Course of
"Automatic Control Systems"
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Nyquist stability criterion

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Phase variation formula

- The previous lesson the concept of phase variation has been introduced.
- The phase variation is related to the number and sign of poles/zeros of the transfer function $F(s)|_{s=j\omega}$.

Given a transfer function $F(s)|_{s=j\omega}$, said:

- ***n the total number of poles***
- ***m the total number of zeros***
- ***n_p (n_n) the number of poles with positive (negative) real part***
- ***m_p (m_n) the total number of zeros with positive (negative) real part***

$$\boldsymbol{n = n_n + n_p} \qquad \boldsymbol{m = m_n + m_p}$$

$$\underset{-\infty\ \ \boldsymbol{\omega}\ \ \infty}{\Delta\angle F(j\omega)} = \pi(m_n - n_n) - \pi(m_p - n_p)$$
$$= \pi(m - n) - 2\pi(m_p - n_p)$$

Phase variation formula

- The previous phase variation formula doesn't consider the case of poles and zeros on the imaginary axis.

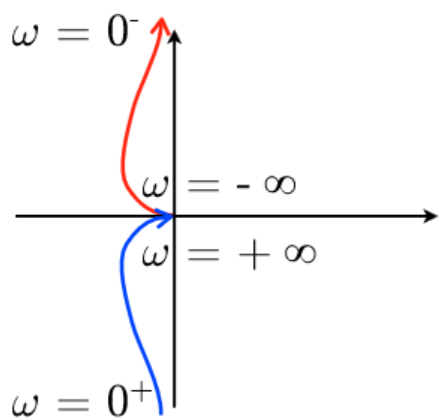
- Indeed, in case of poles and zeros on the imaginary axis the phase variation can not be defined

- In the following slides we will consider these two critical cases and we will illustrate how to extend the definition of phase variation

Phase variation with poles on the imaginary axis

- Open loop poles on the imaginary axis (null real part), can be due to:
  - One or more integrators $1/s^h$
  - Resonance $1/(1+s^2/\omega_n^2)^h$
- In both the cases we have a ***discontinuity in the phase margin***:
  - ***Integrator***: passing ***from*** $\boldsymbol{\pi/2}$ ***to*** $\boldsymbol{-\pi/2}$ with infinite magnitude ***at*** $\boldsymbol{\omega = 0}$
  - ***Resonance***: passing ***from*** $\boldsymbol{0}$ ***to*** $\boldsymbol{-\pi}$ with infinity magnitude ***at*** $\boldsymbol{\omega = \omega_n}$ and ***from*** $\boldsymbol{\pi}$ ***to*** $\boldsymbol{0}$ with infinity magnitude ***at*** $\boldsymbol{\omega = -\omega_n}$

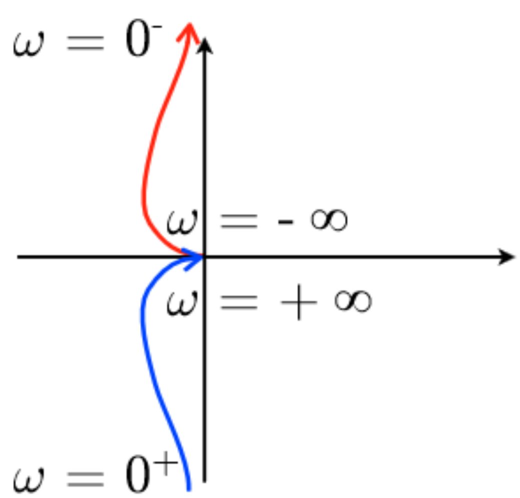
$$F(s) = \frac{1}{s(s+1)}$$

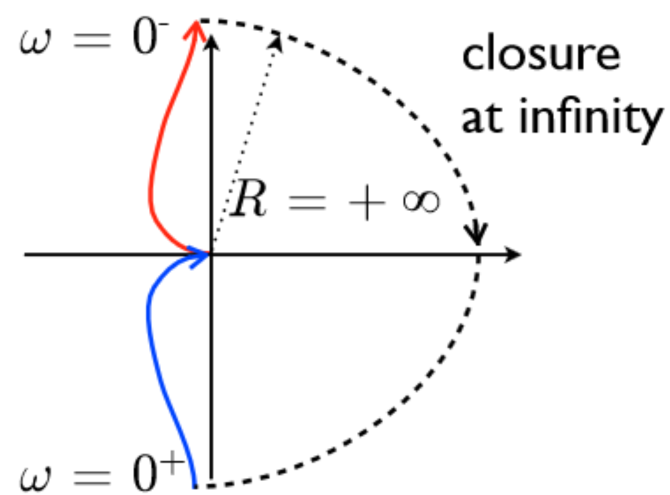


Phase variation with poles on the imaginary axis

- In order to obtain a closed polar plot and to extend the definition of phase variation, we introduce ***the closures at infinity***.
- The closures at infinity consists in rotating clockwise the Nyquist plot of the $F(j\omega)$ in the discontinuity frequency with an infinite radius.
- With this manipulation, the contribution to the phase variation of poles on the imaginary axis will be the same as the poles with negative real part.

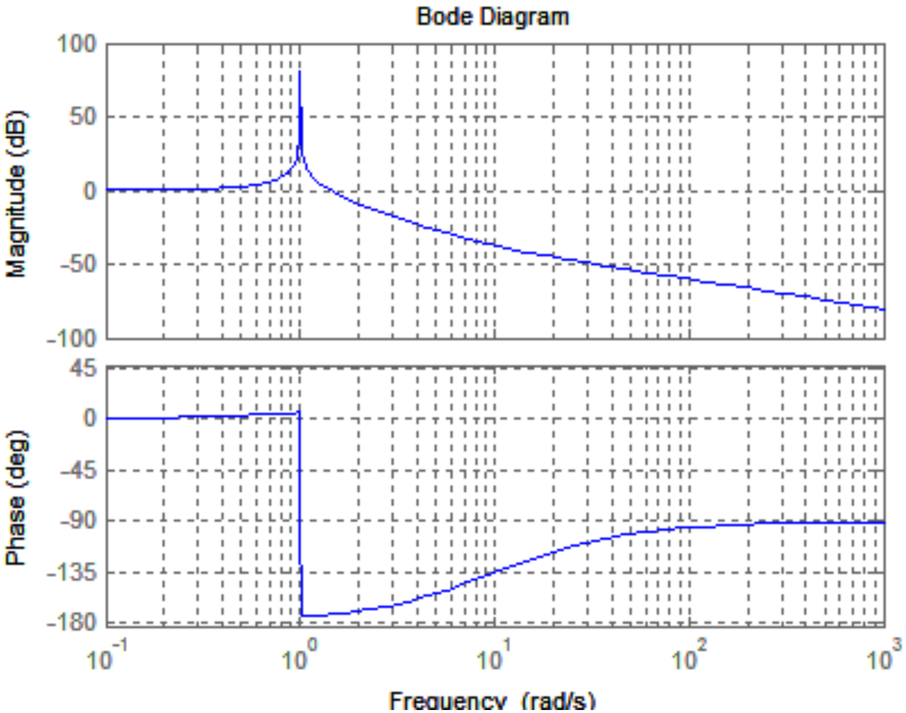
$$F(s) = \frac{1}{s(s+1)}$$

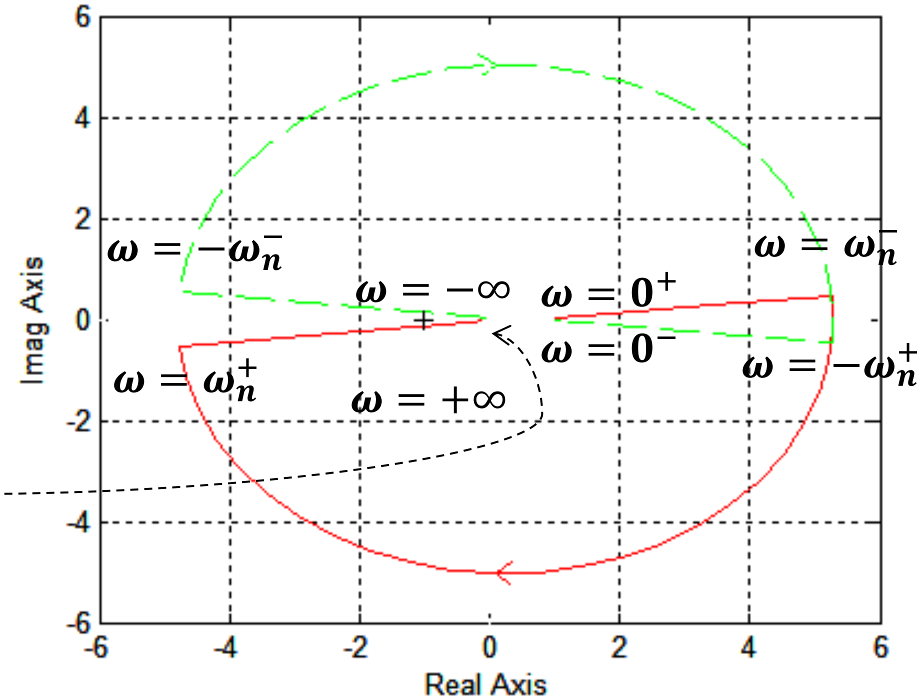




- Transfer function with resonance

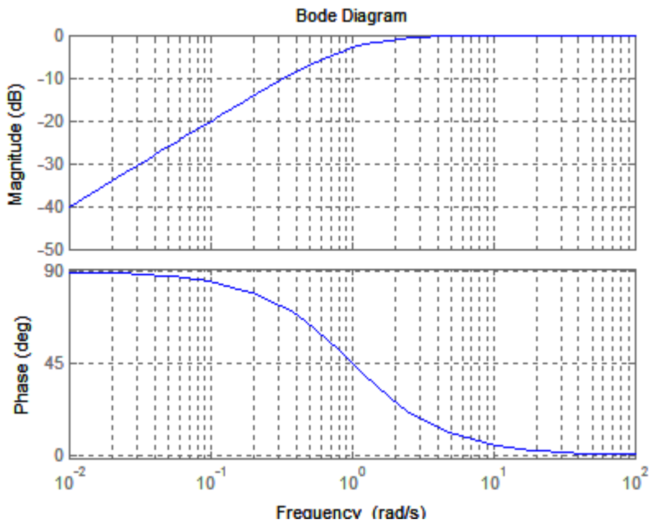
$$F(s) = \frac{1 + 0.1s}{1 + s^2}$$



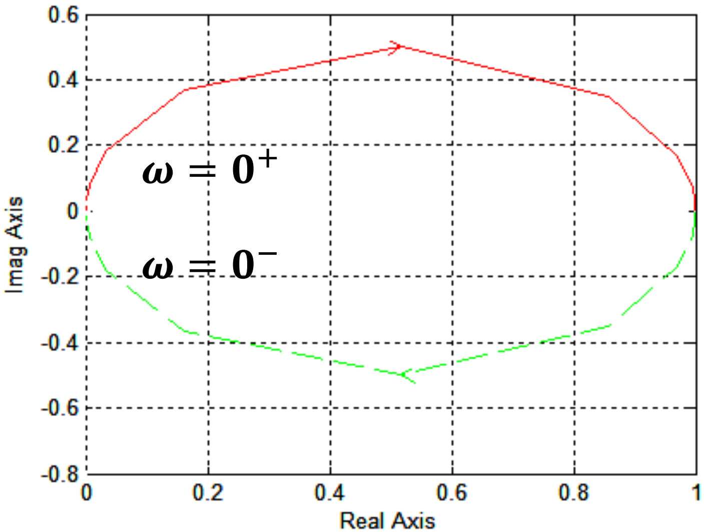


$$\underset{-\infty \;\; \omega \;\; \infty}{\Delta\angle F(j\omega)} = \pi(m-n) - 2\pi(m_p - n_p) = \pi(1-2) = -\pi$$

- Open loop zeros on the imaginary axis (null real part), can be due to:
  - One or more derivative s^h
  - Anti-resonance $(1+s^2/\omega_n^2)^h$
- In both the cases we have a ***discontinuity in the phase margin***:
  - ***Derivate***: passing ***from*** $-\pi/2$ ***to*** $\pi/2$ with zero magnitude ***at*** $\omega = 0$
  - ***Anti-Resonance***: passing ***from*** 0 ***to*** π with zero magnitude ***at*** $\omega = \omega_n$ and ***from*** $-\pi$ ***to*** 0 with zero magnitude ***at*** $\omega = -\omega_n$



$$F(s) = \frac{s}{1+s}$$

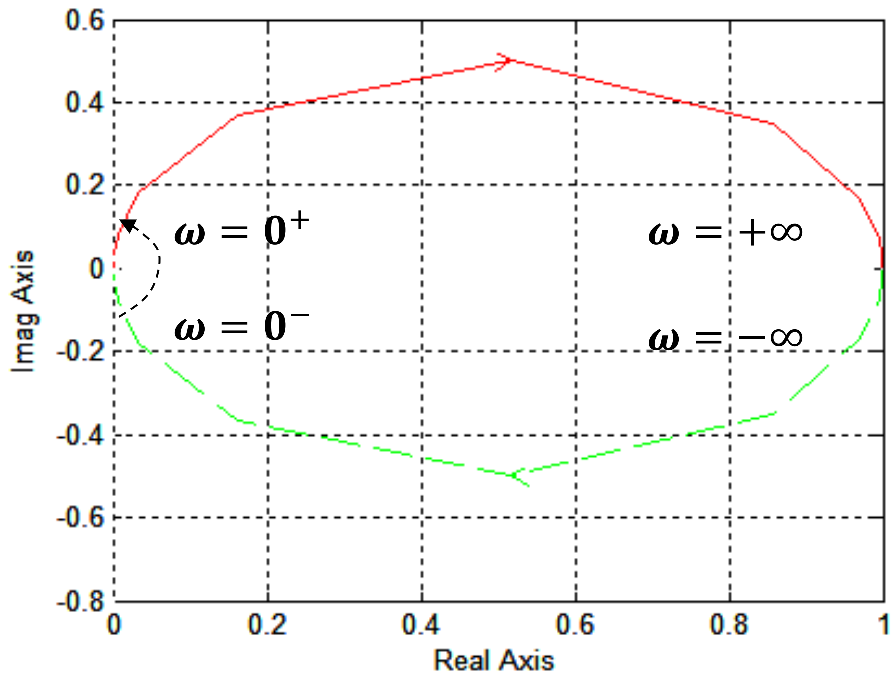


- In order to extend the definition of phase variation, we will assume that ***in*** $\boldsymbol{\omega = 0}$ ***and*** $\boldsymbol{\omega = \omega_n}$ the Nyquist plot of the frequency response $F(j\omega)$ will rotate ***counterclockwise*** with infinitesimal magnitude.

- With this manipulation, the contribution to the phase variation of zeros on the imaginary axis will be the same as the zeros with negative real part.

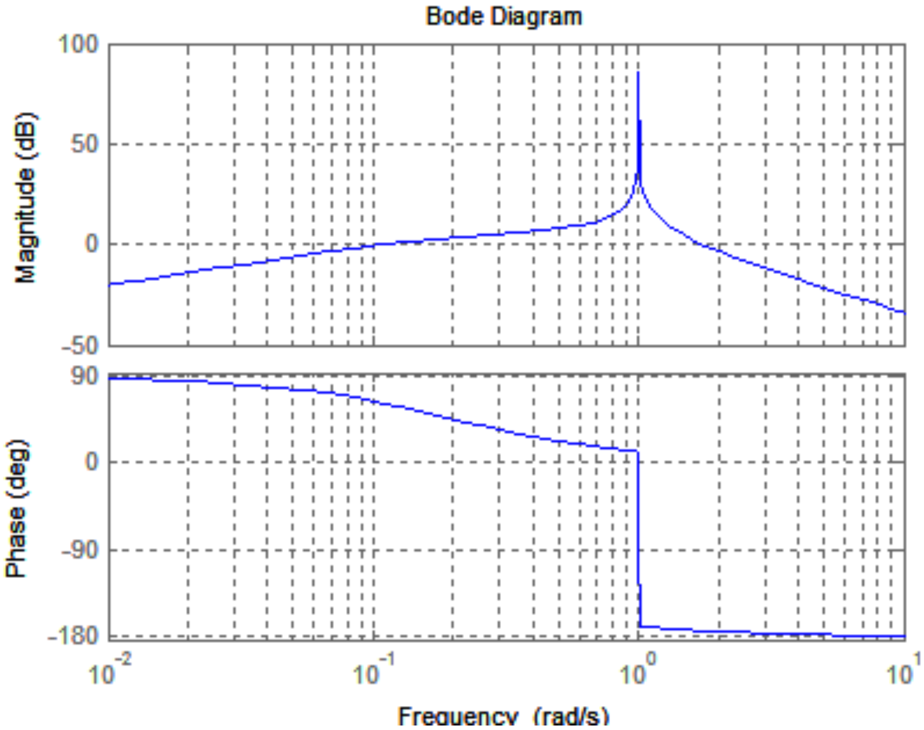
$$\boldsymbol{F(s) = \frac{s}{1+s}}$$

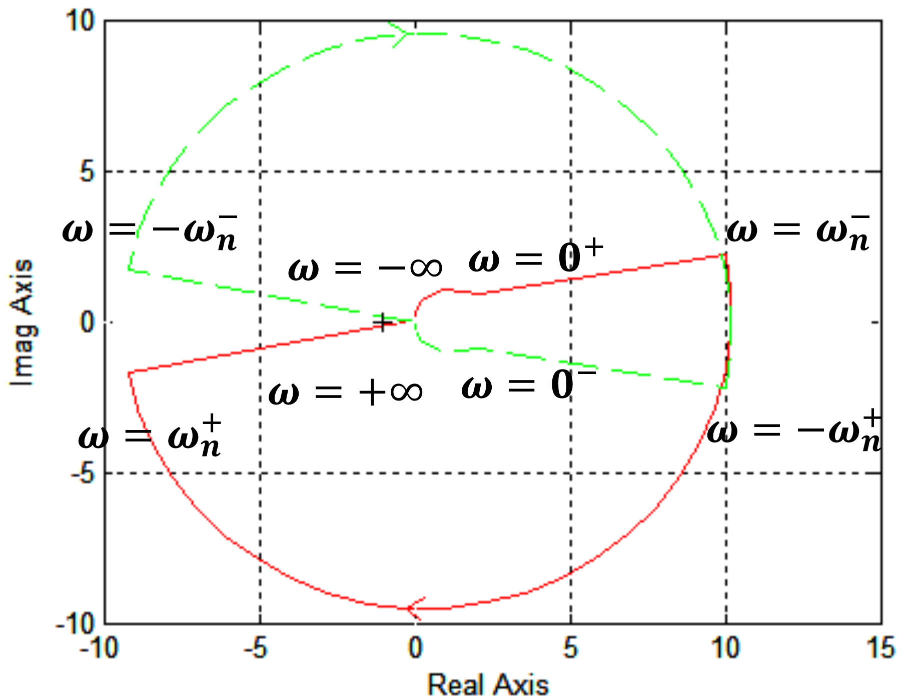
$$\underset{-\infty\ \omega\ \infty}{\Delta\angle F(j\omega)} = 0$$



- Transfer function with resonance

$$F(s) = \frac{10s}{(1+5s)(1+s^2)}$$

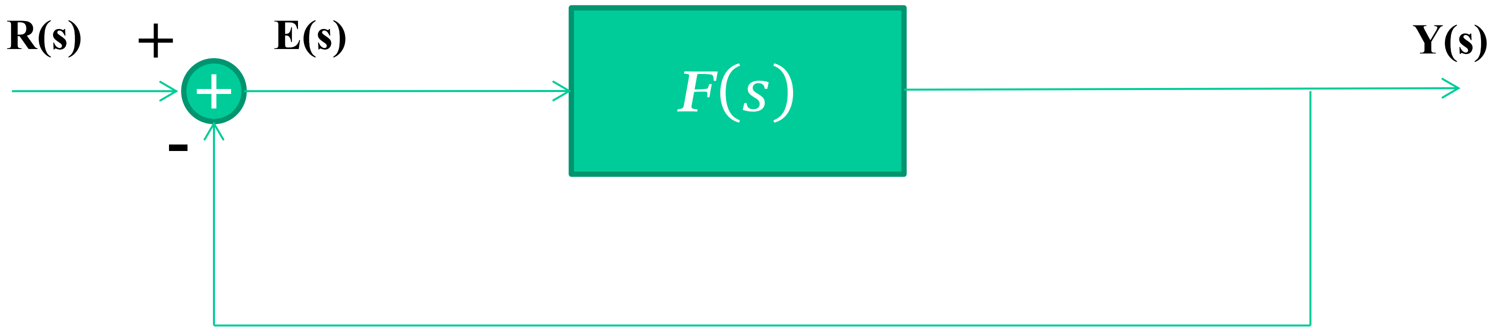




$$\underset{-\infty\ \omega\ \infty}{\Delta\angle F(j\omega)} = \pi(m-n) - 2\pi\left(m_p - n_p\right) = \pi(1-3) = -2\pi$$

Stability of the closed loop system

- Let us consider the $R(s) \rightarrow Y(s)$ closed loop system



- Assume that the hidden modes of the open loop function $F(s) = K(s)G(s)$ are asymptotically stable

- The stability of the closed loop system depends on the poles of the transfer function

$$\mathrm{T}(s) = \frac{F(s)}{1+F(s)}$$

Stability of the closed loop system

- Indicate with $N_F(s)$ and $D_F(s)$ the numerator and the denominator of the open loop function

$$\mathbf{F}(s) = \frac{N_F(s)}{D_F(s)} = \frac{N_K(s)N_G(s)}{D_K(s)D_G(s)},$$

- The closed loop function can be written as

$$\mathbf{T}(s) = \frac{\frac{N_F(s)}{D_F(s)}}{1 + \frac{N_F(s)}{D_F(s)}} = \frac{N_F(s)}{D_F(s)+N_F(s)}.$$

- The denominator of $T(s)$ is given by the sum of $N_F(s)$ and $D_F(s)$; therefore, the design problem of a controller able to guarantee the stability of the closed loop system is to be very demanding.

- By means of the Nyquist plots and the ***Nyquist criteria***, we are going ***to determine the stability of the closed loop*** system ***from the open loop*** system features

Nyquist Stability Criterion

Let us consider a strictly proper open-loop function $F(s)$ *and assume that the Nyquist diagram of* $F(s)$ *doesn't intersect the critical point* $-1+j0$.

Said

- $\overleftarrow{\mathcal{N}}$ *the number of counter-clockwise encirclements of the critical point* $-1+j0$ *of the Nyquist plot of* $F(s)$
- $n_{p^+}(F(s))$ *the number of unstable poles of* $F(s)$

the closed loop function $T(s)$ *is asymptotically stable if and only if*

$$\overleftarrow{\mathcal{N}} = n_{p+}(F(s)).$$

Moreover, if $\overleftarrow{\mathcal{N}} \neq n_{p^+}$, *the number of unstable poles of the closed loop function* $T(s)$ *is equal to* $n_{p^+}(F(s)) - \overleftarrow{\mathcal{N}}$.

Nyquist Stability Criterion: proof

- Let us define the so-called ***Difference Function***

$$D(s) = 1 + F(s) = \frac{N_F(s) + D_F(s)}{D_F(s)}$$

- It is straightforward to notice that :

  - The poles of $D(s)$ are the open loop control system poles , i.e. $D_D(s) = D_F(s)$

  - The zeros of $D(s)$ are the closed loop control system poles, i.e. $N_D(s) = D_T(s)$

Nyquist Stability Criterion: proof

- Said
  - $\boldsymbol{n_p(D(s))}$ the number of poles of $D(s)$
  - $\boldsymbol{n_{p^+}(D(s))}$ the number of poles with positive real part of $D(s)$
  - $\boldsymbol{n_z(D(s))}$ the number of zeros of $D(s)$
  - $\boldsymbol{n_{z^+}(D(s))}$ the number of zeros with positive real part of $D(s)$

The phase variation of the difference function is

$$\underset{-\infty\ \ \boldsymbol{\omega}\ \ \infty}{\Delta\angle D(j\omega)} = \pi\Big(\boldsymbol{n_z(D(s))} - \boldsymbol{n_p(D(s))}\Big) - 2\pi\Big(\boldsymbol{n_{z^+}(D(s))} - \boldsymbol{n_{p^+}(D(s))}\Big)$$

Nyquist Stability Criterion: proof

- However,

1. $F(s)$ strictly proper $\rightarrow$ $D(s) = 1 + F(s) = \frac{N_F(s)+D_F(s)}{D_F(s)}$ proper and than

$$\boldsymbol{n_p(D(s)) = n_z(D(s))}$$

2. Taking into account that $N_D(s) = D_T(s)$ and it is required the closed loop stability of the system, than

$$\boldsymbol{n_{z^+}\big(D(s)\big) = 0}$$

- Hence the phase variation of the difference function is

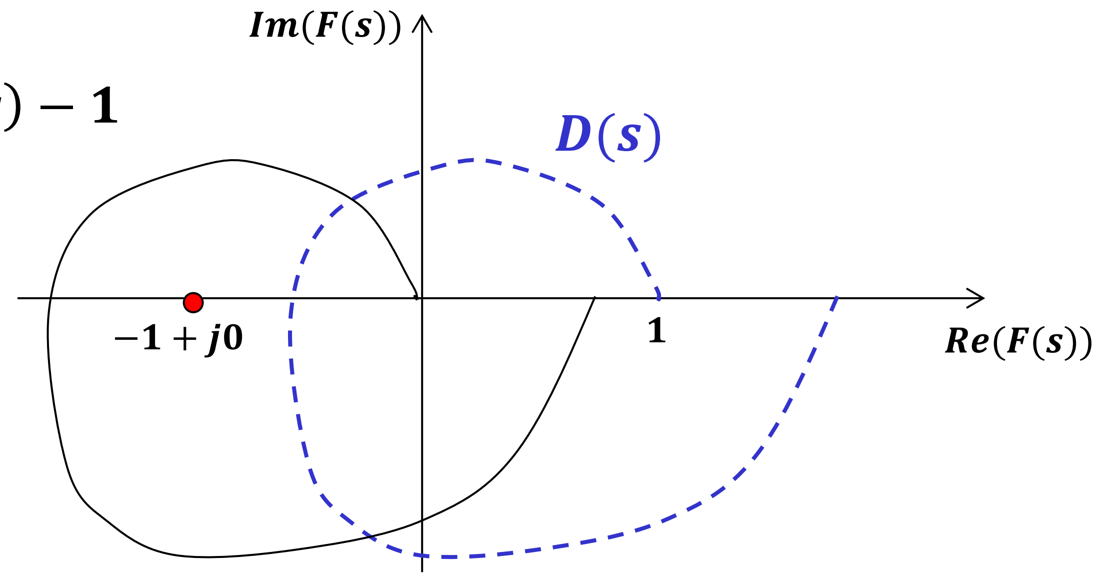
$$\underset{-\infty\ \boldsymbol{\omega}\ \infty}{\Delta\angle D(j\omega)} = 2\pi \cdot \boldsymbol{n_{p^+}(D(s))} = 2\pi \cdot \boldsymbol{n_{p^+}(F(s))}$$

- The function $D(s)$ will encircle counter-clockwise the origin of the Nyquist plane a number of times given by

$$n_{p^+}\left(F(s)\right)$$

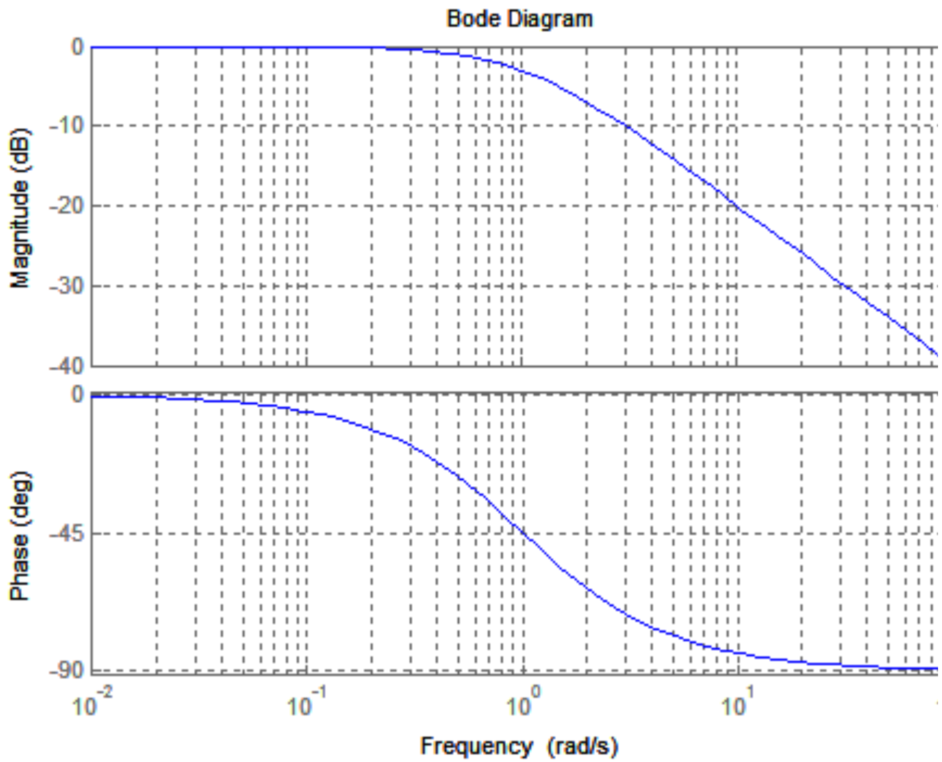
- The proof is concluded taking into account that the encirclements of the origin of the $D(s)$ Nyquist plot correspond to the encirclements of the critical point $-1+j0$ of the $F(s)$ Nyquist plot

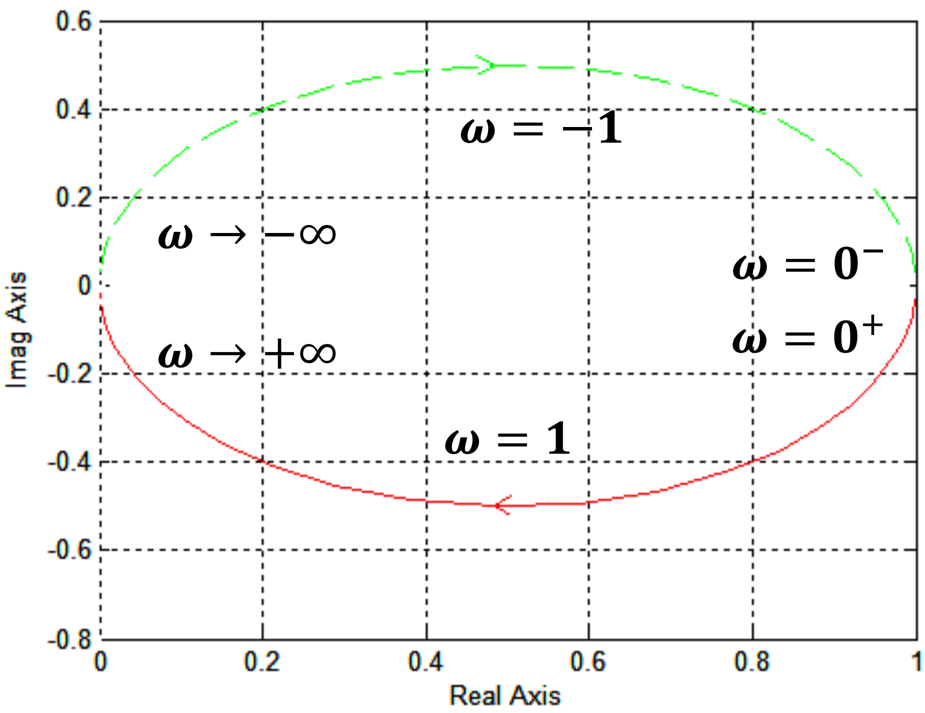
$$\mathbf{F}(s) = D(s) - 1$$



Nyquist stability criterion: example 1

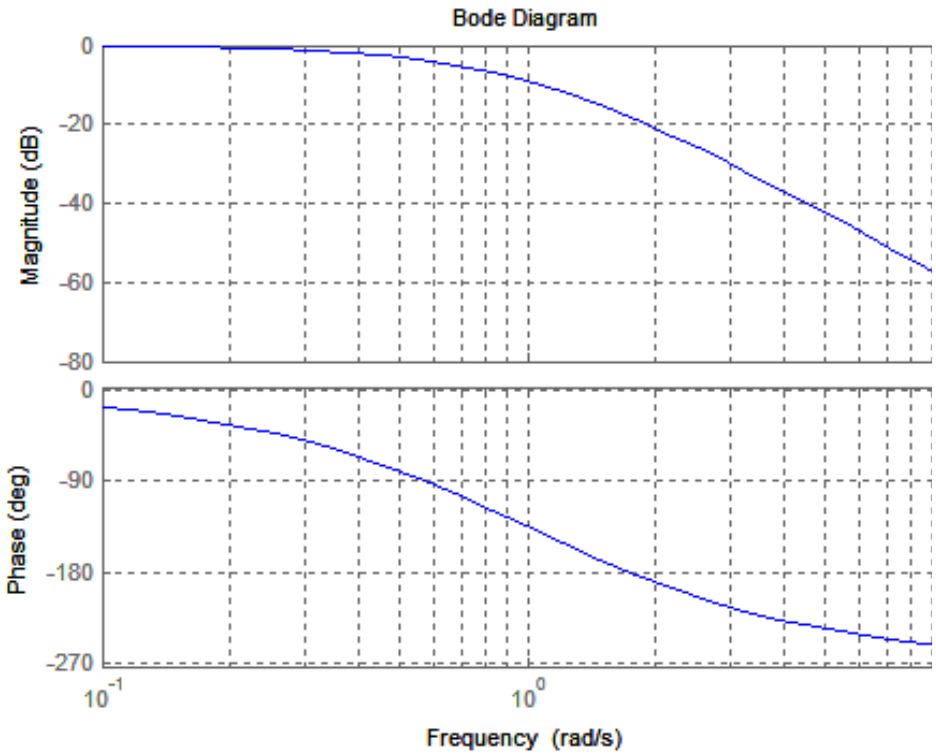
- Let us consider again the frequency response $F(s) = \frac{1}{1+s}$

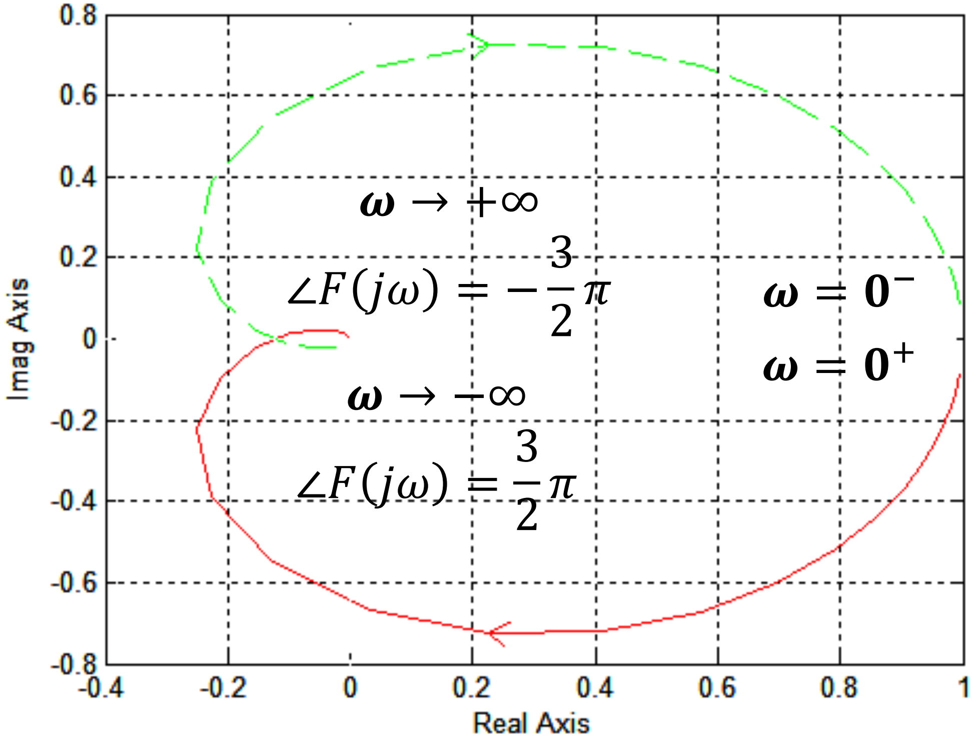




$$\overleftarrow{\mathcal{N}} = n_{p+}(F(s)) = 0 \;\rightarrow \textit{asymptotically stable closed loop function}$$

- Let us consider the frequency response $F(s) = \frac{1}{(1+s)^3}$





$$\overleftarrow{\mathcal{N}} = n_{p+}(F(s)) = 0 \ \rightarrow \textit{asymptotically stable closed loop function}$$

However the two examples have an important difference in terms of robust stability ….