UNIVERSITÀ DEGLI STUDI DI NAPOLI
PARTHENOPE

Artificial Intelligence

First-Order Logic (Predicate Logic)

LESSON 13

prof. Antonino Staiano

M.Sc. In ''Machine Learning e Big Data'' - University Parthenope of Naples

Pros and Cons of Propositional Logic

✓Propositional logic is declarative
- pieces of syntax correspond to facts

✓Propositional logic allows dealing with partial information using disjunction and negation

✓Propositional logic is compositional
- meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

✓Meaning in propositional logic is context-independent (unlike natural language, where meaning depends on context)

✓Propositional logic has very limited expressive power (unlike natural language)
- E.g., cannot say "*pits cause breezes in adjacent squares*" except by writing one sentence for each square

First-Order Logic

- Whereas propositional logic assumes the world contains facts, first-order logic (like natural language) assumes the world contains
  - Objects: people, houses, numbers, theories, Pinocchio, colors, football games, wars, centuries . . .
  - Relations:
    - Unary (also called **properties**)
      - red, round, bogus, prime, multistoried . . .
    - n-ary
      - brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
  - Functions (relations with one *value* for a given *input*)
    - father of, best friend, third goal of, one more than, end of
  - . . .

Logics in General

- Ontological commitment
  - What a language assumes about the nature of reality
- Epistemological commitment
  - The possible states of knowledge that a logic allows with respect to each fact

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax: Basic Elements

- The basic elements are symbols that are used to represent domain elements (a set of objects), relations, and functions
  - Constant symbols denote objects
    - *One, Two, Three, John, Mary*
  - Predicate symbols denote relations
    - *GreaterThan, Prime, Sum, Father*
  - Functions symbols denote functions
    - *Plus, FatherOf, LeftLegOf*

Syntax: Basic Elements

- Variables
  - $x, y, a, b, \ldots$
- Connectives
  - $\land \quad \lor \quad \neg \quad \Rightarrow \quad \Leftrightarrow$
- Equality
  - $=$
- Quantifiers
  - $\forall \ \exists$

Atomic Sentences

- An atomic sentence is formed from a predicate symbol optionally followed by a parenthesized list of terms

- Atomic sentence = *predicate(term$_1$,..., term$_n$)*
  or *term$_1$=term$_2$*
  - Term = *function(term$_1$, ..., term$_n$)*
    or *constant* or *variable*
- Example

Brother(KingJohn, RichardTheLionheart)

GreaterThan (Length(LeftLegOf (Richard)), Length(LeftLegOf(KingJ ohn)))

Complex Sentences

- Complex sentences are made up of atomic sentences with the use of connectives
  - $\neg S,\ S_1 \wedge S_2,\ S_1 \vee S_2,\ S_1 \Rightarrow S_2,\ S_1 \Leftrightarrow S_2$
- Example
  - $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$
  - $GreaterThan(1,2) \vee LessOrEqual(1,2)$
  - $GreaterThan(1,2) \wedge \neg\ GreaterThan(1,2)$

Syntax

- A formal grammar in Backus-Naur Form (BNF)

$$
\begin{array}{rcl}
\textit{Sentence} & \rightarrow & \textit{AtomicSentence} \mid \textit{ComplexSentence} \\
\textit{AtomicSentence} & \rightarrow & \textit{Predicate} \mid \textit{Predicate}(\textit{Term}, \ldots) \mid \textit{Term} = \textit{Term} \\
\textit{ComplexSentence} & \rightarrow & (\ \textit{Sentence}\ ) \\
& \mid & \neg\, \textit{Sentence} \\
& \mid & \textit{Sentence} \land \textit{Sentence} \\
& \mid & \textit{Sentence} \lor \textit{Sentence} \\
& \mid & \textit{Sentence} \;\Rightarrow\; \textit{Sentence} \\
& \mid & \textit{Sentence} \;\Leftrightarrow\; \textit{Sentence} \\
& \mid & \textit{Quantifier}\ \textit{Variable}, \ldots\ \textit{Sentence} \\
\\
\textit{Term} & \rightarrow & \textit{Function}(\textit{Term}, \ldots) \\
& \mid & \textit{Constant} \\
& \mid & \textit{Variable} \\
\\
\textit{Quantifier} & \rightarrow & \forall \mid \exists \\
\textit{Constant} & \rightarrow & A \mid X_1 \mid \textit{John} \mid \cdots \\
\textit{Variable} & \rightarrow & a \mid x \mid s \mid \cdots \\
\textit{Predicate} & \rightarrow & \textit{True} \mid \textit{False} \mid \textit{After} \mid \textit{Loves} \mid \textit{Raining} \mid \cdots \\
\textit{Function} & \rightarrow & \textit{Mother} \mid \textit{LeftLeg} \mid \cdots \\
\textsc{Operator Precedence} & : & \neg, =, \land, \lor, \Rightarrow, \Leftrightarrow
\end{array}
$$

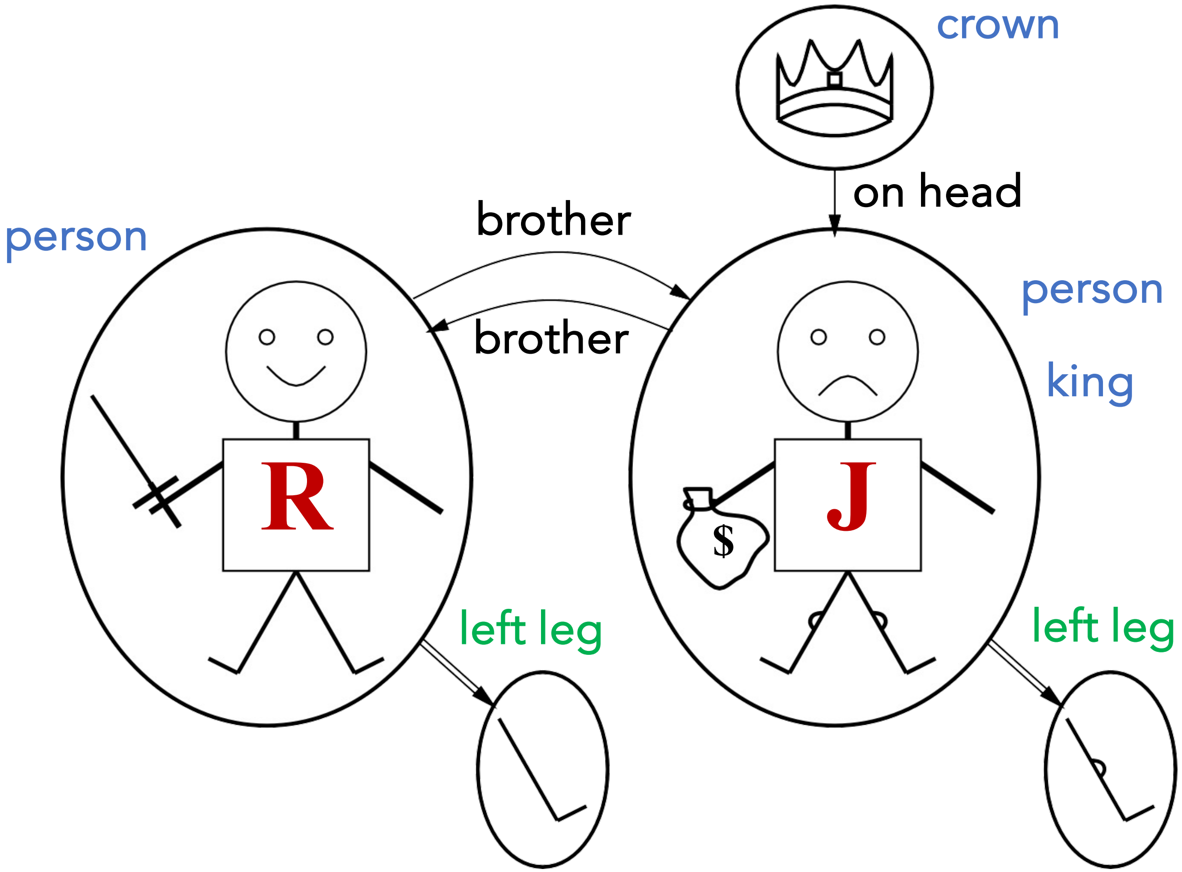
Truth in First-Order Logic

- Sentences are true with respect to a model and an interpretation
- A model contains objects (domain elements) and relations among them
- An interpretation specifies referents for
  - Constant symbols -> objects
  - Predicate symbols -> relations
  - Function symbols -> functional relations
- An atomic sentence predicate($term_1$,…, $term_n$) is true iff
  - the objects referred to by $term_1$,…, $term_n$ are in the relation referred to by predicate

Models in Practice

- In predicate logic, a model consists of
  - A domain of discourse, i.e., the set of all objects or individuals mentioned in the propositions, e.g.
    - The set of natural numbers
    - A set of individuals: Socrates, Plato, …
  - Relations between domain elements, explicitly represented as the set of tuples among which a relation holds, e.g.
    - Being greater than (binary relation): {(2,1), (3,1), …}
    - Being a prime number (unary relation): {1, 2, 3, 5, 7, 11, 13, …}
      - Unary relations are also called properties
    - Being the sum of (ternary relation): {(1, 1, 2), (1, 2, 3), …}
    - Being the father of (binary relation): {(John, Mary), …}
  - Functions mapping tuples of domain elements to a single one, e.g.
    - Plus: (1,1) -> 2, (1, 2) ->3, …
    - Father of: John -> Mary, …

Models: Example



Semantics: Interpretations

- Remember that semantics defines the truth of well-formed sentences, related to a particular model
- In predicate logic, this requires an interpretation:
  - Defining which domain elements, relations, and functions are referred to by symbols
- Examples
  - *One*, *Two*, and *Three* denote the natural numbers 1, 2, 3
  - *John* and *Mary* denote the individuals John and Mary
  - *GreaterThan* denotes the binary relation "to be greater than" (>) between numbers
  - *Father* denotes the fatherhood relation between individuals
  - *Plus* denotes the function mapping a pair of numbers to their sum

Semantics: Terms

- Terms are logical expressions denoting domain elements
- A term can be
  - Simple: a constant symbol, e.g., *One*, *Two*, *Three*
  - Complex: a function symbol applied (possibly, recursively) to other terms
    - *FatherOf(Mary)*
    - *Plus(One, Two)*
    - *Plus(One, Plus(One, One))*
- Worth noting
  - It is not necessary to assign a constant symbol to every domain element (domains can even be infinite): only elements explicitly mentioned in propositions (e.g., *Socrates*) should be assigned a constant symbol
  - A domain element can be denoted by more than one symbol

Semantics: Atomic sentences

- Atomic sentences are the simplest kind of propositions
  - A predicate symbol applied to a list of terms
- Examples
  - *GreaterThan(Two, One)*
  - *Prime(Two)*
  - *Prime(Plus(Two, Two))*
  - *Sum(One, One, Two)*
  - *Father(John, Mary)*
  - *Father(FatherOf(John), FatherOf(Mary))*

Semantics: Atomic sentences

- Definition
  - *An atomic sentence is true, in a given model and under a given interpretation, if the relation referred to by its predicate symbol holds between the objects referred to by its argument (terms)*
- Example
  - According to the above model and interpretation
  - *GreaterThan(Two, One)* is true
  - *Prime(Two)* is true
  - *Prime(Plus(Two, Two))* is false
  - *Sum(One, One, Two)* is true
  - *Father(John, Mary)* is true

Truth Example

- Consider the interpretation in which
  - Richard → Richard the Lionheart
  - John → the evil King John
  - Brother → the brotherhood relation
- Under this interpretation
  - *Brother(Richard, John)* is true as Richard the Lionheart and the evil King John are in the brotherhood relation in the model

Semantics: Complex sentences

- Complex sentences are obtained as in propositional logic, using logical connectives
- Examples
  - $Prime(Two) \land Prime(Three)$
  - $\neg Sum(One, One, Two)$
  - $GreaterThan(Two, One) \Longrightarrow (\neg GreaterThan(One, Two))$
  - $Father(John, Mary) \lor Father(Mary, John)$
- The truth value is determined as in propositional logic
  - The second sentence is false, the others are true

Semantics: Quantifiers

- Quantifiers allow one to express propositions involving collections of domain elements, without enumerating them explicitly
- Two main quantifiers are used in predicate logic:
  - Universal quantifier, e.g.:
    - *All men are mortal*
    - *All rooms neighboring the wumpus are smelly*
    - *All even numbers are not prime*
  - Existential quantifier, e.g.:
    - *Some numbers are prime*
    - *Some rooms contain pits*
    - *Some men are philosophers*
- Quantifiers require a new kind of term: variable symbols, usually denoted with lowercase letters

Semantics: Universal quantifiers

- Example
  - Let's pretend that the domain is the set of natural numbers
    - All natural numbers are greater or equal to one

$$\forall x \, \mathit{GreaterOrEqual}(x, One)$$

Semantics: Universal quantifier

- The semantics of a sentence ∀x α(x), where α(x) is a sentence containing the variable x, is
  - α(x) is true for each domain element in place of x
- Example
  - If the domain is the set of natural numbers
    - ∀x *GreaterOrEqual*(*x*, *One*) means that the following (infinite) sentences are all true
      - GreaterOrEqual(*One*, *One*)
      - GreaterOrEqual(*Two*, *One*)
      - …
      - …

Universal Quantification

- Example
  - $\forall x\ BelongsTo(x, Hogwarts) \Rightarrow Wizard(x)$

- $\forall x\ P$ is true in a model m iff P is true with x being each possible object in the model
  - Equivalent to the conjunction of instances of P

$$(BelongsTo(Dumbledore, Hogwarts) \Rightarrow Wizard(Dumbledore))$$
$$\wedge\ (BelongsTo(Piton, Hogwarts) \Rightarrow Wizard(Piton))$$
$$\wedge\ \ldots$$

A Mistake to Avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- A common mistake is
  - Using $\land$ as the main connective with $\forall$
    - $\forall x$ *BelongsTo*(*x*, *Hogwarts*) $\land$ *Wizard*(*x*) means
      - "Everyone is at Hogwarts, and everyone is a wizard"

Mistake to avoid with ∀, in general

- Let's take the proposition: *all even numbers greater than two are not prime*
- A common mistake is to represent it as follows:

$$\forall x\ Even(x) \land GreaterThan(x, Two) \land (\neg Prime(x))$$

- That sentence means
  - *all numbers are even, greater than two, and are not prime*, which is different from the original one (and is also false)
- The correct sentence can be obtained by noting that the original proposition can be restated as
  - ***for all*** *x*, ***if*** *x is even and greater than two*, ***then*** *it is not prime*, which is represented by an implication:

$$\forall x\ (Even(x) \land GreaterThan(x, Two)) \Rightarrow (\neg Prime(x))$$

- In general, propositions where "all" refers to all domain elements that satisfy some condition must be represented using an implication

Semantics: Universal quantifier

- Consider again this sentence:

$$\forall x\ (\mathit{Even}(x) \land \mathit{GreaterThan}(x, \mathit{Two})) \Rightarrow (\neg \mathit{Prime}(x))$$

- Saying it is true means that sentences like these are true:

$$(\mathit{Even}(\mathit{One}) \land \mathit{GreaterThan}(\mathit{One}, \mathit{Two})) \Rightarrow (\neg \mathit{Prime}(\mathit{One}))$$

- Note
  - the antecedent of the implication is false (the number 'one' is not even, nor is greater than the number 'two')
  - This is not contradictory, since implications with false antecedents are true by definition

Semantics: Existential quantifier

- Assume that the domain is the set of natural numbers
  - *Some numbers are prime*

$$\exists x\ Prime(x)$$

    - This is read as *there exists some x such that x is prime*
  - *Some numbers are not greater than three*, *and are even*

$$\exists x\ \neg GreaterThan(x, Three) \land \mathrm{Even}(x)$$

Existential Quantification

- Someone at Hogwarts is a wizard
  - $\exists x$ *BelongsTo*(x,Hogwarts) ∧ Wizard(x)

- $\exists x\ P$ is true in a model *m* iff *P* is true with *x* being some possible object in the model
  - Equivalent to the disjunction of instances of P

(*BelongsTo*(*Dumbledore*, *Hogwarts*) ∧ *Wizard*(*Dumbledore*))
∨ (*BelongsTo*(*Piton*, *Hogwarts*) ∧ *Wizard*(*Piton*))

Yet Another Mistake to Avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$
  - $\exists x$ *BelongsTo*(x, Hogwarts) $\Rightarrow$Wizard(x)
  - is true if there is anyone who is not at Hogwarts!

Mistake to avoid with ∃, in general

- Consider a proposition like the following: *some odd numbers are prime*
- A common mistake is to represent it using an implication:

$$\exists x\ Odd(x) \Rightarrow Prime(x)$$

- That sentence means:
  - *there exists some number such that, if it is odd, then it is prime*
  - The latter proposition is weaker than the original since it is true (by definition of ⇒) also if there were no odd numbers (i.e., if the antecedent Odd(x) is false for all domain elements)
- The correct sentence can be obtained by noting that the original proposition can be restated as:
  - ***there exists*** *some x such that x is odd* ***and*** *x is prime*

$$\exists x\ \mathrm{Odd}(x) \land \mathrm{Prime}(x)$$

- In general, propositions introduced by "some" must be represented using a conjunction

Semantics: Nested quantifiers

- A sentence can contain more than one quantified variable
- If the quantifier is the same for all variables, e.g.:

$$\forall x(\forall y(\forall z \ldots \alpha[x,y,z,\ldots]\ldots))$$

then the sentence can be rewritten more concisely as:

$$\forall x,y,z \ldots \alpha[x,y,z,\ldots]$$

- For instance, in the domain of natural numbers, the sentence
  - *If a number is greater than another number, then also the successor of the former is greater than the latter*

can be written (using the function Successor) as:

$$\forall x,y\ GreaterThan(x,y) \Rightarrow GreaterThan(Successor(x),y)$$

Semantics: Connections Between Quantifiers

- The quantifiers ∀ and ∃ are related by negation, just as in natural language
- For example, to say that *every natural number is greater than or equal to zero* is the same as saying that *there does not exist some natural number which is not greater than or equal to zero*
- The two propositions can be translated into the following sentences, whose domain is assumed to be the set of natural numbers:

$$\forall x \; GreaterOrEqual(x, Zero)$$

$$\neg(\exists x \; \neg GreaterOrEqual(x, Zero))$$

Semantics: Connections Between Quantifiers

- In general, since ∀ is a conjunction over all domain elements and ∃ is a disjunction, they obey De Morgan's rules
  - shown below on the left, unquantified sentences

$\neg(P \lor Q) \equiv \neg P \land \neg Q$	$\neg\exists x\, P \equiv \forall x\, \neg P$
$\neg(P \land Q) \equiv \neg P \lor \neg Q$	$\neg\forall x\, P \equiv \exists x\, \neg P$
$P \land Q \equiv \neg(\neg P \lor \neg Q)$	$\forall x\, P \equiv \neg\exists x\, \neg P$
$P \lor Q \equiv \neg(\neg P \land \neg Q)$	$\exists x\, P \equiv \neg\forall x\, \neg P$

Propositional Logic vs First-Order Logic

- Propositional Logic
  - Propositional symbols

MinervaGryffindor

MinervaHufflepuff

MinervaRavenclaw

MinervaSlytherin

...

First-Order Logic

Constant Symbol	Predicate Symbol
Minerva	*Person*
Pomona	*House*
Horace	*BelongsTo*
Gilderoy	
Gryffindor	
Hufflepuff	
Ravenclaw	
Slytherin	

First-Order Logic

$Person(Minerva)$ Minerva is a person.

$House(Gryffindor)$ Gryffindor is a house.

$\neg House(Minerva)$ Minerva is not a house.

$BelongsTo(Minerva, Gryffindor)$

Minerva belongs to Gryffindor.

Universal Quantification

$$\forall x.\ BelongsTo(x, Gryffindor) \rightarrow \neg BelongsTo(x, Hufflepuff)$$

For all objects x, if x belongs to Gryffindor, then x does not belong to Hufflepuff.

Anyone in Gryffindor is not in Hufflepuff.

Existential Quantification

$$\exists x.\ House(x) \wedge BelongsTo(Minerva, x)$$

There exists an object x such that
x is a house and Minerva belongs to x.

Minerva belongs to a house.

Existential Quantification

$$\forall x.\ Person(x) \rightarrow (\exists y.\ House(y) \wedge BelongsTo(x, y))$$

For all objects x, if x is a person, then there exists an object y such that y is a house and x belongs to y.

Every person belongs to a house.

Exercises

- Represent the following propositions using sentences in predicate logic, including the definition of the domain
  1. All men are mortal; Socrates is a man; Socrates is mortal
  2. All rooms neighboring a pit are breezy (Wumpus game)
  3. Peano-Russell's axioms of arithmetic that define natural numbers (nonnegative integers)

     P1 zero is a natural number

     P2 the successor of any natural number is a natural number

     P3 zero is not successor of any natural number

     P4 no two natural numbers have the same successor

     P5 any property which belongs to zero, and to the successor of every natural number which has the property, belongs to all natural numbers

Solutions

1. Model and symbols
   - Domain: any set including all men
   - Constant symbols: *Socrates*
   - Predicate symbols: *Man* and *Mortal*, unary predicates, e.g., *Man(Socrates)* means that Socrates is a man

- The sentences are:
  - $\forall x\ Man(x) \Rightarrow Mortal(x)$
  - *Man(Socrates)*
  - *Mortal(Socrates)*

All men are mortal; Socrates is a man; Socrates is mortal

Solutions

2. A possible choice of model and symbols
   - Domain: row and column coordinates
   - Constant symbols: 1, 2, 3, 4
   - Predicate symbols:
     - *Pit*, binary predicate; e.g., P(1,2) means that there is a pit in room (1,2)
     - *Adjacent*, predicate with four terms; e.g., Adjacent(1,1,1,2) means that room (1,1) is adjacent to room (1,2)
     - *Breezy*, binary predicate; e.g., Breezy(2,2) means that there is a breeze in room (2,2)

Solutions

- One possible sentence is the following:
  - $\forall x,y\ (Breezy(x,y) \Leftrightarrow (\exists p,q\ Adjacent(x,y,p,q) \wedge Pit(p,q)))$
  - Note that the sentence above also expresses the fact that rooms with no adjacent pits are not breezy
- Another possible sentence:
  - $\forall x, y\ (Pit(x, y) \Rightarrow (\forall p, q\ Adjacent(x, y, p, q) \Rightarrow Breezy(p, q)))$
  - In this case, there is no logical equivalence: if all the rooms adjacent to a given one are breezy, the latter does not necessarily contain a pit

All rooms neighboring a pit are breezy (Wumpus game)

Solutions

3. A possible choice of models and symbols
   - domain: any set including all natural numbers (e.g., the set of real numbers)
   - constant symbols: Z, denoting the number zero
   - predicate symbols:
     - N, unary predicate denoting the fact of being a natural number; e.g., N(Z) means that zero is a natural number
     - Eq, binary predicate denoting equality; e.g., Eq(Z,Z) means that zero equals zero
     - P denoting any given property
   - function symbols: S, mapping a natural number to its successor; e.g., S(Z) denotes one, S(S(Z)) denotes two

Solutions

P1 $N(Z)$

P2 $\forall x\ N(x) \Rightarrow N(S(x))$

P3 $\neg(\exists x\ Eq(Z,S(x)))$

P4 $\forall x, y\ Eq(S(x), S(y)) \Rightarrow Eq(x, y)$

P5 $(P(Z) \wedge \forall x((N(x) \wedge P(x)) \Rightarrow P(S(x)))) \Rightarrow (\forall x\ (N(x) \Rightarrow P(x)))$

- "If the property P holds for zero and for any natural number such that if the property holds for that number, it also holds for its successor, then the property holds for all natural numbers"

P1 zero is a natural number
P2 the successor of any natural number is a natural number
P3 zero is not successor of any natural number
P4 no two natural numbers have the same successor
P5 any property which belongs to zero, and to the successor of every natural number which has the property, belongs to all natural numbers

Equality

- The equality symbol is used to signify that two terms refer to the same object
  - Father (John) = Henry
    - The object referred to by Father (John) and the object referred to by Henry are the same
- The equality symbol can be used to state the facts about a given function and can also be used with negation
  - Richard hs at least two brothers
    - $\exists x,y$ Brother $(x, Richard) \land$ Brother $(y, Richard) \land \neg(x = y)$
- Observation
  - The sentence $\exists x,y$ Brother $(x, Richard) \land$ Brother $(y, Richard)$ would not have the intended meaning
    - It is true in the model where Richard has only one brother
      - $x \neq y$ could be used as an abbreviation for $\neg(x = y)$

Assertion and queries

- To add a sentence in a KB, the TELL function can be used
  - Assertion
- To assert that *John is a King*, *Richard is a person* and *all Kings are persons*
  - *TELL(KB, King(John))*
  - *TELL(KB,Person(King))*
  - *TELL(KB, $\forall x$ King(x) $\Rightarrow$ Person(x))*
- For queries, the ASK interface can be used
  - ASK(KB, King(John))
  - Any query that is logically entailed by the KB should be answered affirmatively
  - ASK(KB, $\exists$ x Person(x))
    - If we want to know what value x makes the sentence true, we get two possible answers
      - {x/John} and {x/Richard}
        - Substitution or binding list

Examples

- Brothers are siblings
  - $\forall x, y \ Brother(x, y) \Rightarrow Sibling(x, y)$
- "Sibling" is symmetric
  - $\forall x, y \ Sibling(x, y) \Leftrightarrow Sibling(y, x)$
- One's mother is one's female parent
  - $\forall x, y \ Mother(x, y) \Leftrightarrow (Female(x) \land Parent(x, y))$
- A first cousin is a child of a parent's sibling
  - $\forall x, y \ FirstCousin(x, y) \Leftrightarrow \exists p, ps \ Parent(p, x) \land Sibling(ps, p) \land Parent(ps, y)$