects on the steady-state behavior of the closed-loop system

- However integrators in the open-loop system have a destabilizing effect on the closed-loop because of the $-\pi/2$ phase lag that can reduce the phase margin.

- Therefore, a rule of thumb for the control system design is to use the minimal number of integrators able to satisfy the steady-state requirements.