Course of
"Automatic Control Systems"
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Nyquist plots

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

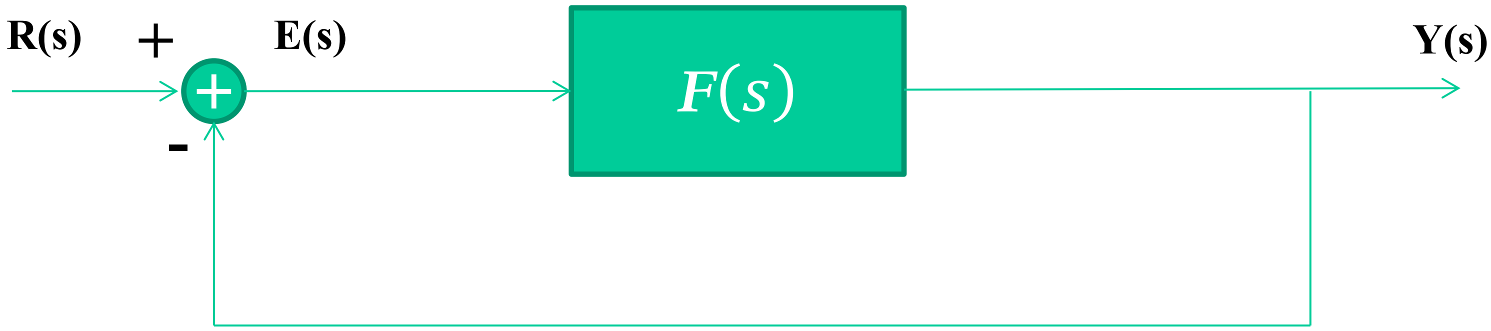
Università degli Studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code: **mfs9zfr**

Stability of the closed loop system

- Let us consider the $R(s) \rightarrow Y(s)$ closed loop system



- Assume that the hidden modes of the open loop function $F(s) = K(s)G(s)$ are asymptotically stable
- The stability of the closed loop system depends on the poles of the transfer function

$$\mathrm{T}(s) = \frac{F(s)}{1+F(s)}$$

Stability of the closed loop system

- Indicate with $N_F(s)$ and $D_F(s)$ the numerator and the denominator of the open loop function

$$\mathbf{F}(s) = \frac{N_F(s)}{D_F(s)} = \frac{N_K(s)N_G(s)}{D_K(s)D_G(s)},$$

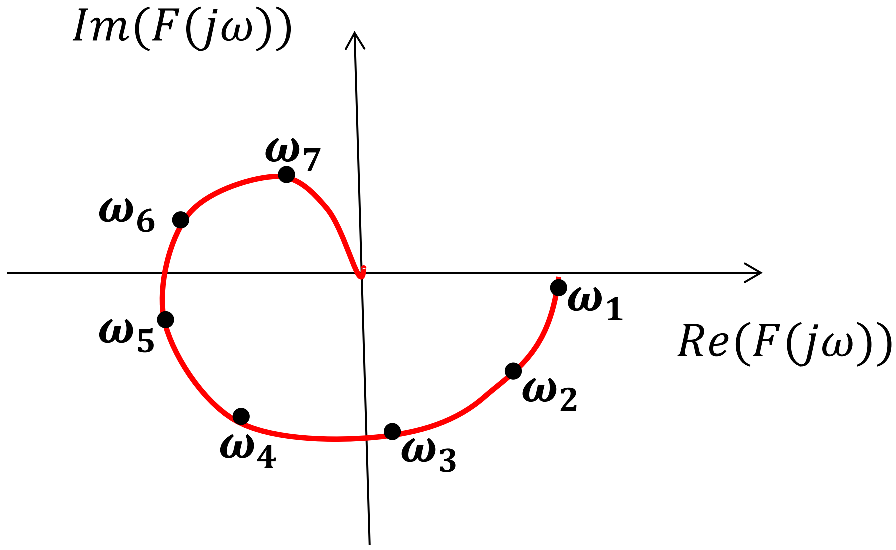
- The closed loop function can be written as

$$\mathbf{T}(s) = \frac{\frac{N_F(s)}{D_F(s)}}{1 + \frac{N_F(s)}{D_F(s)}} = \frac{N_F(s)}{D_F(s)+N_F(s)}.$$

- The denominator of $T(s)$ is given by the sum of $N_F(s)$ and $D_F(s)$; therefore, the design problem of a controller able to guarantee the stability of the closed loop system is to be very demanding.

- By means of the Nyquist plots and the ***Nyquist criteria***, we are going ***to determine the stability of the closed loop*** system ***from the open loop*** system features

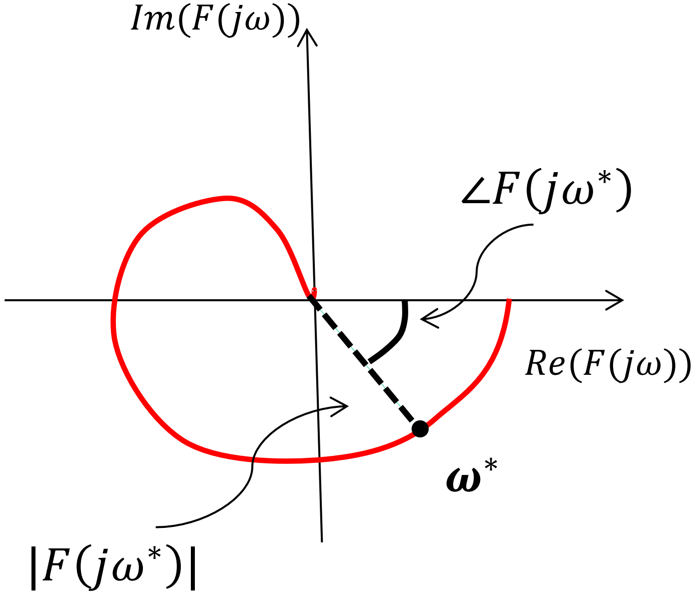
- ***The Nyquist plots are polar*** **diagrams of** the transfer function $\boldsymbol{F(s)|_{s=j\omega}}$



$F(s)$ is represented in the polar plane as a function of $j\omega$ assuming ω moving from 0 to $+\infty$

- They are an alternative solution to the Bode diagrams for the representation of the transfer functions.

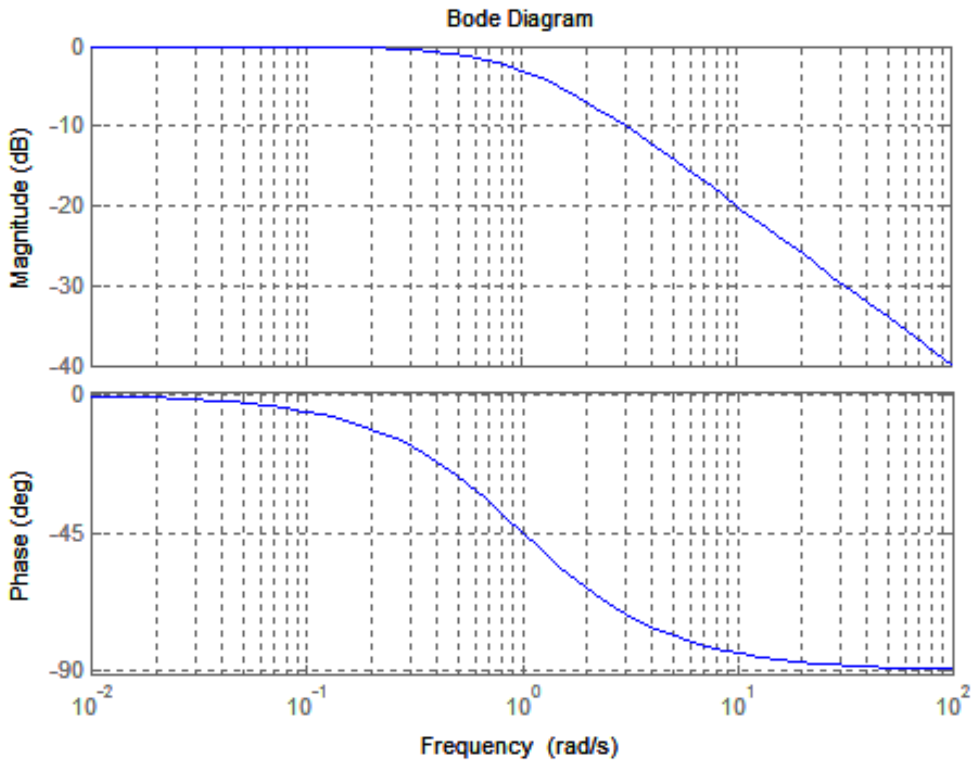
- In a Nyquist plot ***magnitude and phase of $F(j\omega)$ are represented by a curve parametrized in ω.***

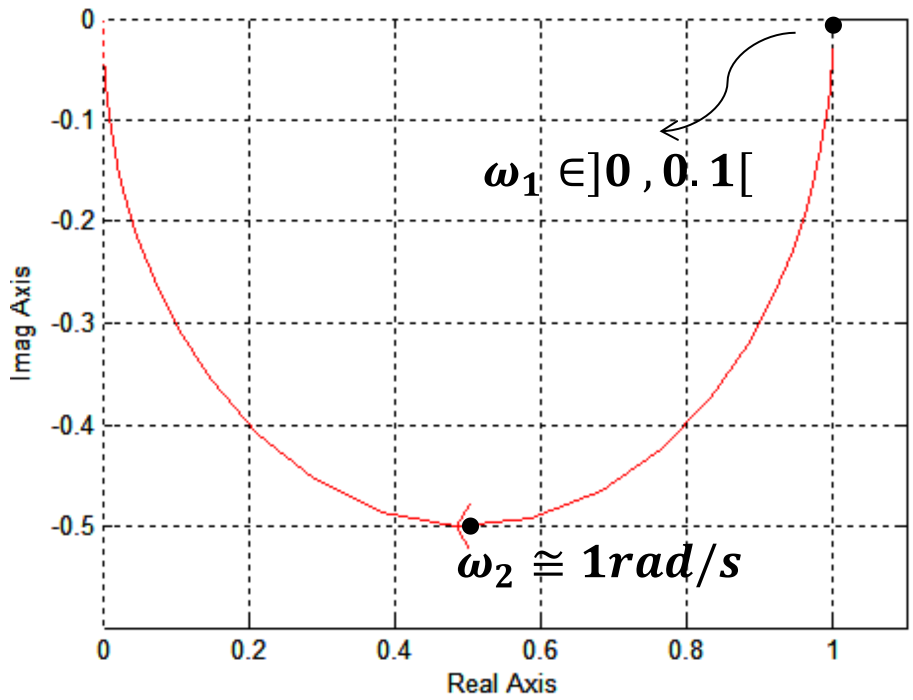


Nyquist plots: example 1

- The Nyquist plots can be obtained from the magnitude and phase Bode plots of

$$F(s) = \frac{1}{1+s}$$



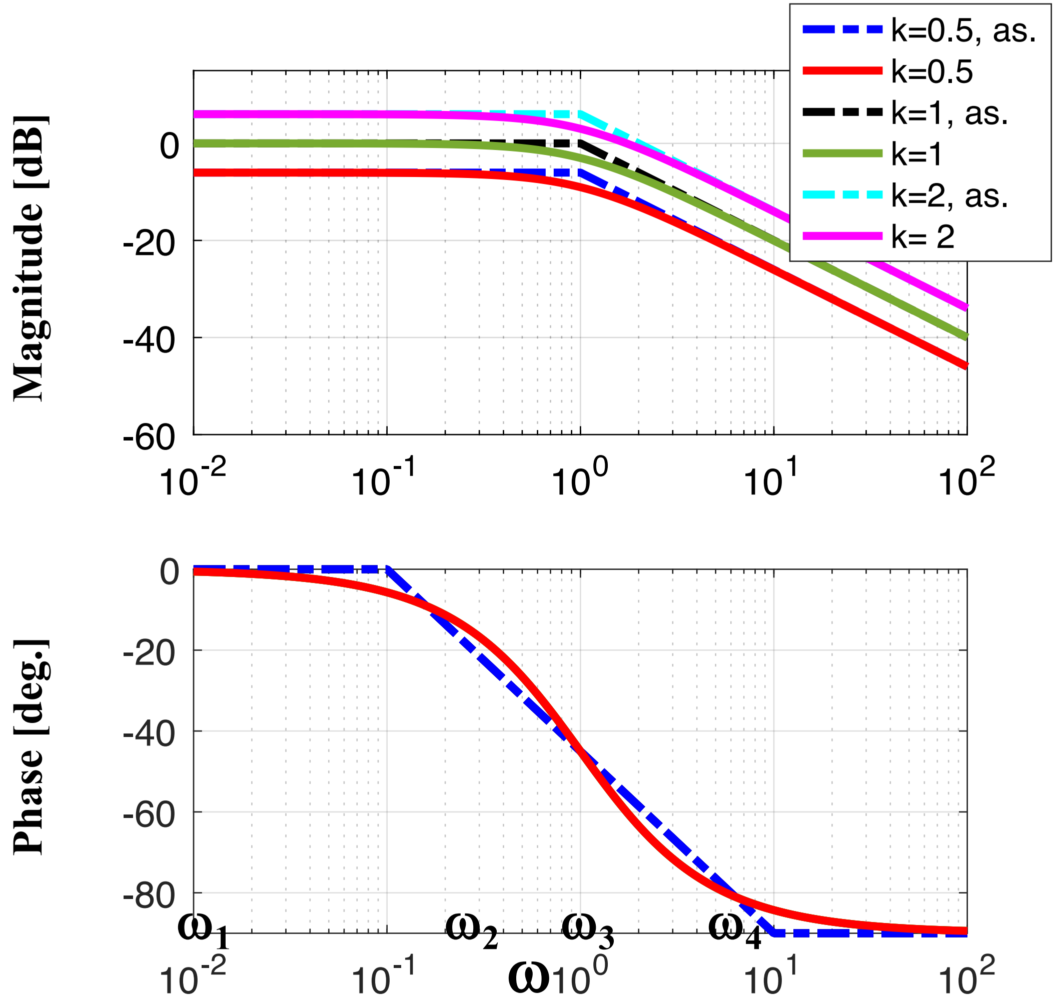


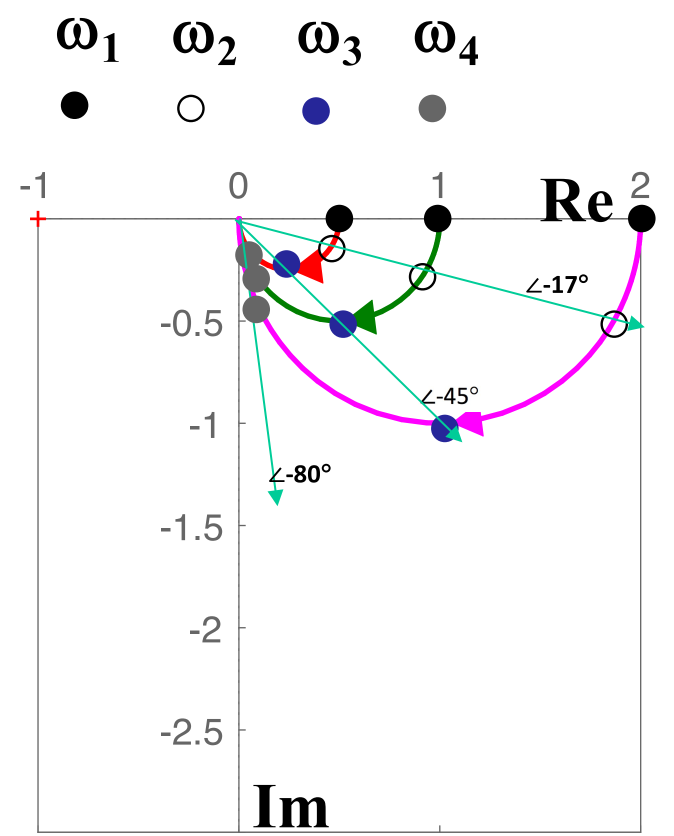
- Note that a single point on the Nyquist plots can also indicate the value of $F(j\omega)$ in a finite interval of ω.

Nyquist plots: first-order open loop system

$F(s) = \frac{k}{1+s\tau}$, $\tau = 1$ s, $F(j\omega) = F(s)|_{s=j\omega}$. $|F(j\omega)| = k\left|\frac{1}{1+j\omega\tau}\frac{1-j\omega\tau}{1-j\omega\tau}\right| = \frac{k\sqrt{1+(\omega\tau)^2}}{1+(\omega\tau)^2} = \frac{k}{\sqrt{1+(\omega\tau)^2}}$;

$|F(j\omega)|_{\mathrm{dB}} = 20\log_{10}|F(j\omega)|$; $\arg F(j\omega) = -\arg(1+j\omega\tau) = -\tan^{-1}(\omega\tau)$.

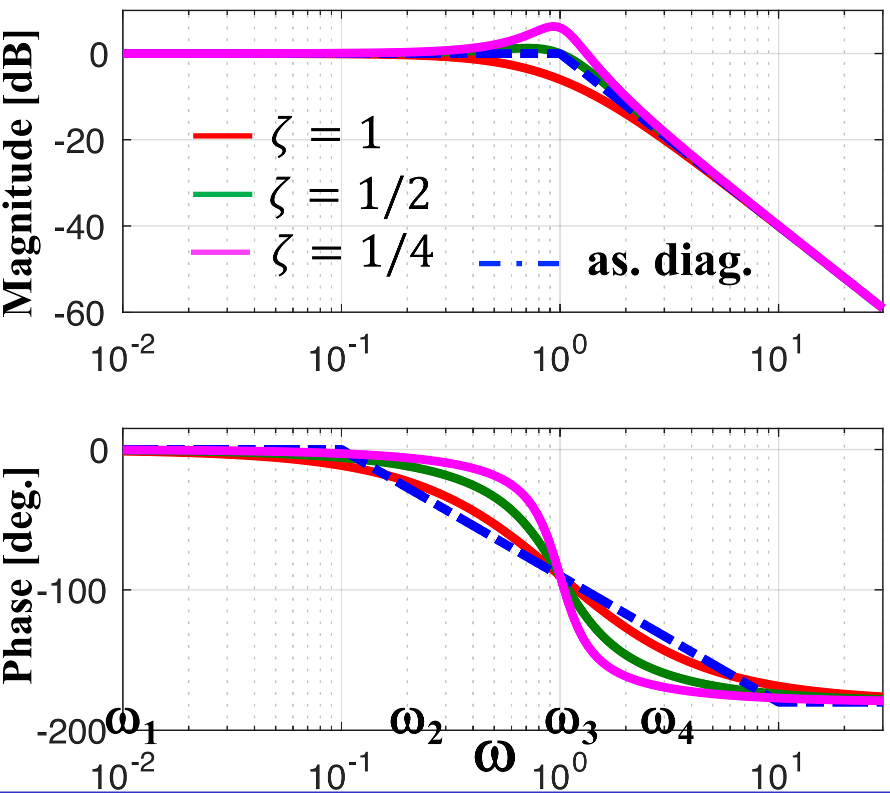


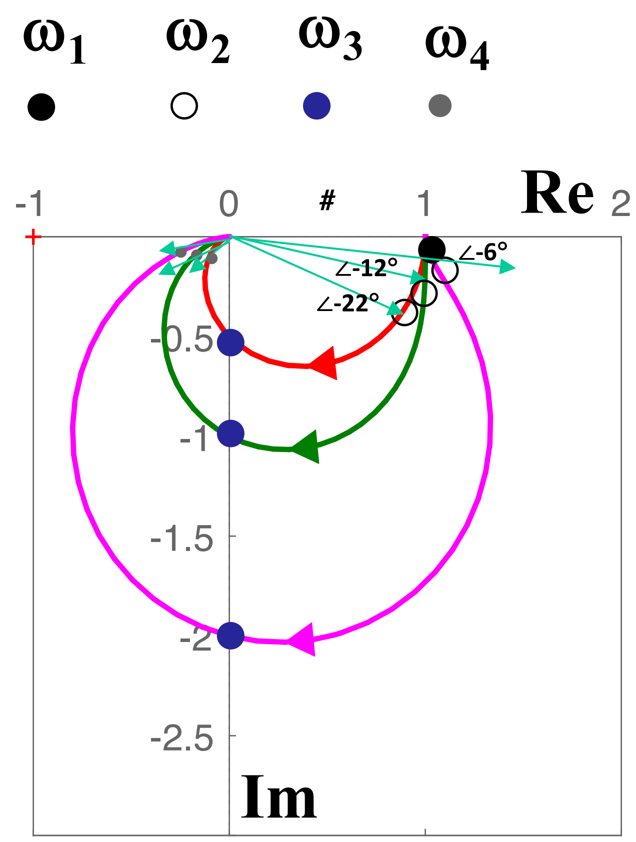


Nyquist plots: second-order open loop system

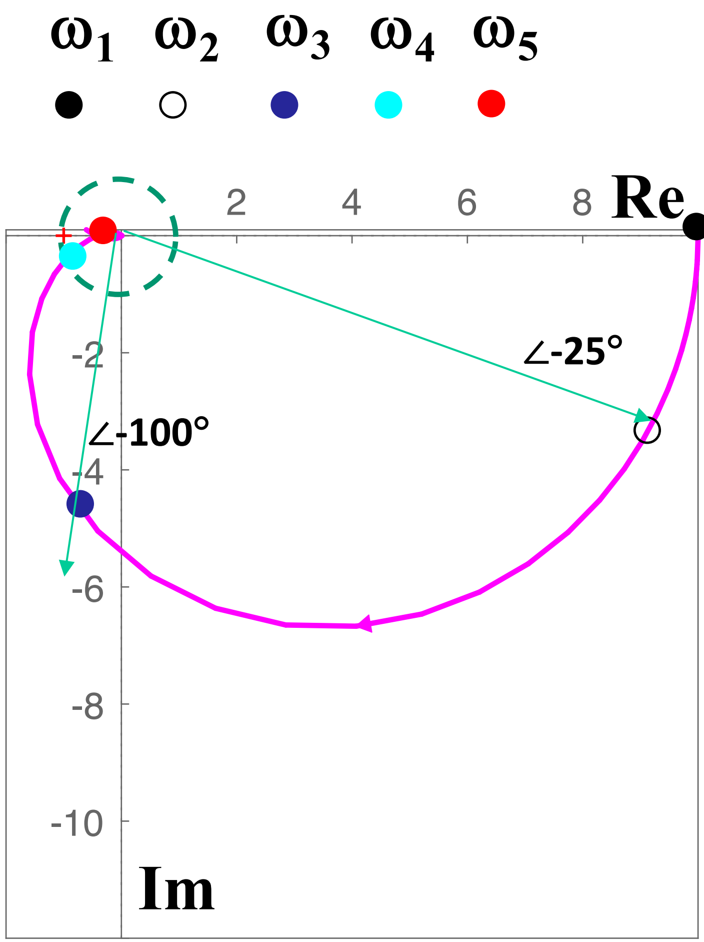
$$F(s)=\frac{1}{\frac{s^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}s+1},\quad \omega_n = 1 \text{ rad/s.} \qquad |F(j\omega)|=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta}{\omega_n}\right)^2}},\qquad \arg F(j\omega)=-\tan^{-1}\frac{\frac{2\zeta}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}.$$

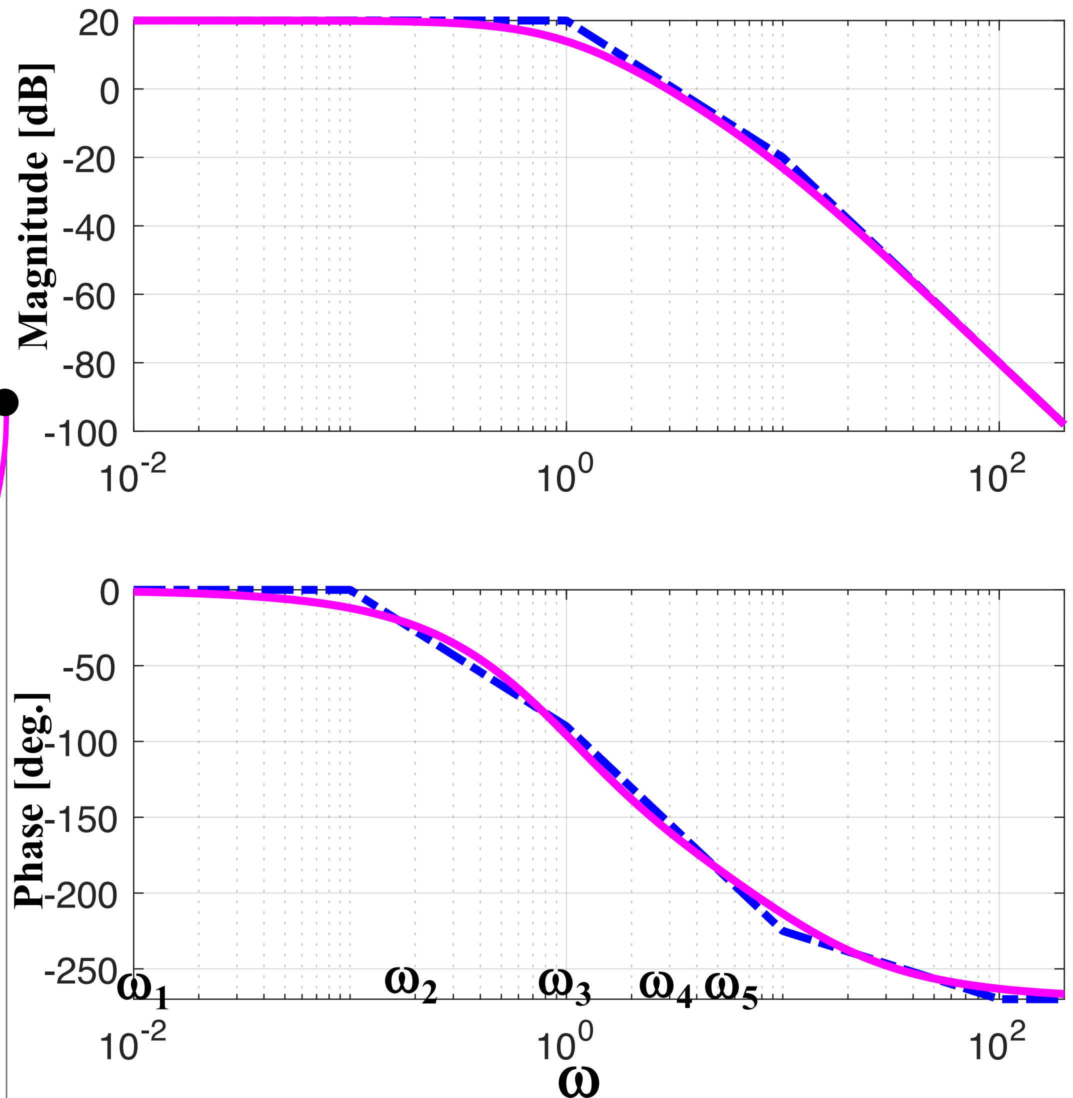
$$|F(j\omega)|_{\text{dB}} = -20\log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta}{\omega_n}\right)^2}$$





$$F(s) = \frac{10}{(s^2 + 2s + 1)\left(\frac{s}{10} + 1\right)}$$



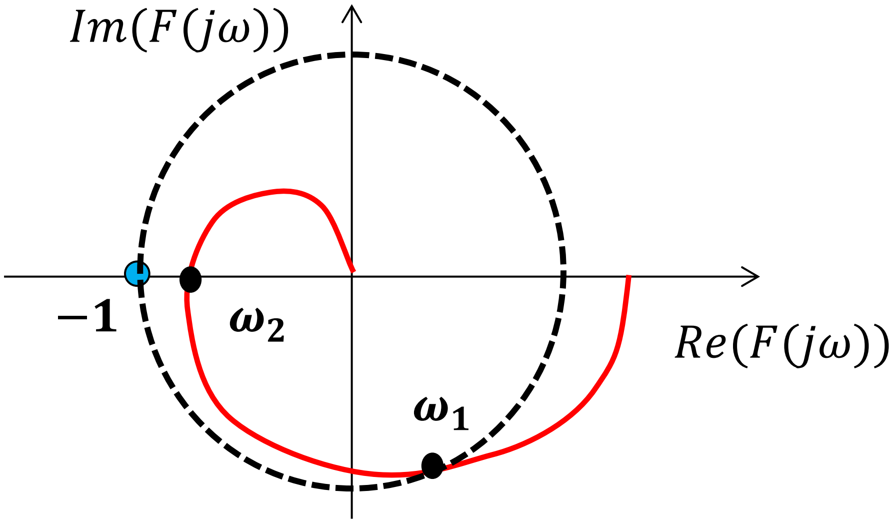


- A precise representation of the Nyquist plots from magnitude and phase Bode plots isn't an easy job.

- However, if we focus on the closed loop stability performance, only a limited set of points on the Nyquist point need to be traced precisely:

  1. ***Intersection of the diagram with the unit circle***
  2. ***Intersection of the diagram with the negative real axis***.

  Indeed, it is of interest to verify if the diagram intersects, encircles the ***Critical point*** $-\mathbf{1}+\boldsymbol{j}\mathbf{0}$



Phase variation

- For the analysis of closed loop system an important parameter to be considered is the ***Phase Variation***

$$\underset{-\infty \;\; \boldsymbol{\omega} \;\; \infty}{\Delta \angle F(j\omega)}$$

defined as the phase variation of $F(j\omega)$ when ω moves from $-\infty$ to ∞ counted positive if counterclockwise.

- In order to evaluate the phase variation, we also need to plot $F(j\omega)$ when ω moves from $-\infty$ to 0.

- For polynomial functions

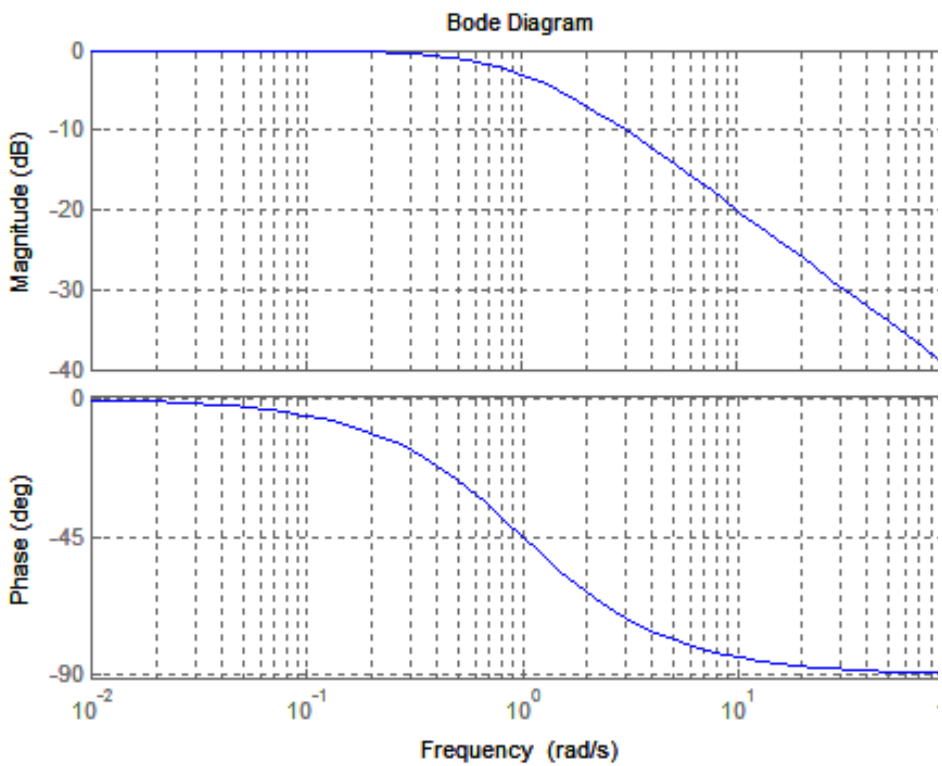
$$Re\big(F(-j\omega)\big) = Re\big(F(j\omega)\big) \qquad \textbf{\textit{Pair function}}$$

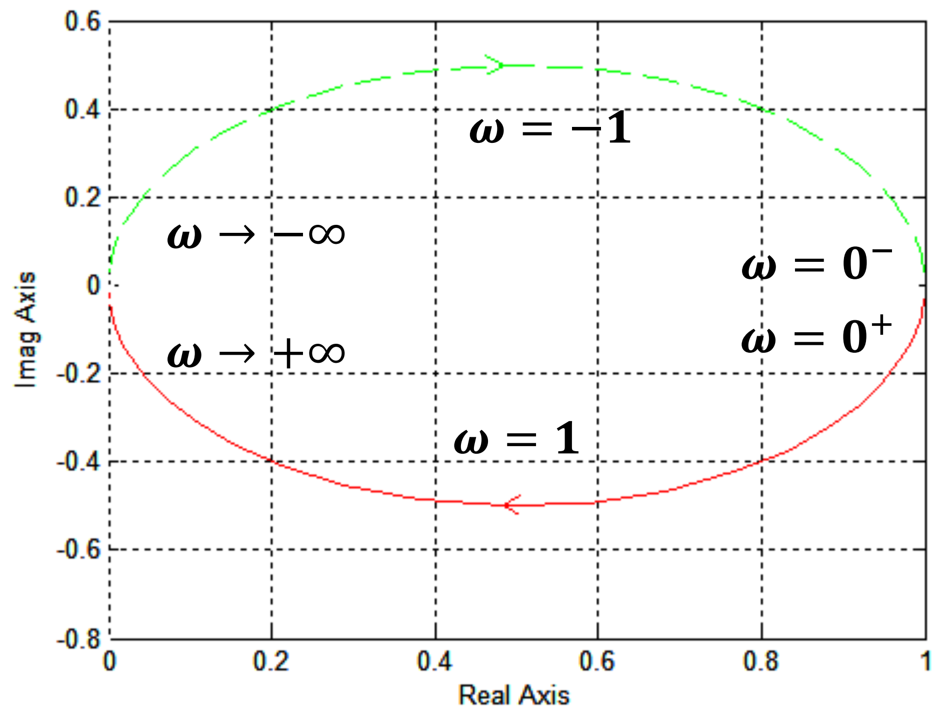
$$Im\big(F(-j\omega)\big) = -Im\big(F(j\omega)\big) \qquad \textbf{\textit{Odd function}}$$

- Hence, the Nyquist plots of $F(j\omega)$ for negative and positive angular frequencies are symmetric wrt the real axis.

Nyquist plot and phase variation: example

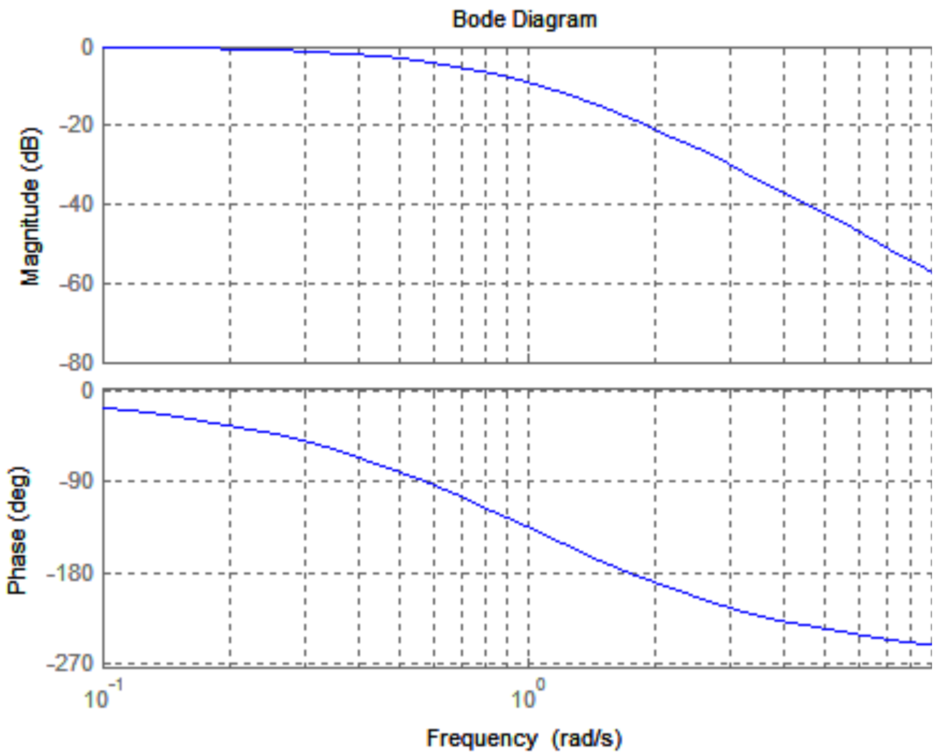
- Let us consider again the transfer function $F(s) = \dfrac{1}{1+s}$

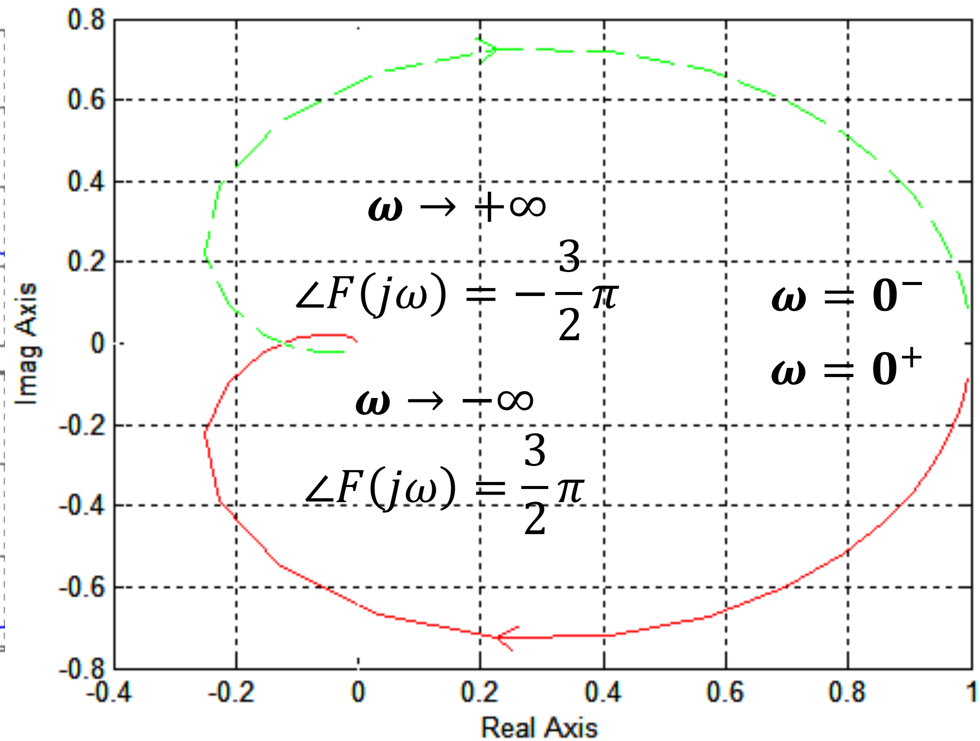




$$\underset{-\infty\ \ \omega\ \ \infty}{\Delta\angle F(j\omega)} = -\pi$$

- Let us consider the transfer function $F(s) = \frac{1}{(1+s)^3}$





$$\Delta\angle F(j\omega) = -3\pi$$
$$-\infty \;\; \boldsymbol{\omega} \;\; \infty$$

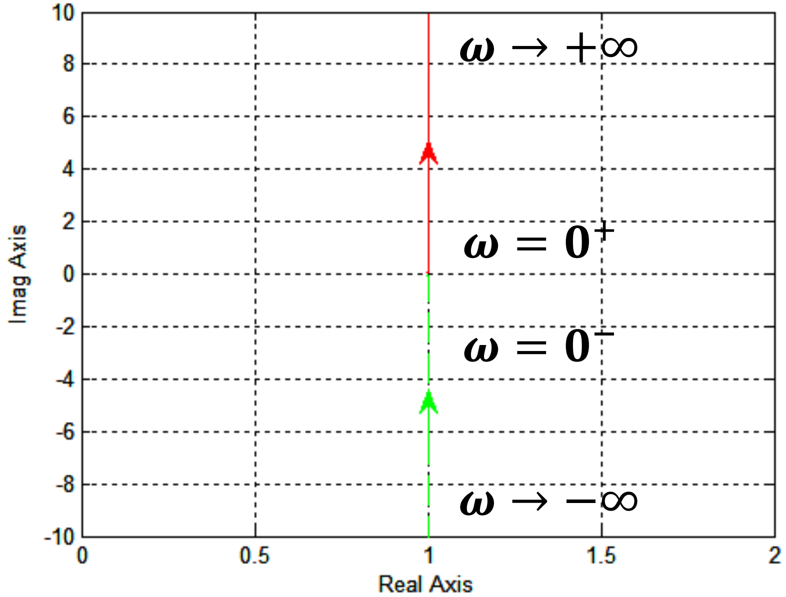
Formula for the phase variation

- In the following we propose a ***formula for the phase variation*** of a transfer function $F(j\omega)$ as a function of the number and sign of $F(j\omega)$ poles and zeros

- We will first evaluate the phase variation due to real no null poles and zeros.

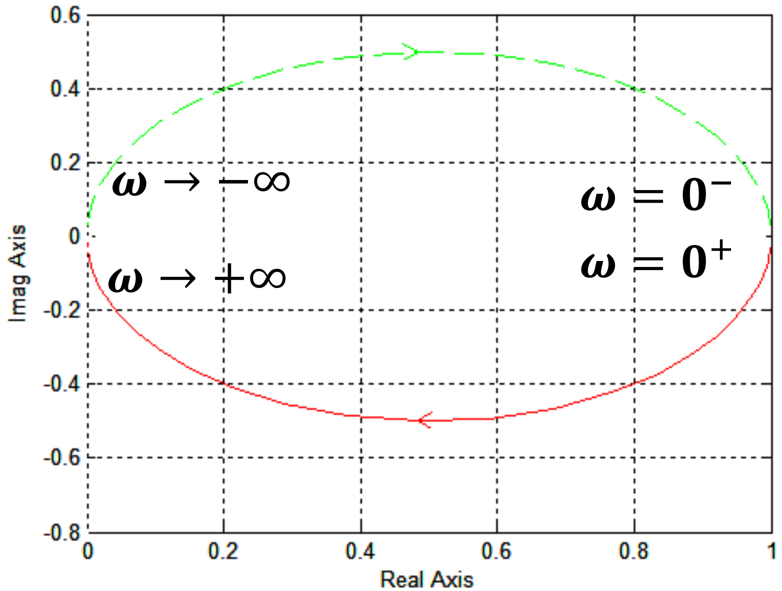
- Then, we will extend the evaluation to the case of complex poles and zeros having a null real part.

Phase variation for negative real poles and zeros



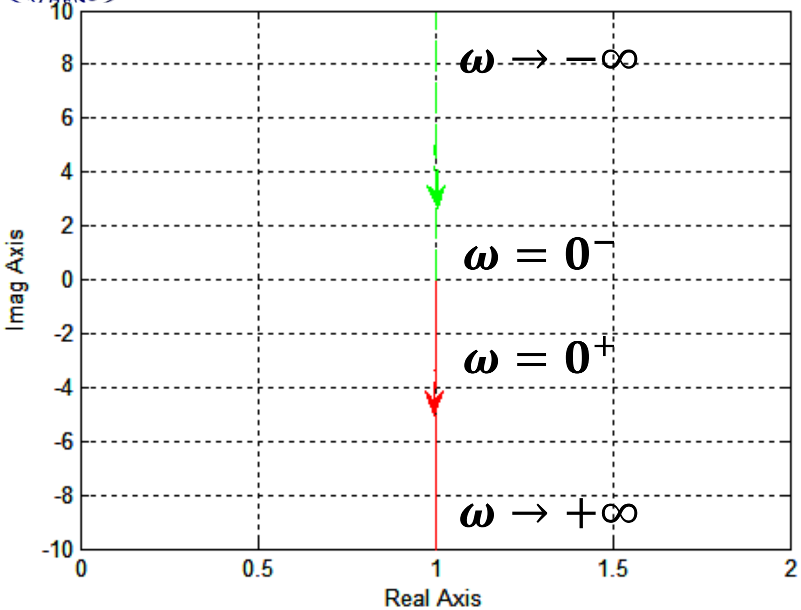
Negative real zero ($\tau > 0$)

$$F(s) = 1 + \tau s \quad \rightarrow \quad \underset{-\infty\ \ \omega\ \ \infty}{\Delta\angle F(j\omega)} = \pi$$



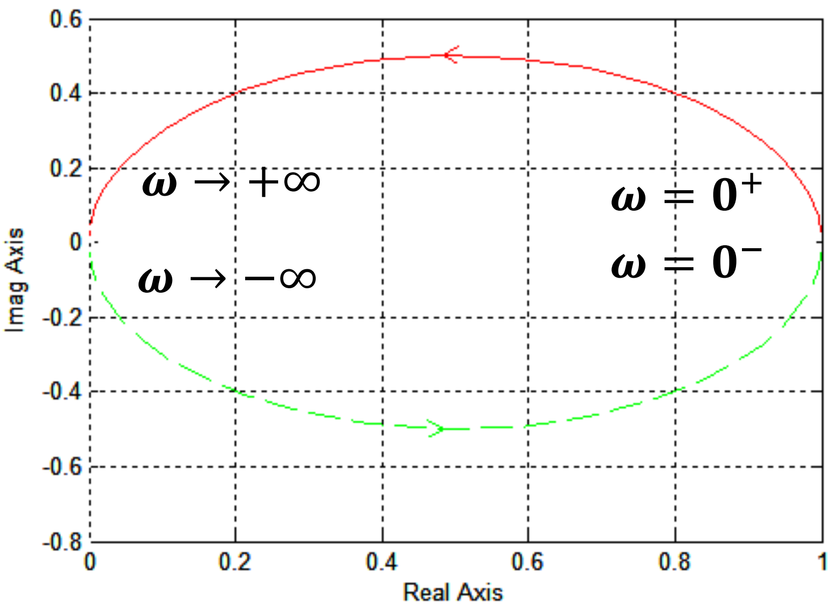
Negative real pole ($\tau > 0$)

$$F(s) = \frac{1}{1+\tau s} \quad \rightarrow \quad \underset{-\infty\ \ \omega\ \ \infty}{\Delta\angle F(j\omega)} = -\pi$$



Positive real zero ($\tau < 0$)

$$F(s) = 1 + \tau s \quad \rightarrow \quad \underset{-\infty\ \omega\ \infty}{\Delta}\angle F(j\omega) = -\pi$$



Positive real pole ($\tau < 0$)

$$F(s) = \frac{1}{1+\tau s} \quad \rightarrow \quad \underset{-\infty\ \omega\ \infty}{\Delta}\angle F(j\omega) = \pi$$

Phase variation for complex poles with $\zeta \neq 0$

Negative complex poles ($\zeta > 0$)

$$F(s) = \frac{1}{1+\frac{2\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}} \quad \rightarrow \quad \underset{-\infty\ \boldsymbol{\omega}\ \infty}{\Delta}\angle F(j\omega) = -2\pi$$

Positive complex poles ($\zeta < 0$)

$$F(s) = \frac{1}{1+\frac{2\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}} \quad \rightarrow \quad \underset{-\infty\ \boldsymbol{\omega}\ \infty}{\Delta}\angle F(j\omega) = 2\pi$$

Negative complex zeros ($\zeta > 0$)

$$F(s) = 1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2} \quad \rightarrow \quad \underset{-\infty\ \boldsymbol{\omega}\ \infty}{\Delta}\angle F(j\omega) = 2\pi$$

Negative complex zeros ($\zeta < 0$)

$$F(s) = 1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2} \quad \rightarrow \quad \underset{-\infty\ \boldsymbol{\omega}\ \infty}{\Delta}\angle F(j\omega) = -2\pi$$

Phase variation formula

- The previous results allows to relate the phase variation to the number and sign of poles/zeros of the transfer function.

Given a transfer function $F(j\omega)$, said:

- ***n the total number of poles***
- ***m the total number of zeros***
- ***n_p (n_n) the number of poles with positive (negative) real part***
- ***m_p (m_n) the total number of zeros with positive (negative) real part***

$$\boldsymbol{n = n_n + n_p} \qquad \boldsymbol{m = m_n + m_p}$$

$$\underset{-\infty \;\; \boldsymbol{\omega} \;\; \infty}{\Delta\angle F(j\omega)} = \pi(m_n - n_n) - \pi(m_p - n_p)$$

$$= \pi(m - n) - 2\pi(m_p - n_p)$$