Artificial Intelligence

Constraint Satisfaction Problems

LESSON 9

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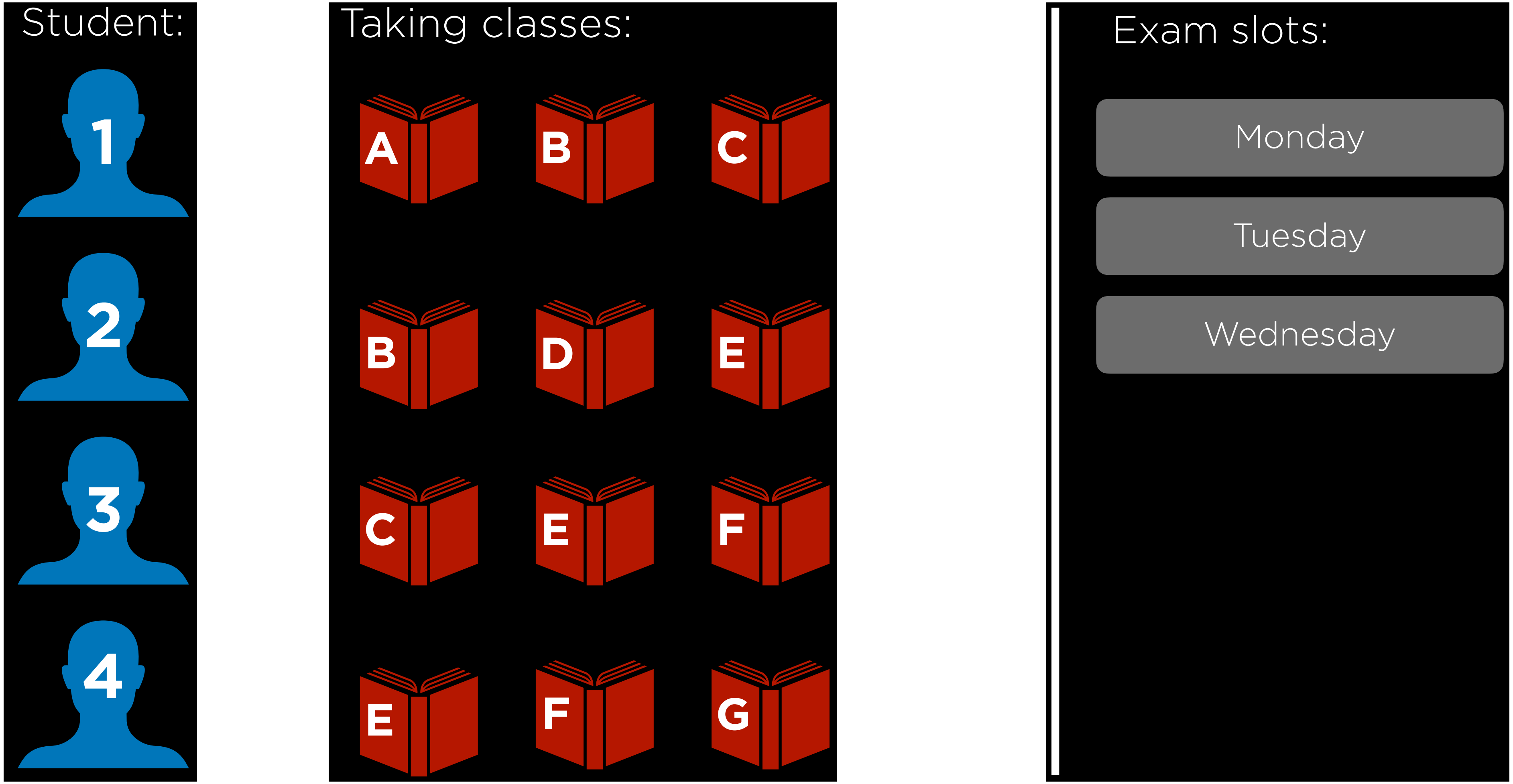
M.Sc. In ''Machine Learning e Big Data'' - University Parthenope of Naples

Constraint Satisfaction Problem (CSP)

- The basic idea of a CSP is
  - having some number of variables that need to take on some values, figure out what values each of those variables should take on
  - However, the variables are subject to particular constraints that are going to limit what values those variables can actually take on
- Let's take a look at a real-world example …

CSP Example: Exam Scheduling



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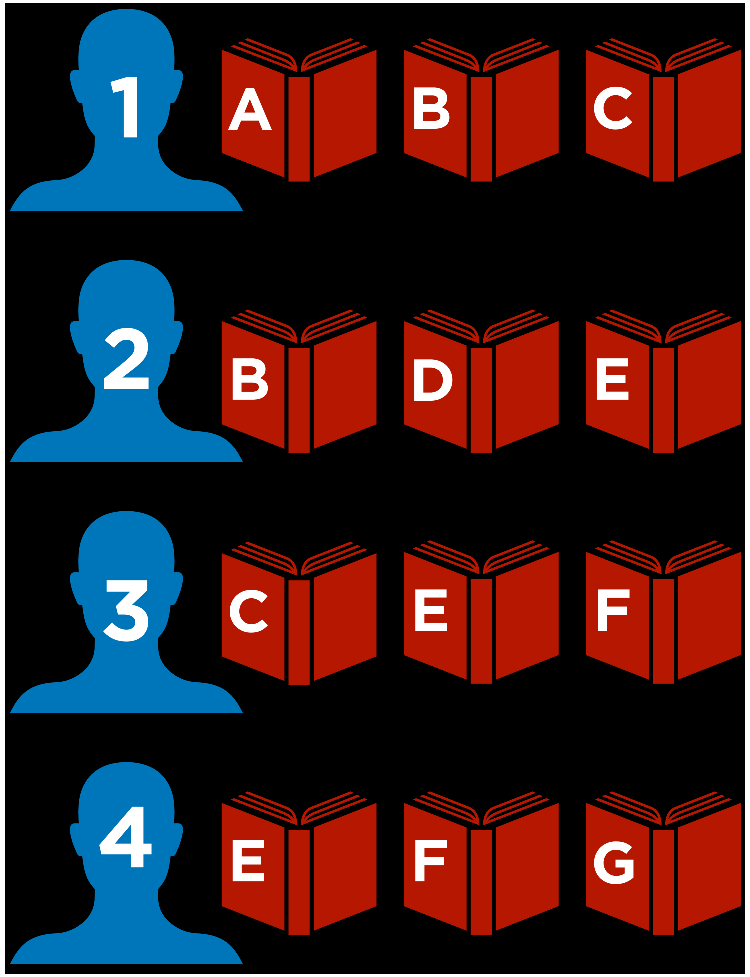

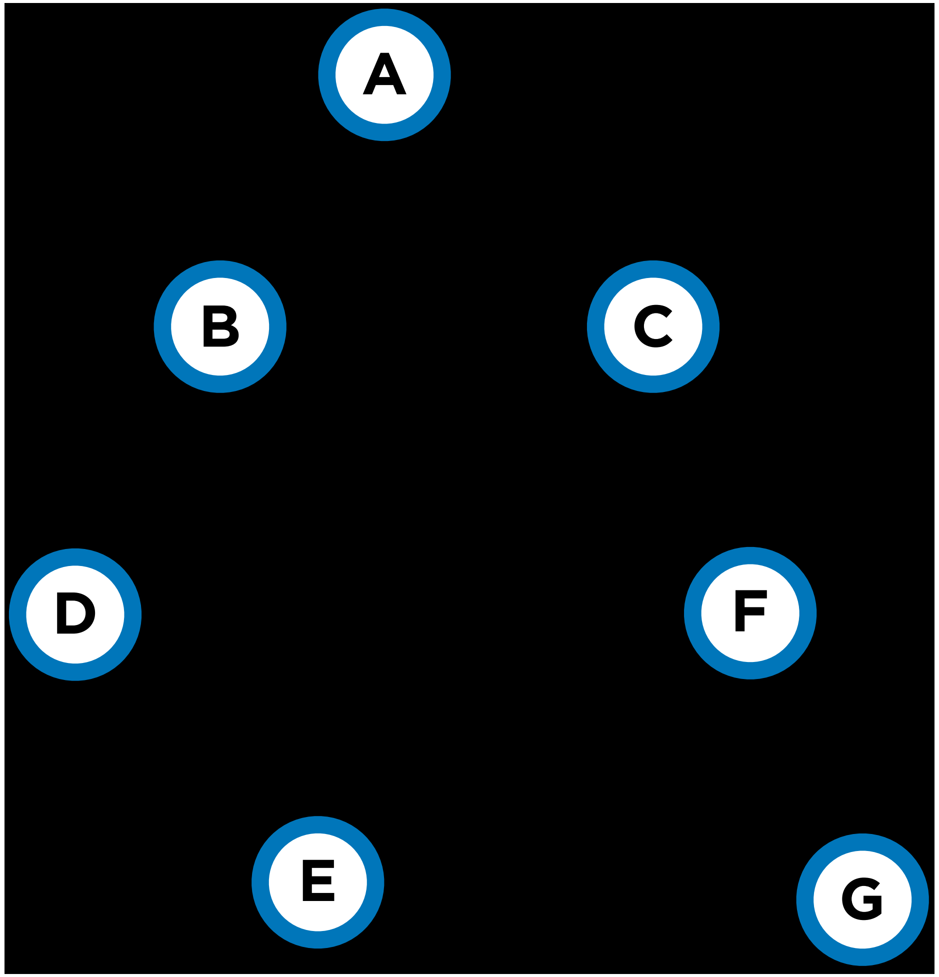

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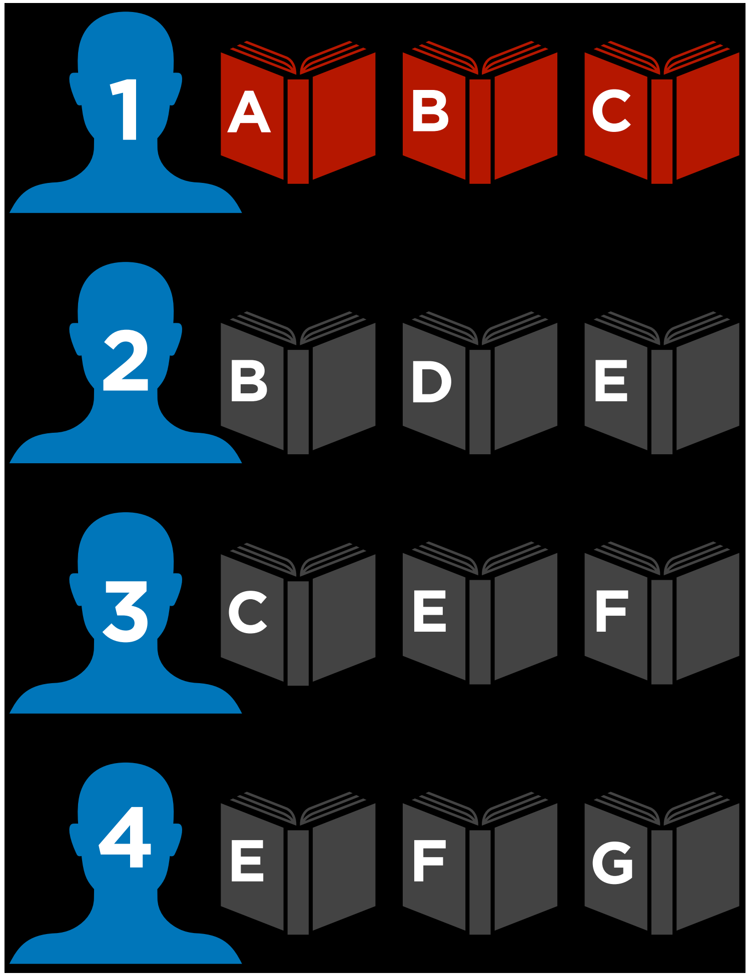

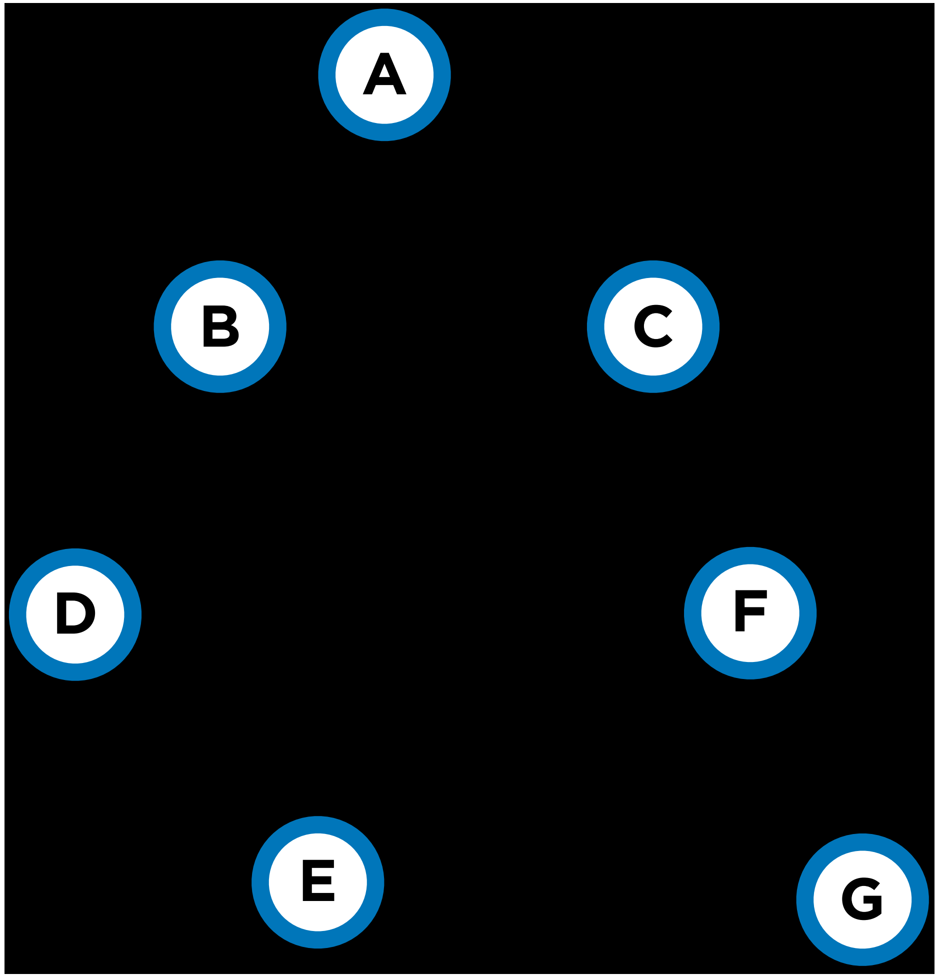

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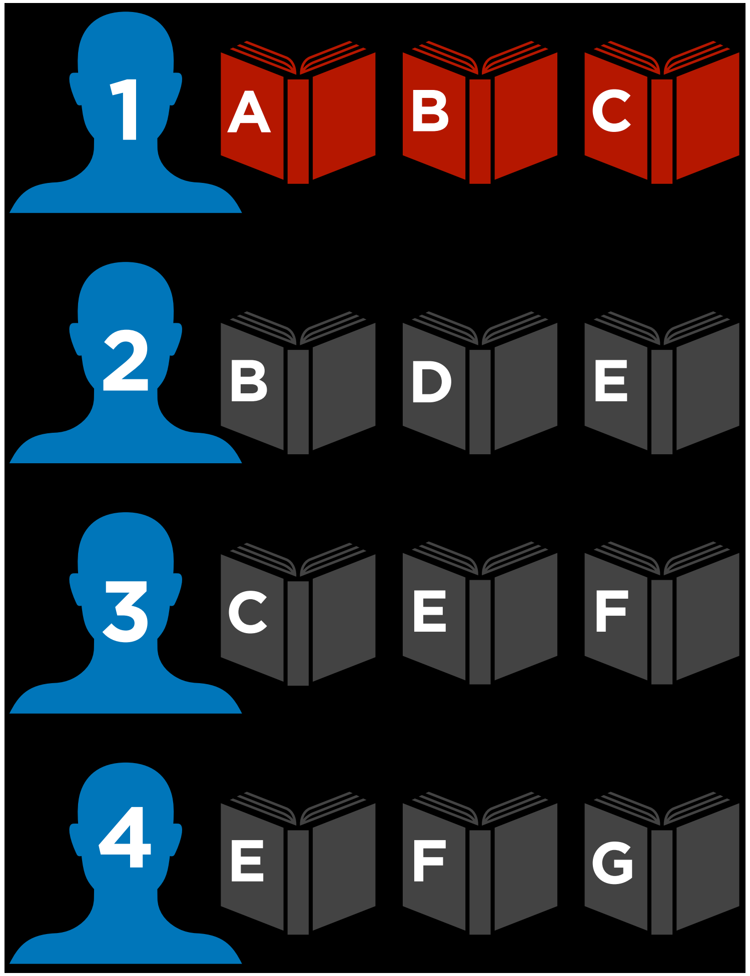

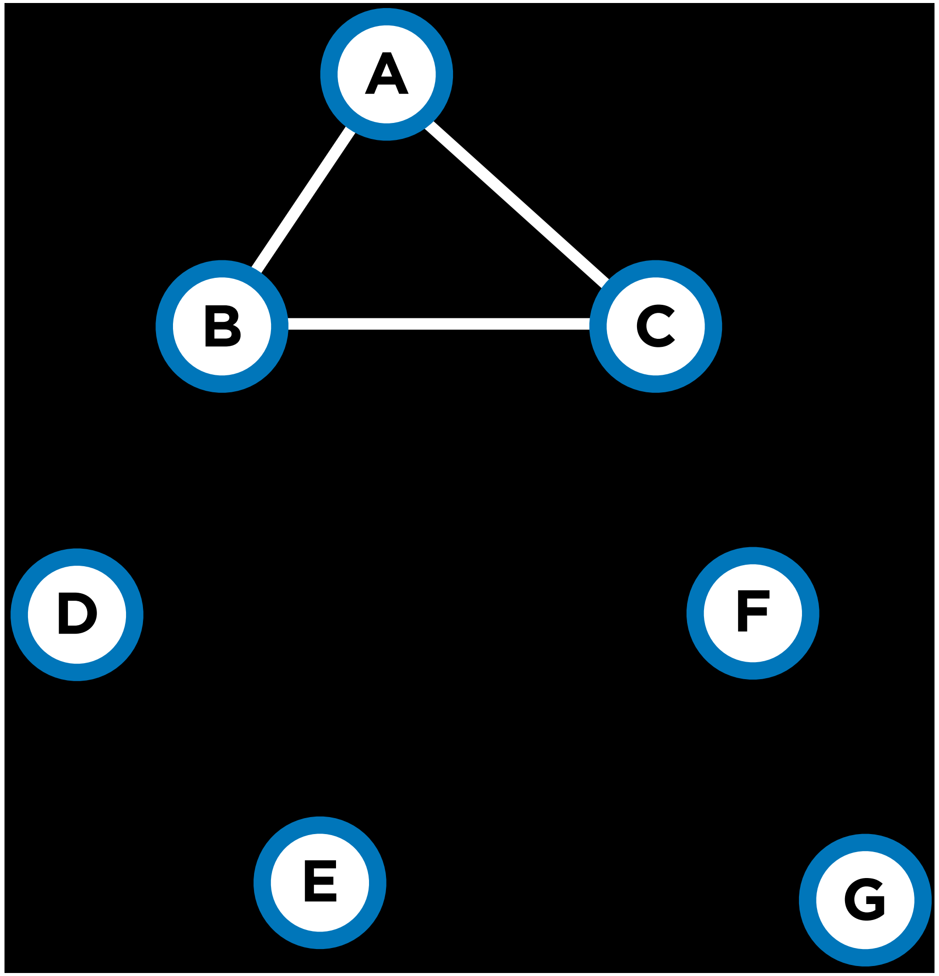

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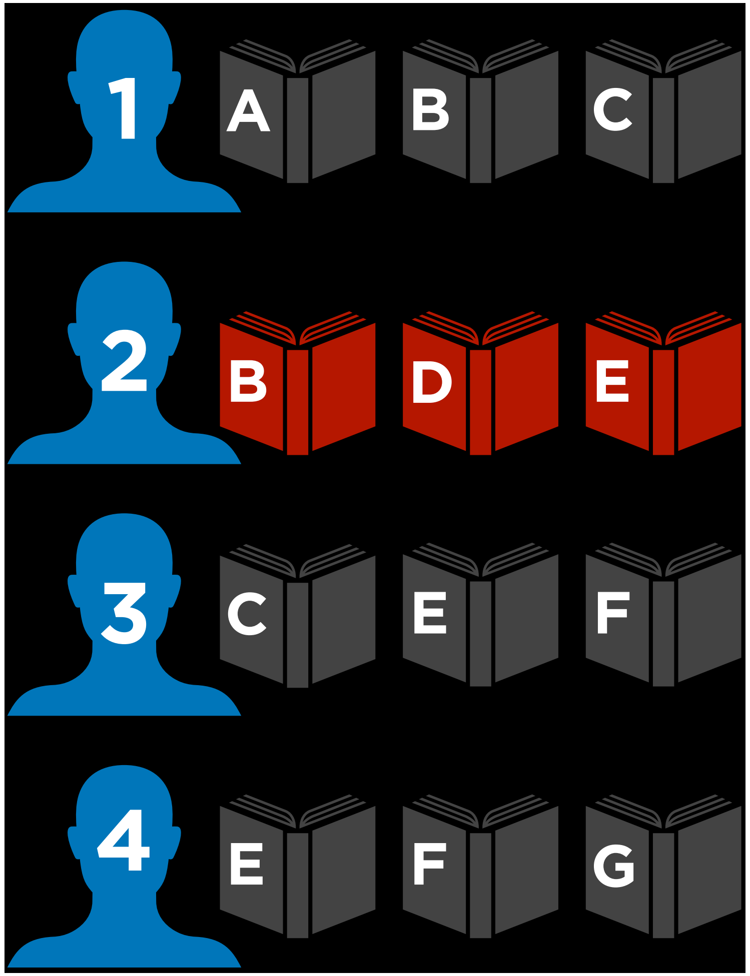

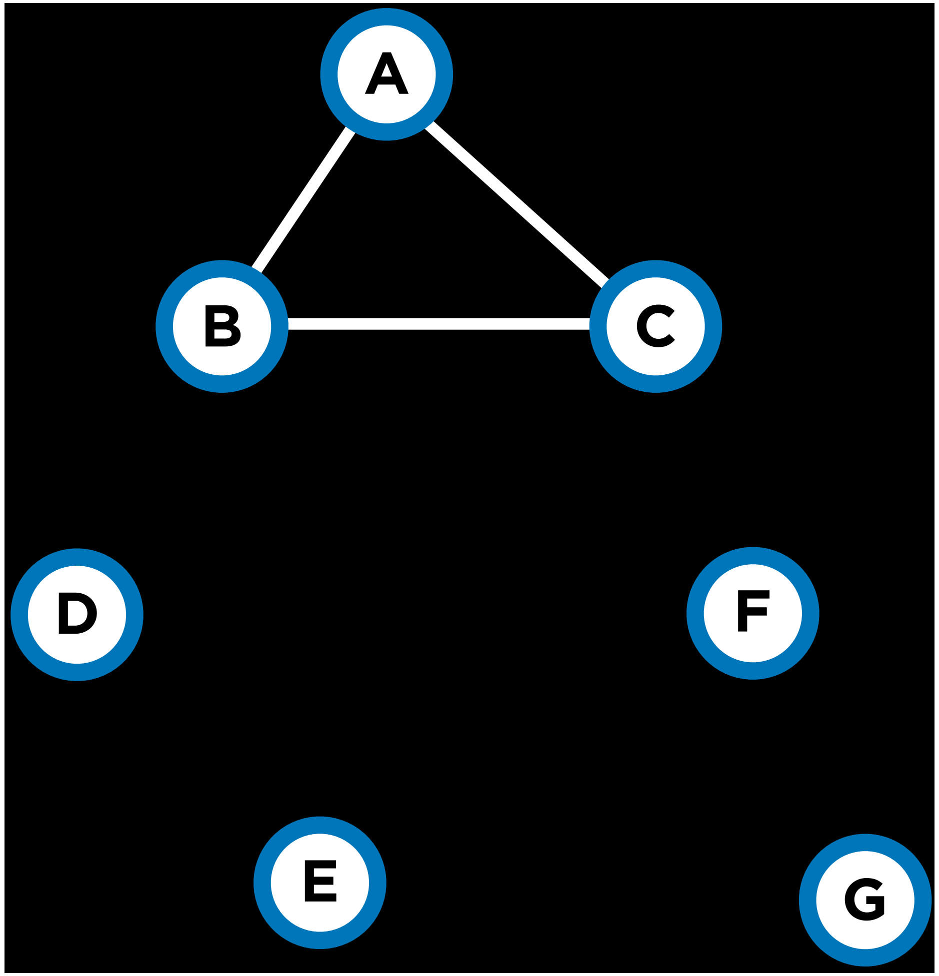

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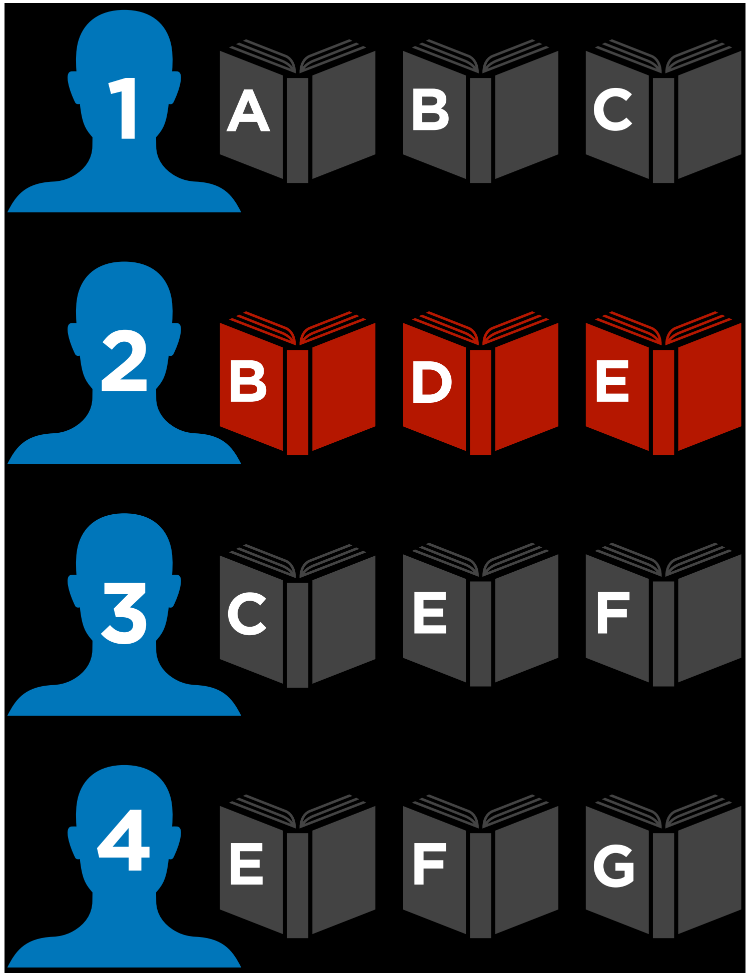

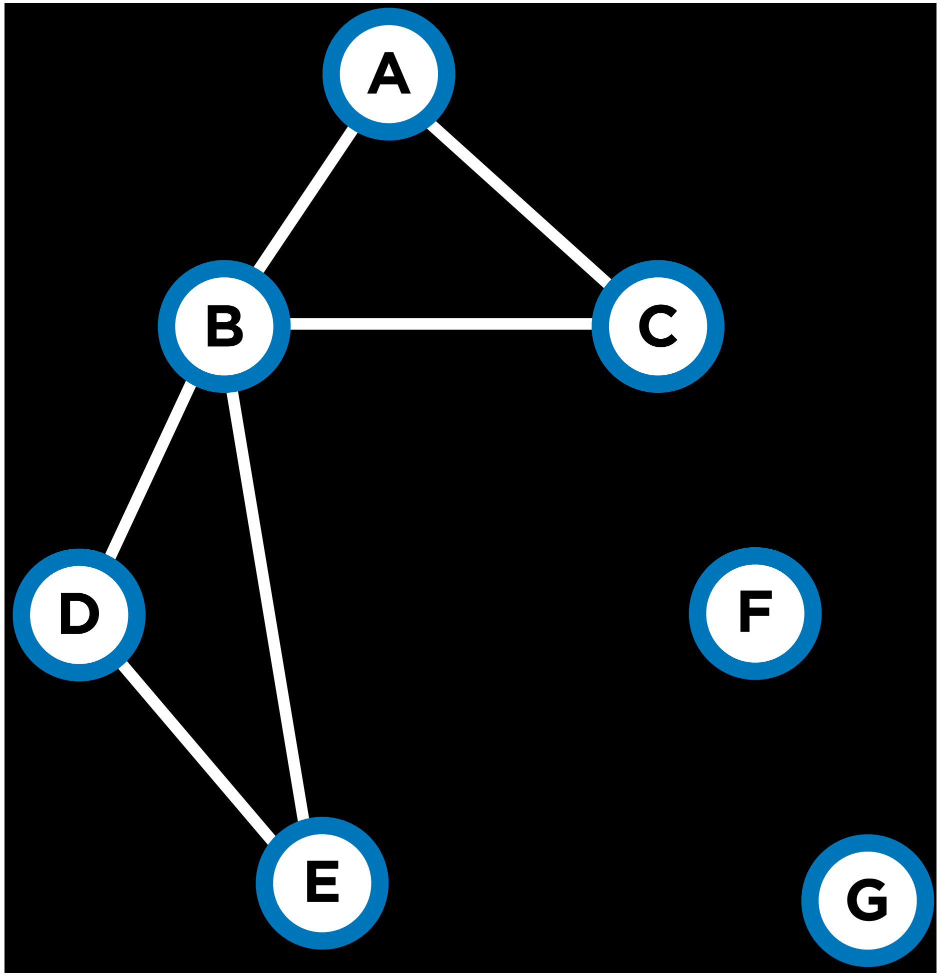

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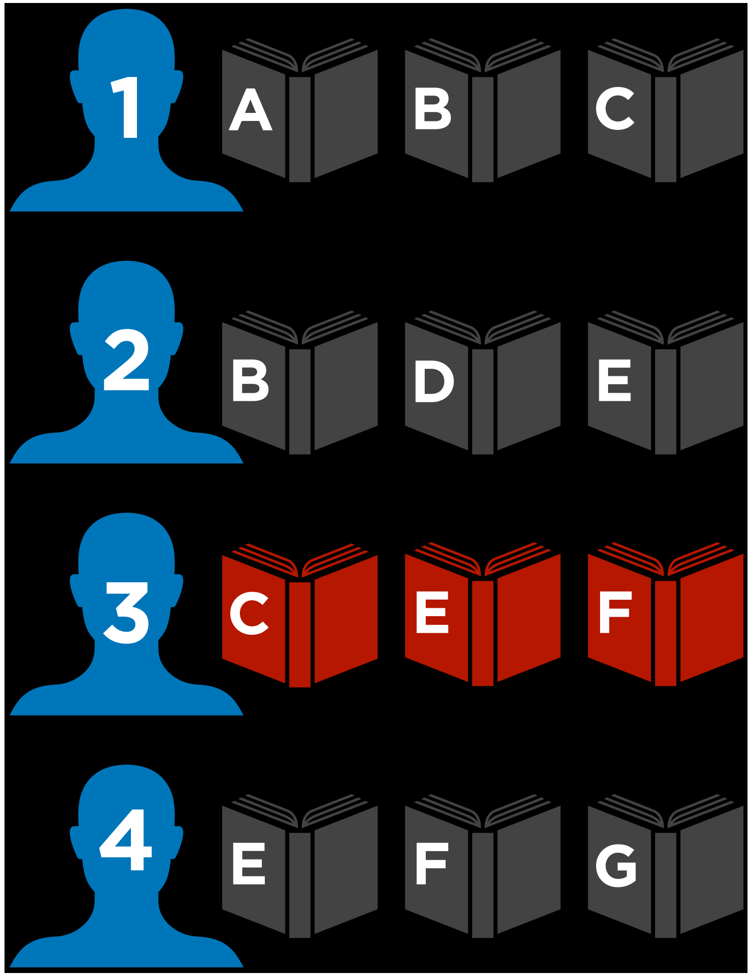

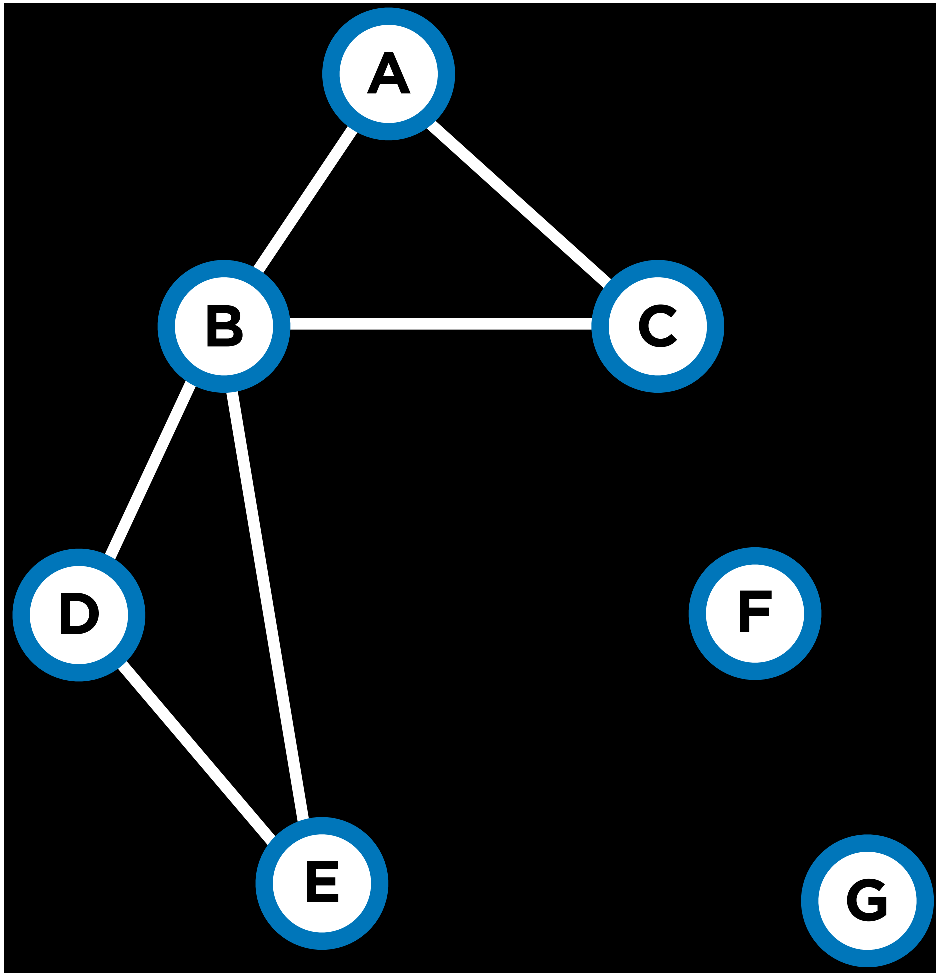

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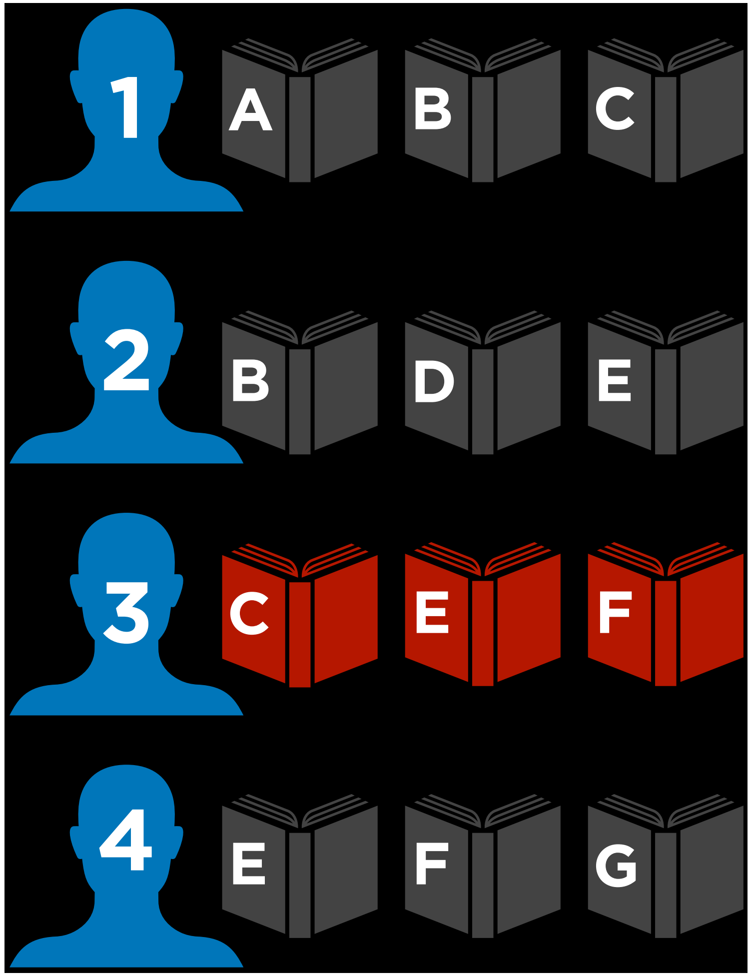

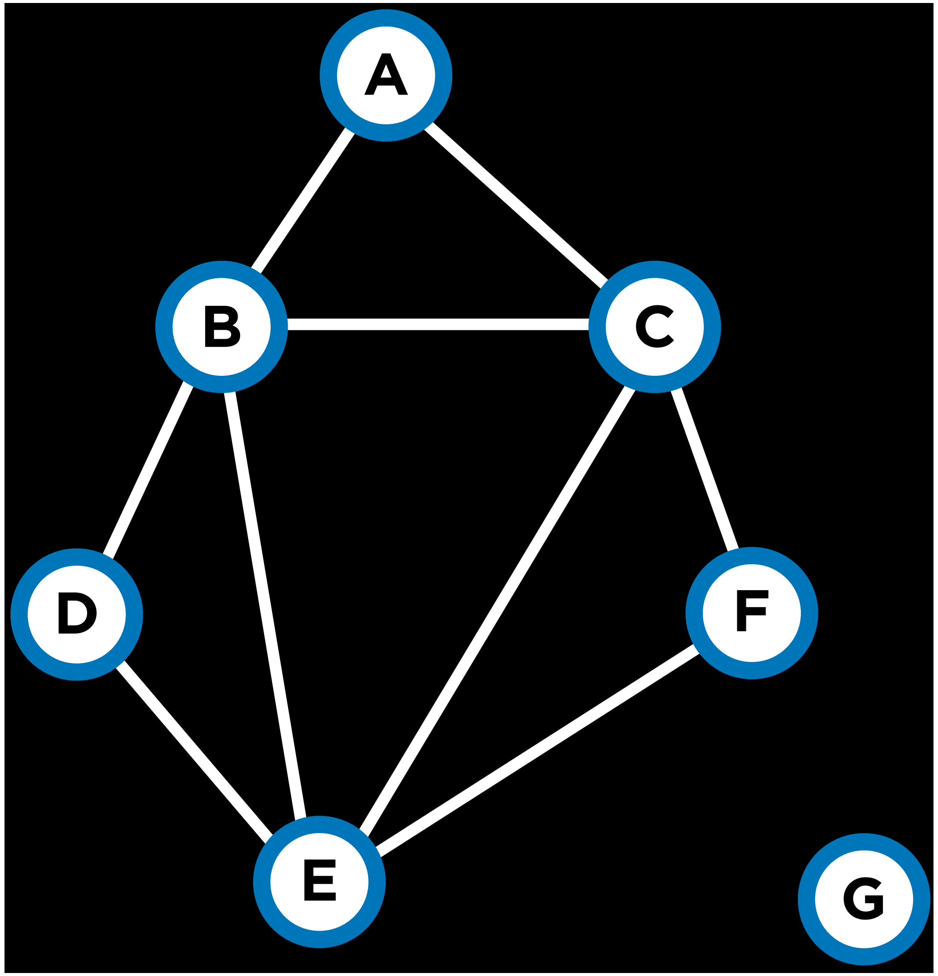

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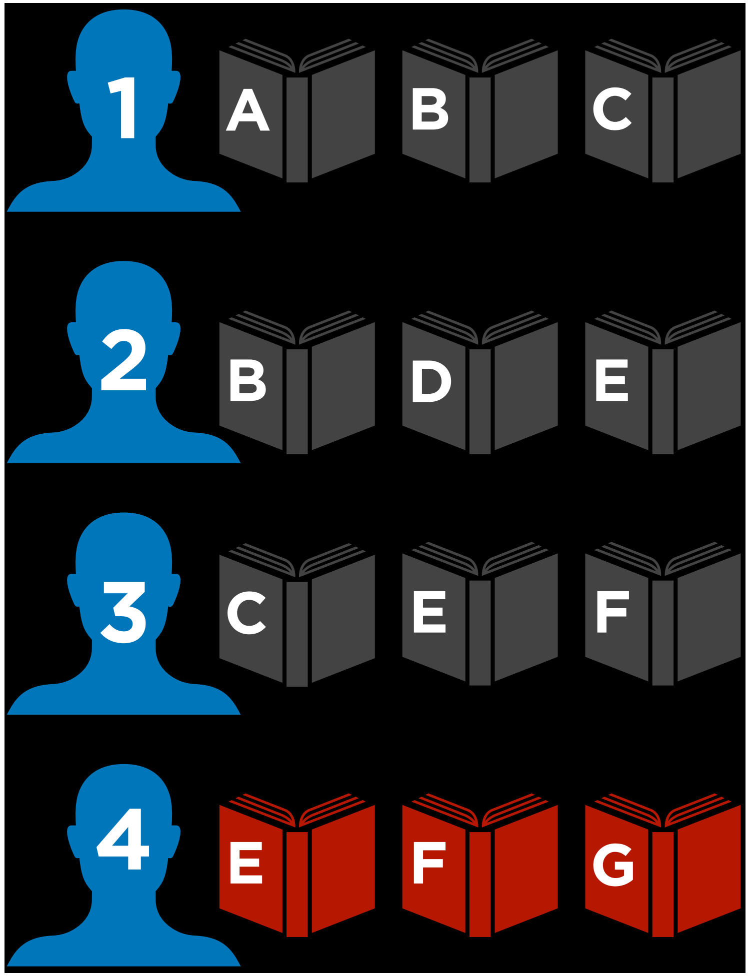

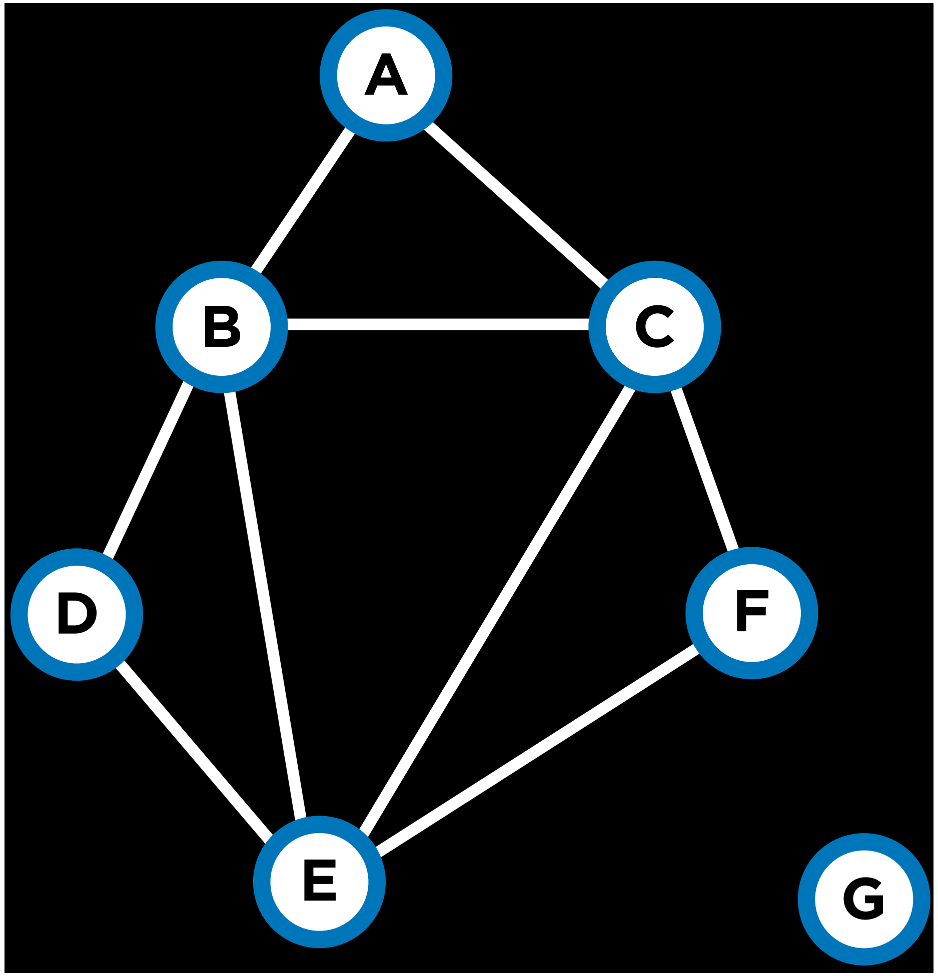

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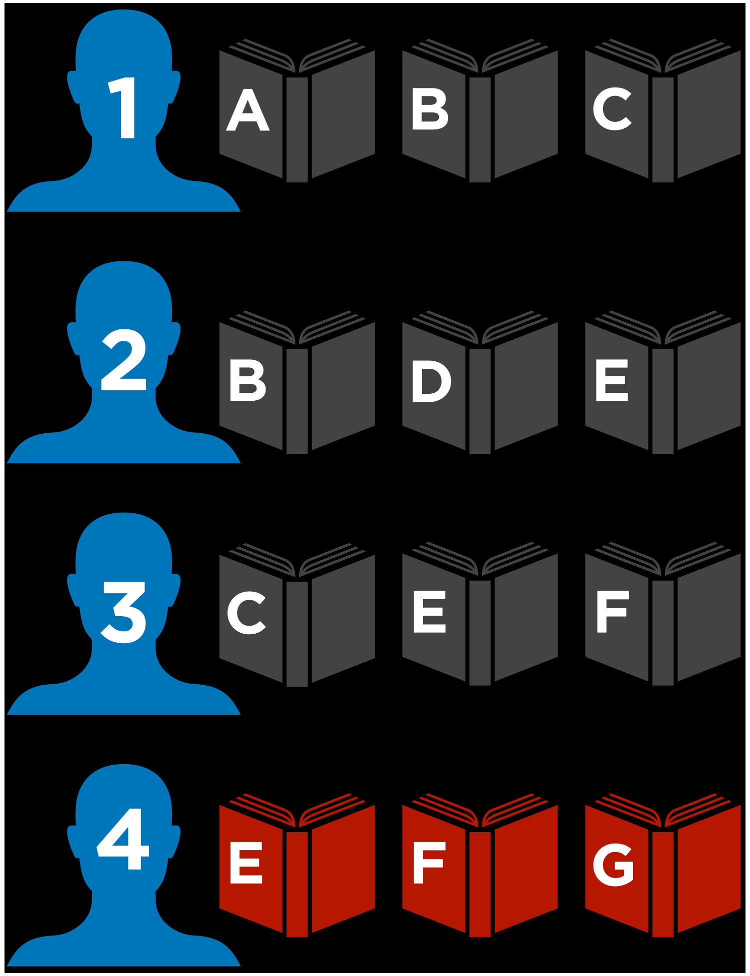
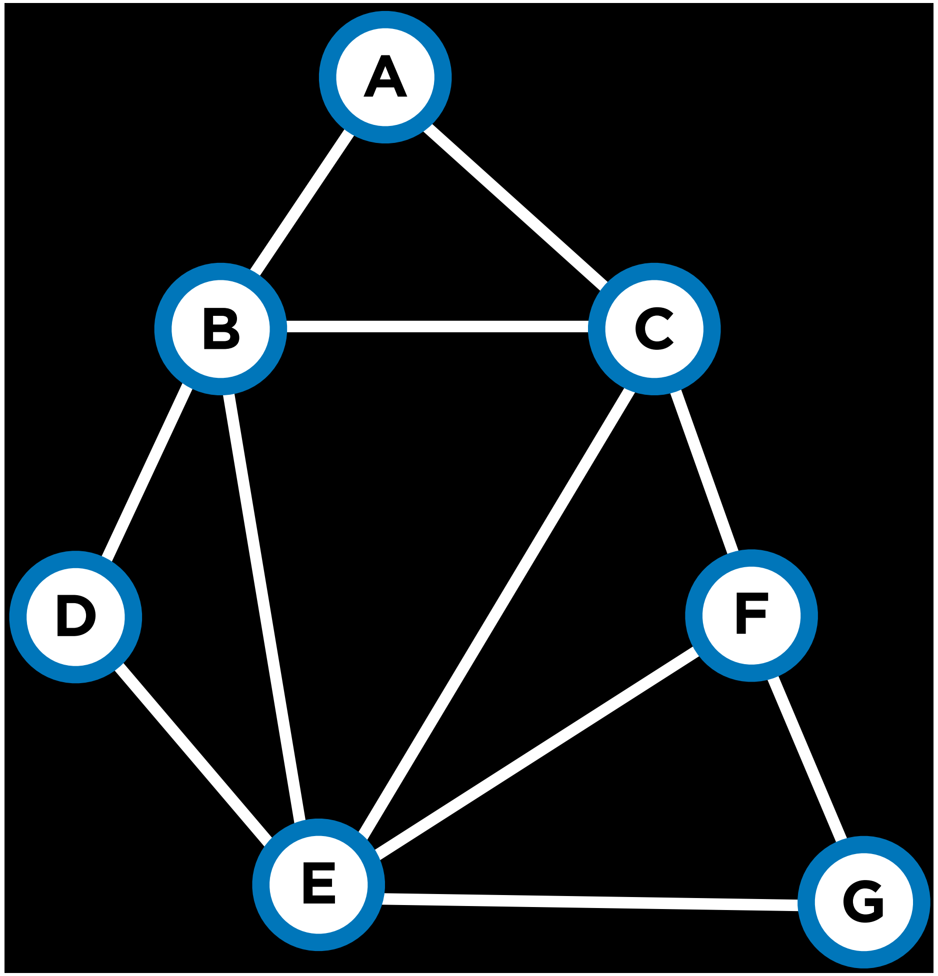
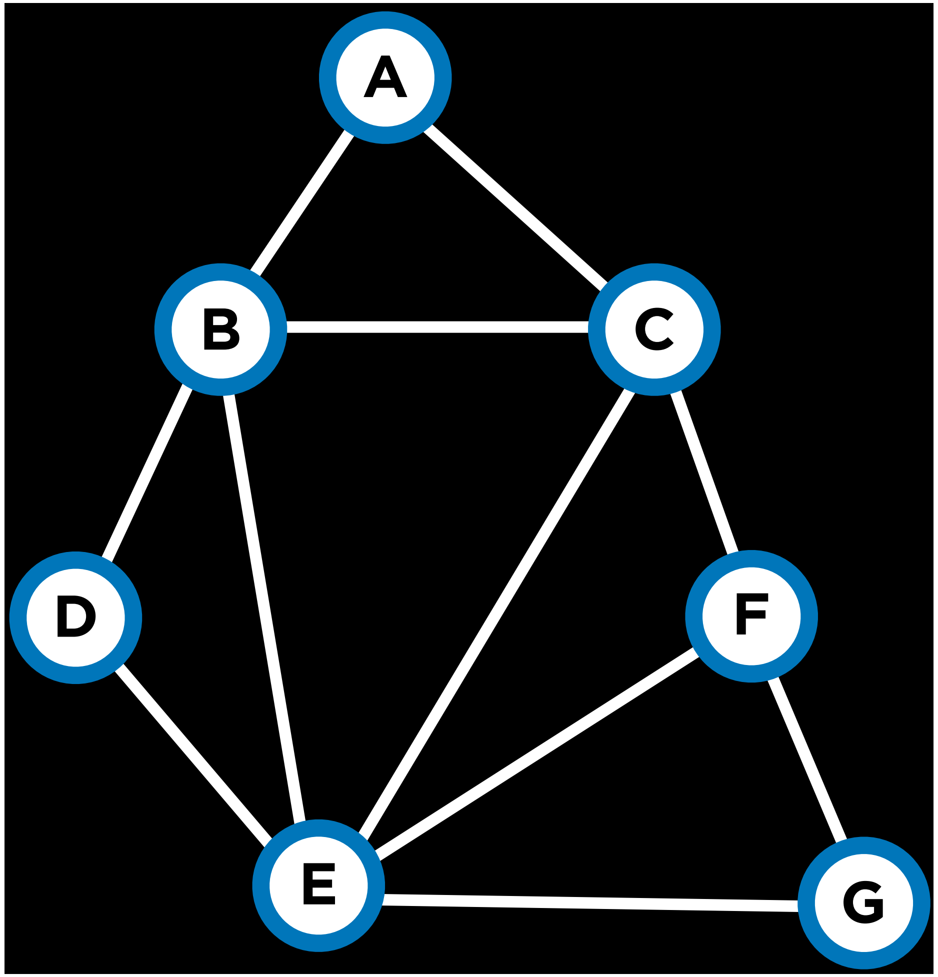




Constraint Graph for Exam Scheduling

- We end up with a graphical representation of all the variables and the constraints between those variables
- In this case, the constraints are inequality constraints
  - e.g., the A-B edge means that the values A takes on cannot be the same as B values



Constraint Satisfaction Problem

- Set of variables $\{X_1, X_2, \ldots, X_n\}$
- Set of domains$\{D_1, D_2, \ldots, D_n\}$, one for each variable
- Set of constraints C

- CSPs deal with assignments of values to variables
  - A complete assignment is one in which every variable is assigned a value, and a solution to a CSP is a consistent, complete assignment
  - A partial assignment leaves some variables unassigned
  - A partial solution is a partial assignment that is consistent

Constraint Satisfaction Problem: Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Variables

$\{(0, 2), (1, 1), (1, 2), (2, 0), \ldots\}$

Domains

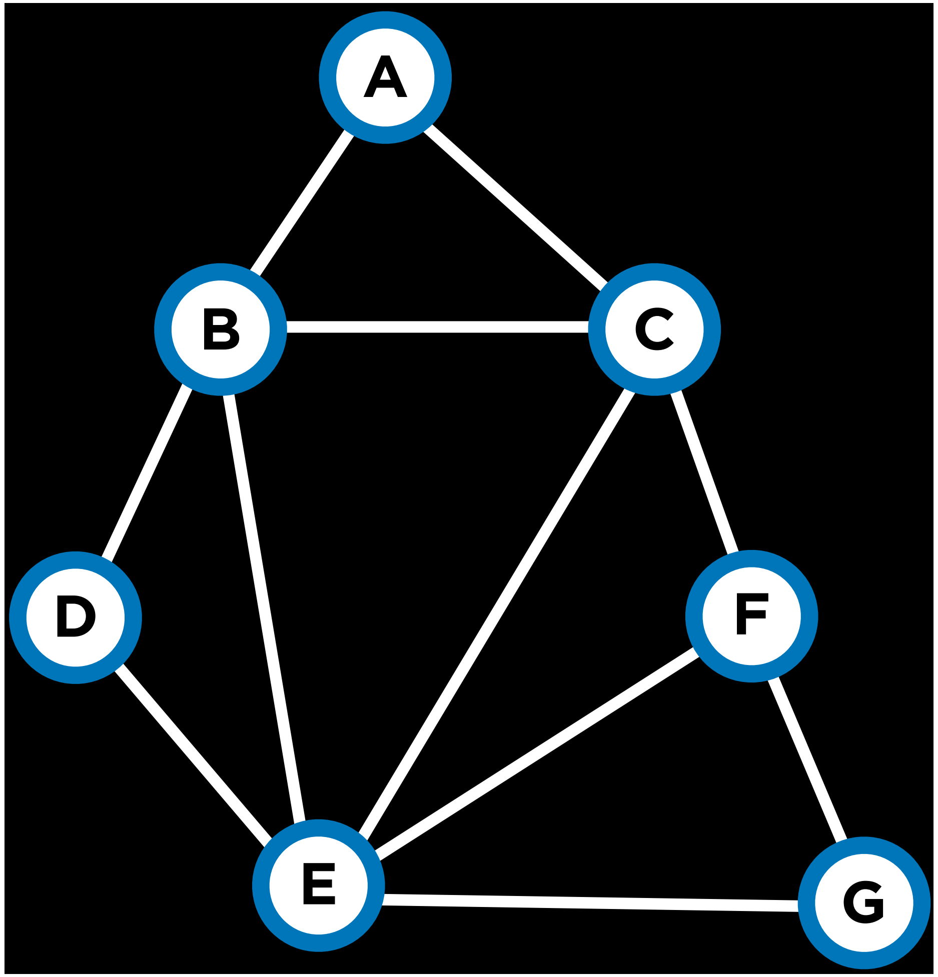
$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

for each variable

Constraints

$\{(0, 2) \neq (1, 1) \neq (1, 2) \neq (2, 0), \ldots\}$

Exam Scheduling Problem Formulation



Variables

$\{A, B, C, D, E, F, G\}$

Domains

$\{Monday, Tuesday, Wednesday\}$

for each variable

Constraints

$\{A \neq B, A \neq C, B \neq C, B \neq D, B \neq E, C \neq E, C \neq F, D \neq E, E \neq F, E \neq G, F \neq G\}$

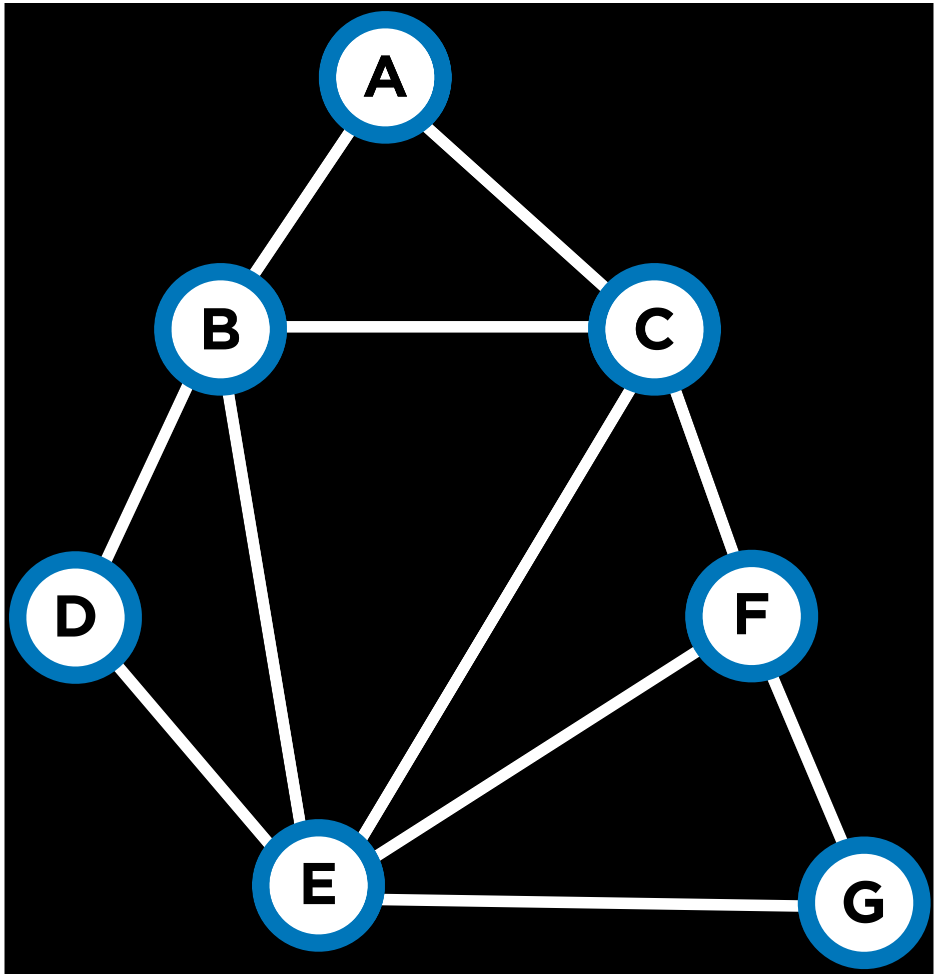
Constraints

- Hard
  - Constraints that must be satisfied in a correct solution

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

- Soft
  - Constraints that express some notion of which solutions are preferred over others
    - The goal is to try to maximize the preference, that is, the preferences should be satisfied as much as possible

Exam Scheduling as a Hard Constraint Problem

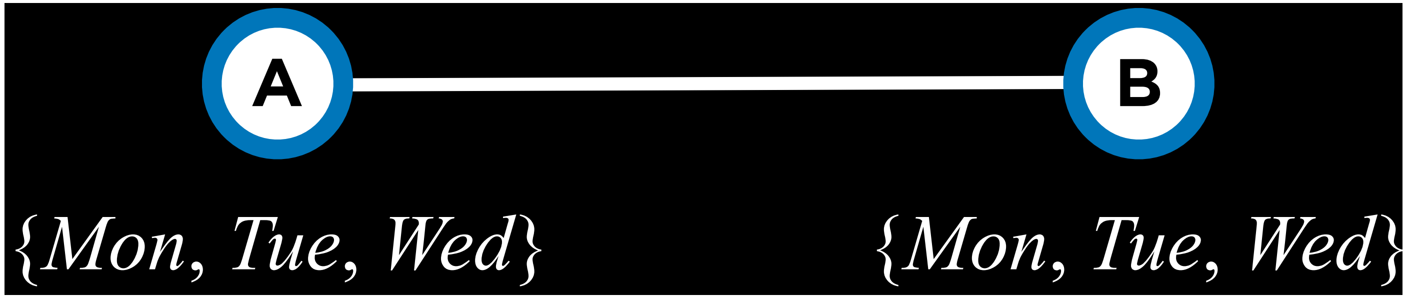


Unary and Binary Constraints

- Constraints in a CSP can be classified into some different categories
  - Unary
    - Constraint involving only one variable
      - {A ≠ Monday}
  - Binary
    - Constraint involving two variables
      - {A ≠ B}

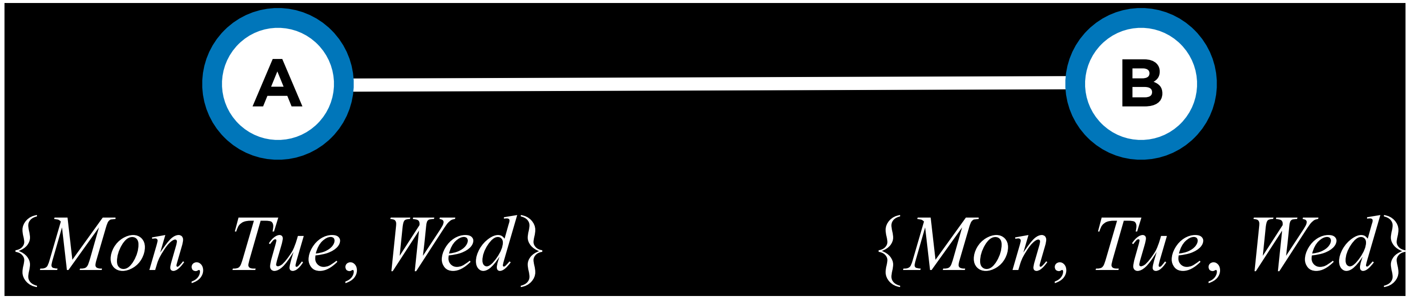
Node Consistency

- Knowing the category of the constraints we can say about a particular CSP, e.g., node consistency
  - When all the values in a variable's domain satisfy the variable's unary constraints



$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

Node Consistency



$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

to make the node consistent, we'll remove Monday from A's domain

Node Consistency

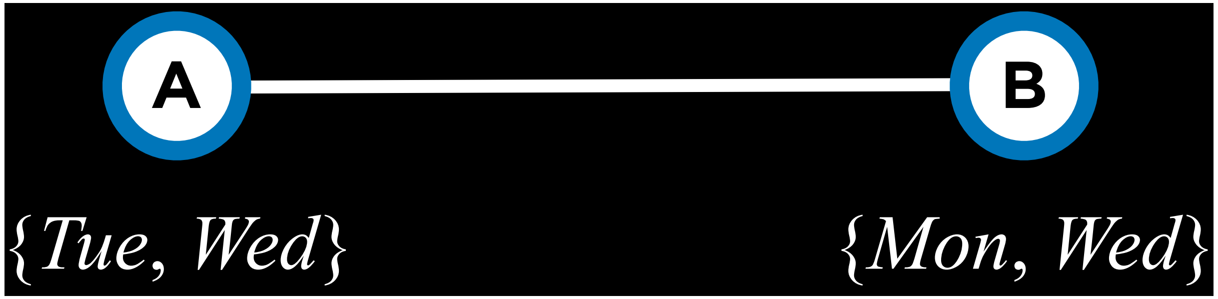
$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

now A is node consistent because, i.e., for each of the A's domain values, there is no unary constraint violated

Node Consistency

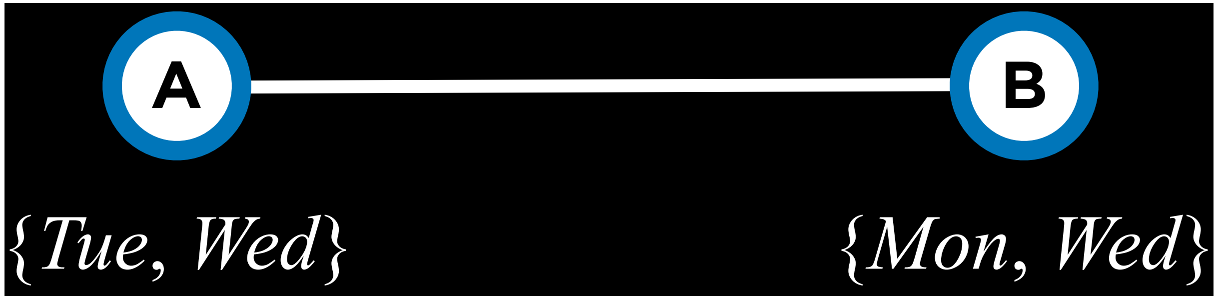
$$\{A \neq Mon,\ B \neq Tue,\ B \neq Mon,\ A \neq B\}$$

Node Consistency



$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

Node Consistency



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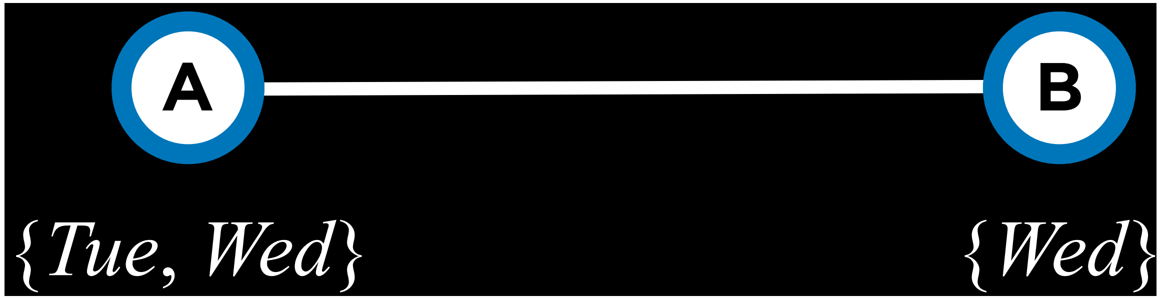
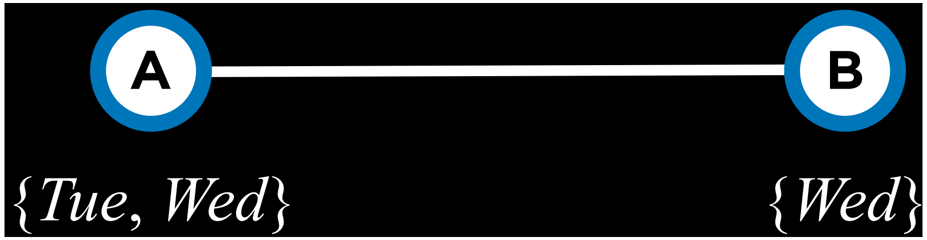
Node Consistency



$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

Node Consistency

- We have easily enforced node consistency



$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

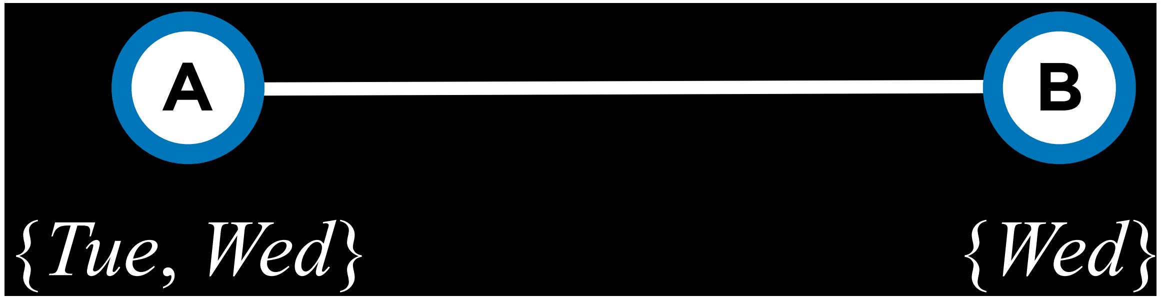
- However, different types of consistency can be considered …

Arc Consistency

- When all the values in a variable's domain satisfy the variable's binary constraints
- To make X arc-consistent with respect to Y, remove elements from X's domain until every choice for X has a possible choice for Y

Arc Consistency

- Is A arc consistent with B?



$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

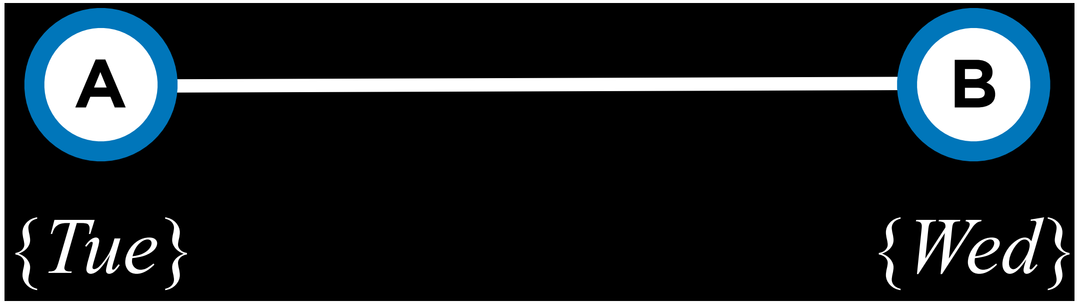
Arc Consistency

- No
  - if we choose Wed for A, then no choice in B's domain satisfies this binary constraint

$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

Arc Consistency

- We can apply arc consistency to a larger graph, not just looking at one pair of arc consistency, solving the whole problem …



$$\{A \neq Mon, B \neq Tue, B \neq Mon, A \neq B\}$$

Arc Consistency

```
function REVISE(csp, X, Y):
    revised = false
    for x in X.domain:
        if no y in Y.domain satisfies constraint for (X,Y):
            delete x from X.domain
            revised = true
    return revised
```

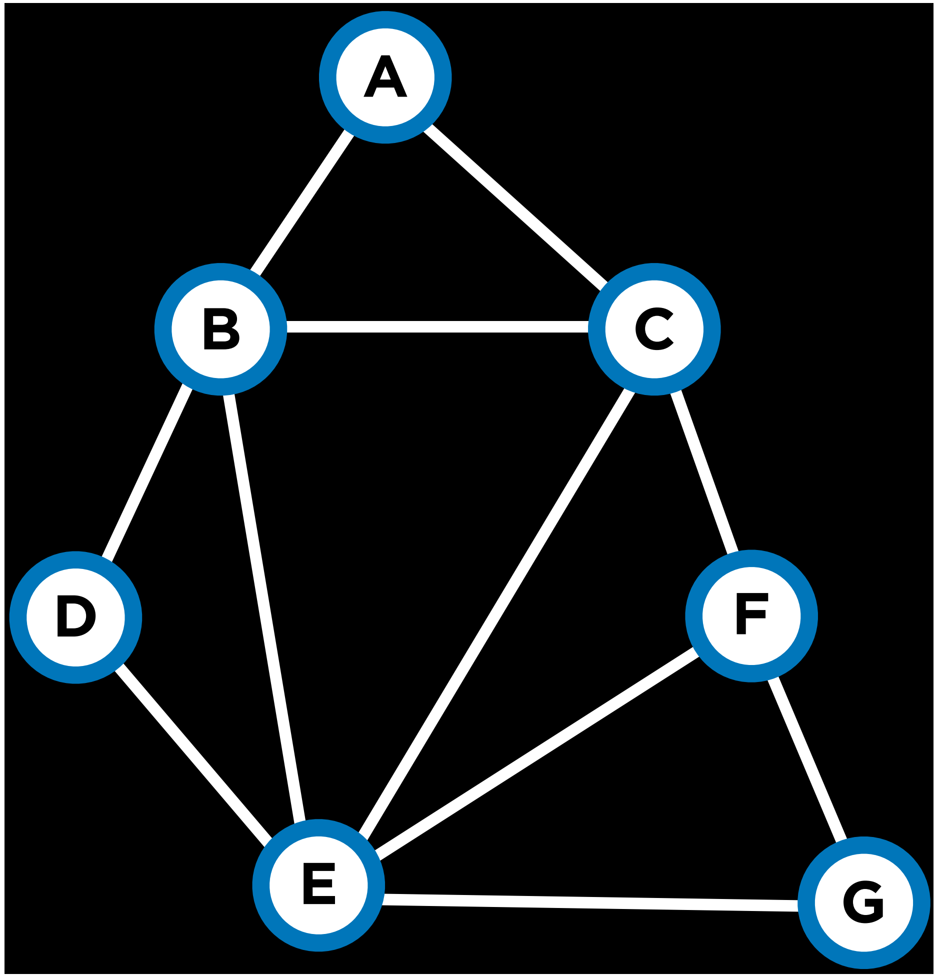
AC-3 Algorithm for Arc Consistency

```
function AC-3(csp):
    queue = all arcs in csp
    while queue non-empty:
        (X,Y)= DEQUEUE(queue)
         if REVISE(csp, X, Y):
             if size of X.domain == 0:
                 return false
             for each Z in X.neighbors - {Y}:
                 ENQUEUE(queue, (Z, X))
    return true
```

$O(n^2d^3)$. Can be reduced to $O(n^2d^2)$,
n = #variables and d = domain size

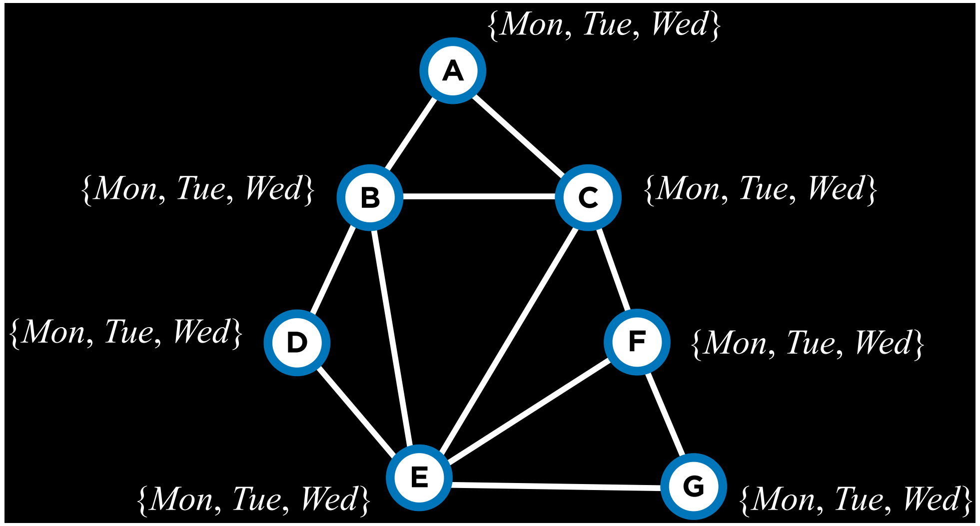
Arc Consistency on a Graph

- What happens here with AC-3?
  - Nothing change here



Arc Consistency on a Graph

- AC-3 can reduce the domains of variables, making the problem more manageable
- However, it doesn't guarantee a solution for all cases
- Sometimes, additional search methods are necessary to find a valid solution



Let's Recall Search Problems

- Initial state
- Actions
- Transition model
- Goal test
- Path cost function

CSPs as Search Problems

- Initial state:
  - empty assignment (no variables)
- Actions:
  - add a {*variable* = *value*} to assignment
- Transition model:
  - shows how adding an assignment changes the assignment
- Goal test:
  - check if all variables assigned and constraints all satisfied
- Path cost function:
  - all paths have the same cost

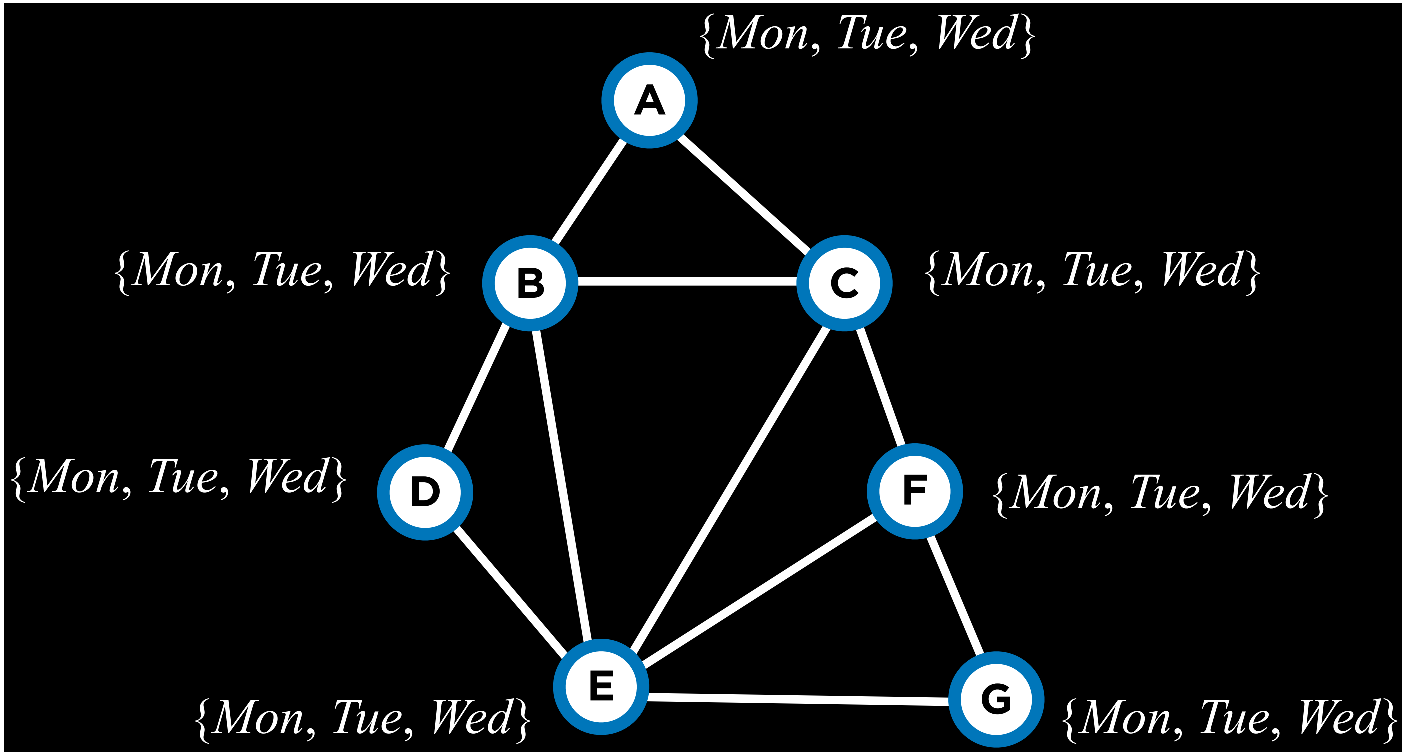
Backtracking Search

- If we implement this search algorithm by using implementations like BFS or DFS, this will be very inefficient
- The search algorithm generally used for CSPs is Backtracking Search
- Idea
  - Go ahead and make assignments from variables to values
    - When we get to a place where there's no way to move forward while still maintaining the constraints we need to enforce, we backtrack and try something else instead

Backtracking Search

```
function BACKTRACK(assignment, csp):
   if assignment complete: return assignment
   var = SELECT-UNASSIGNED-VAR(assignment, csp)
   for value in DOMAIN-VALUES(var, assignment):
      if value consistent with assignment:
         add {var = value} to assignment
         result = BACKTRACK(assignment, csp)
         if result ≠ failure: return result
      remove {var = value} from assignment
   return failure
```

Backtracking in Practice



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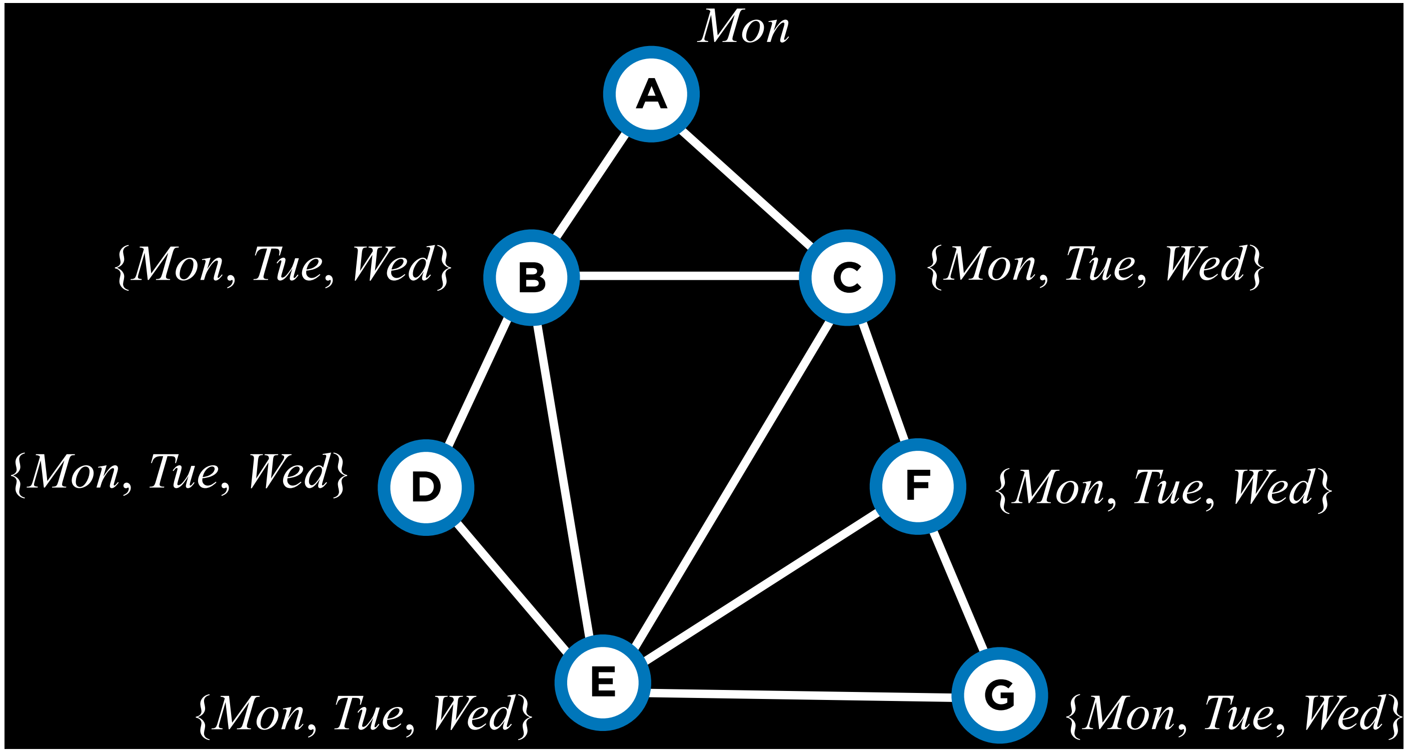

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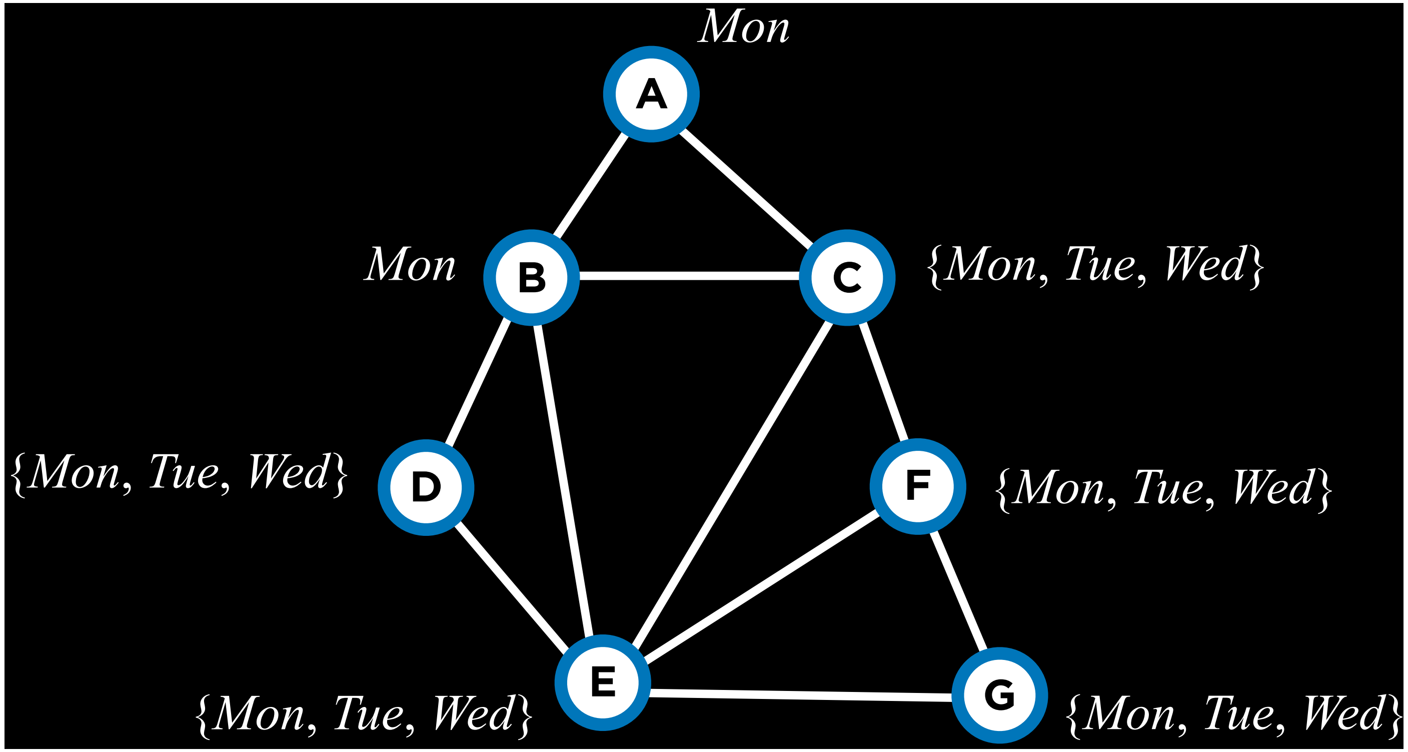

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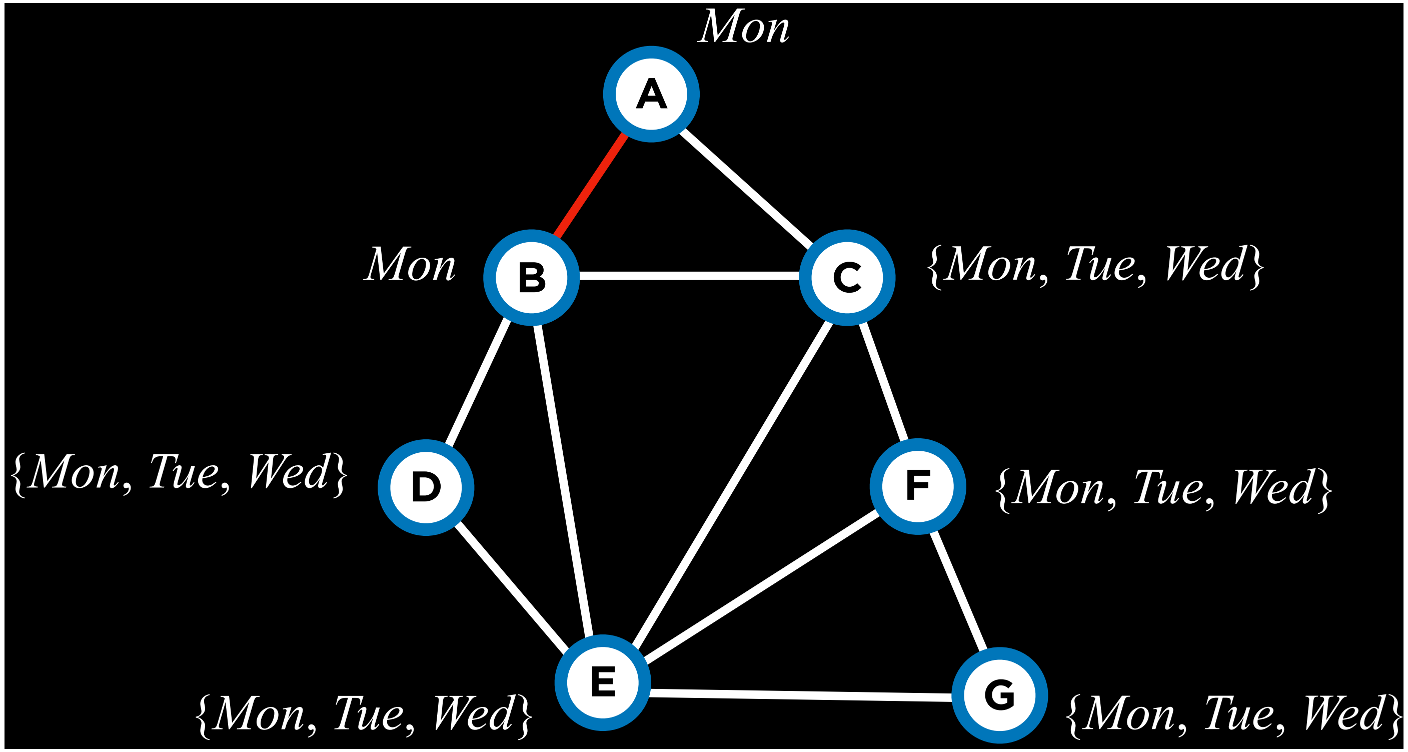

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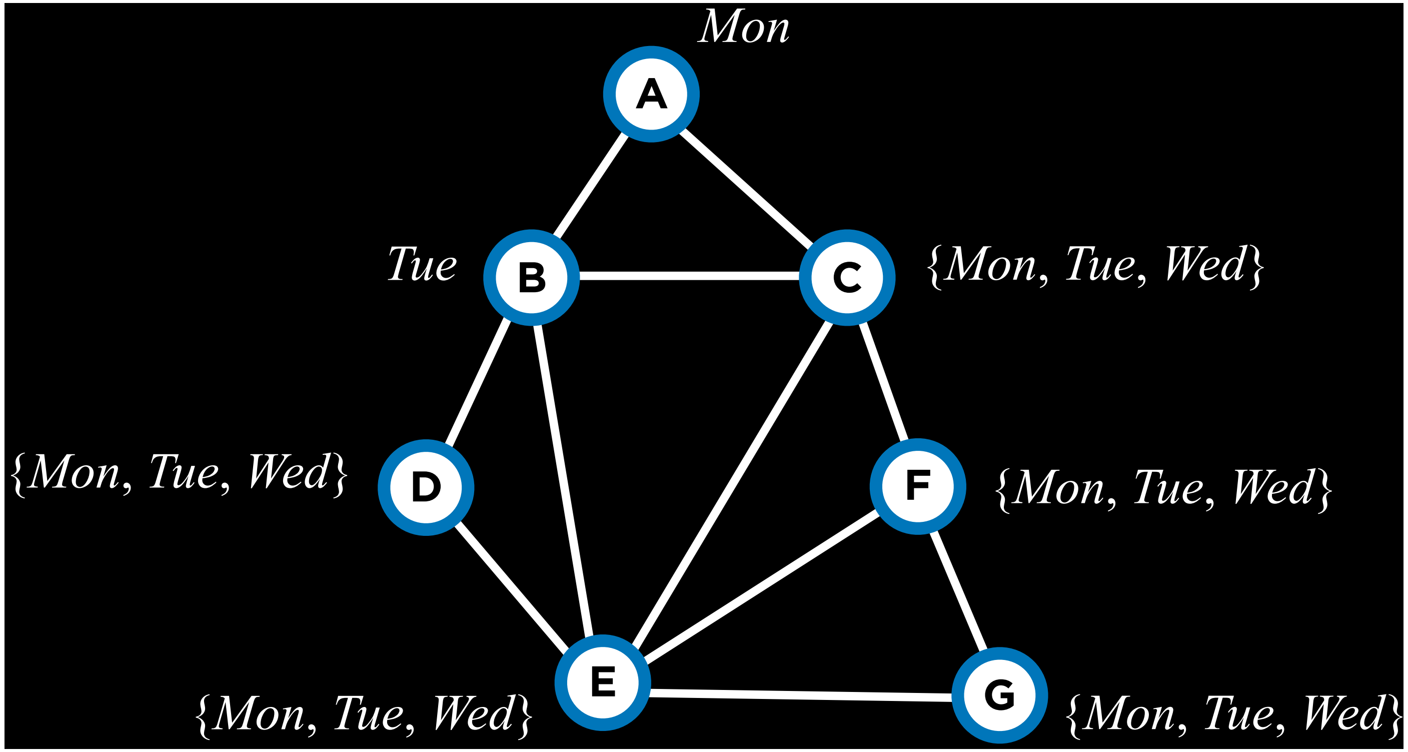

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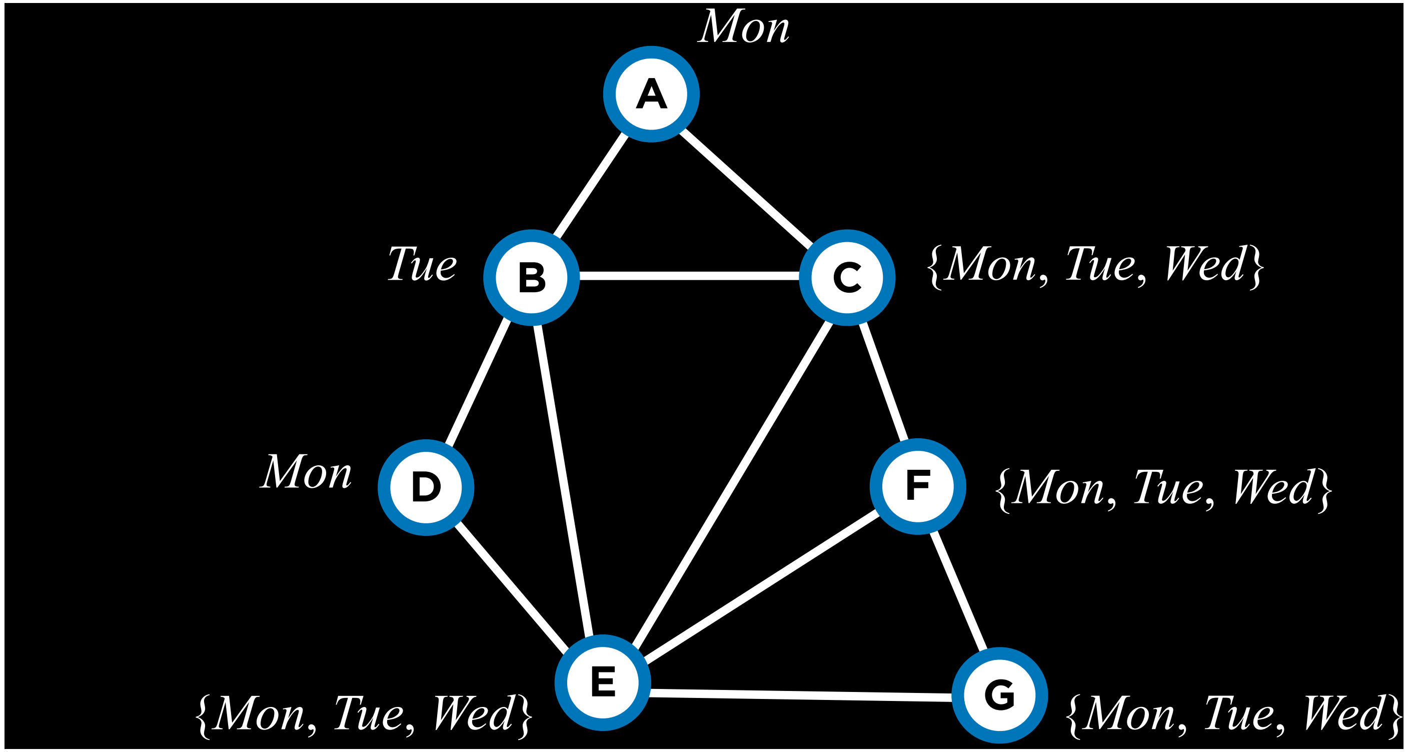

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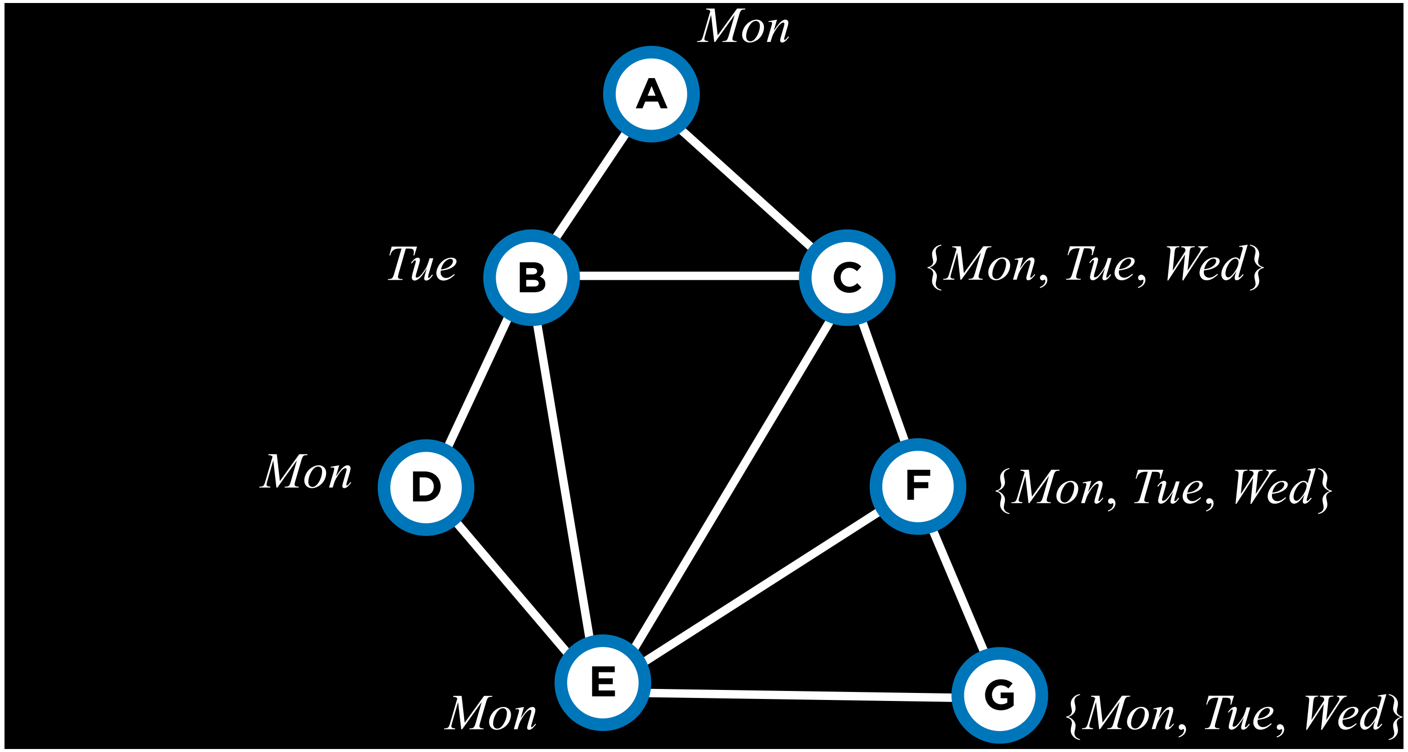

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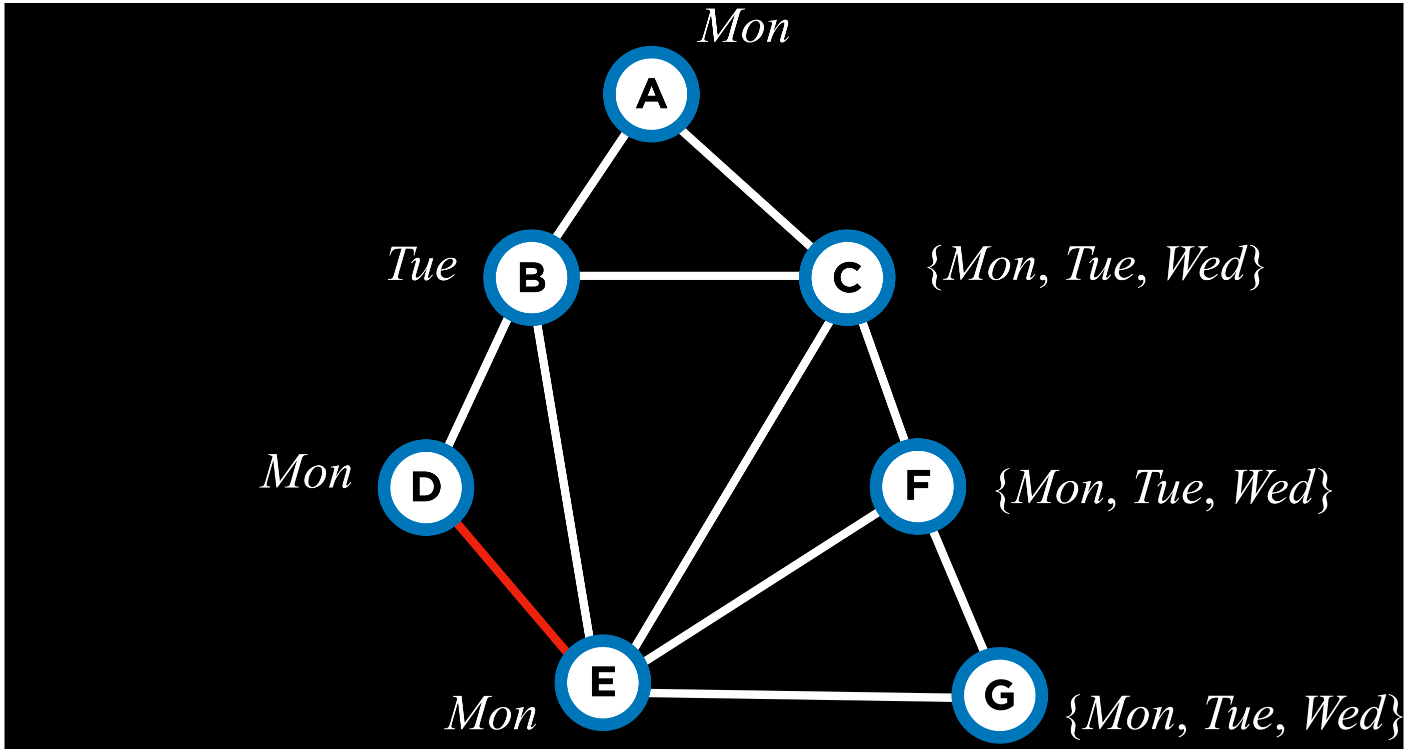

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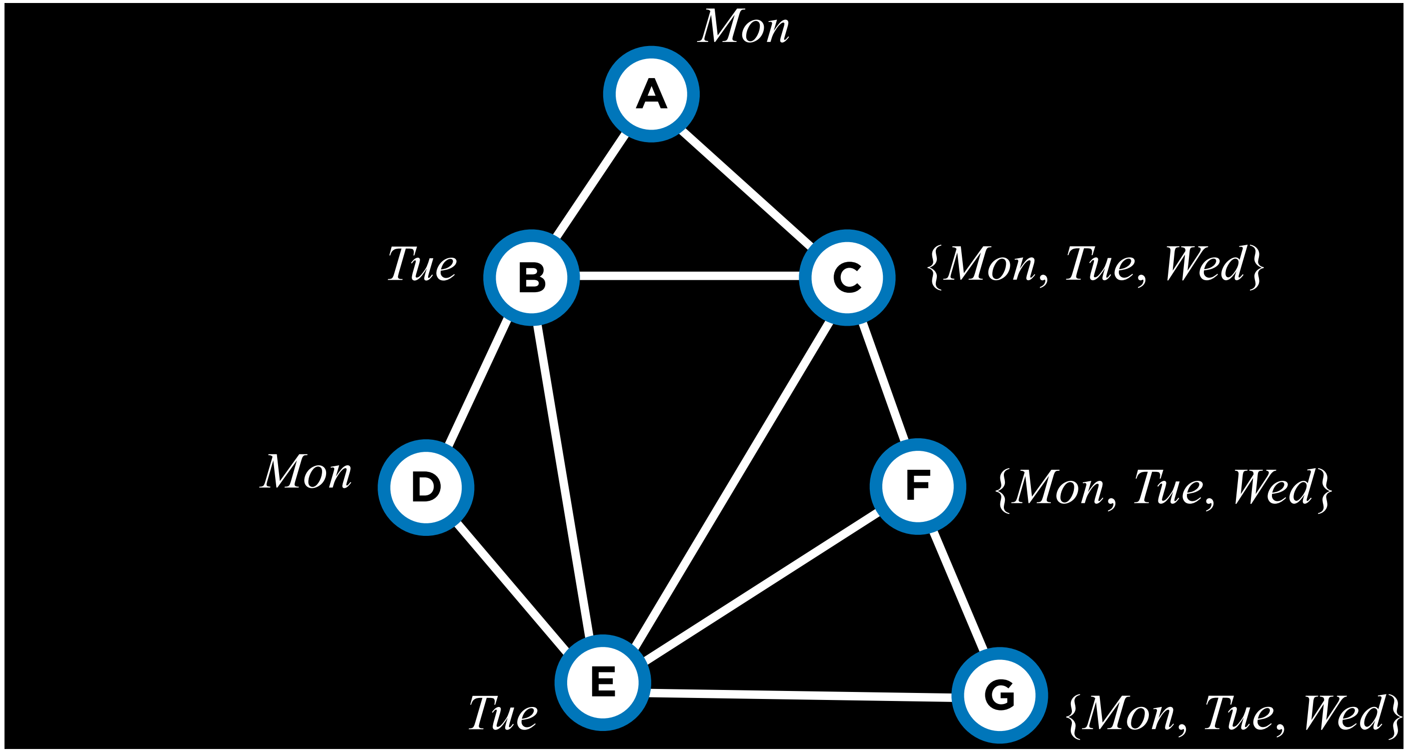

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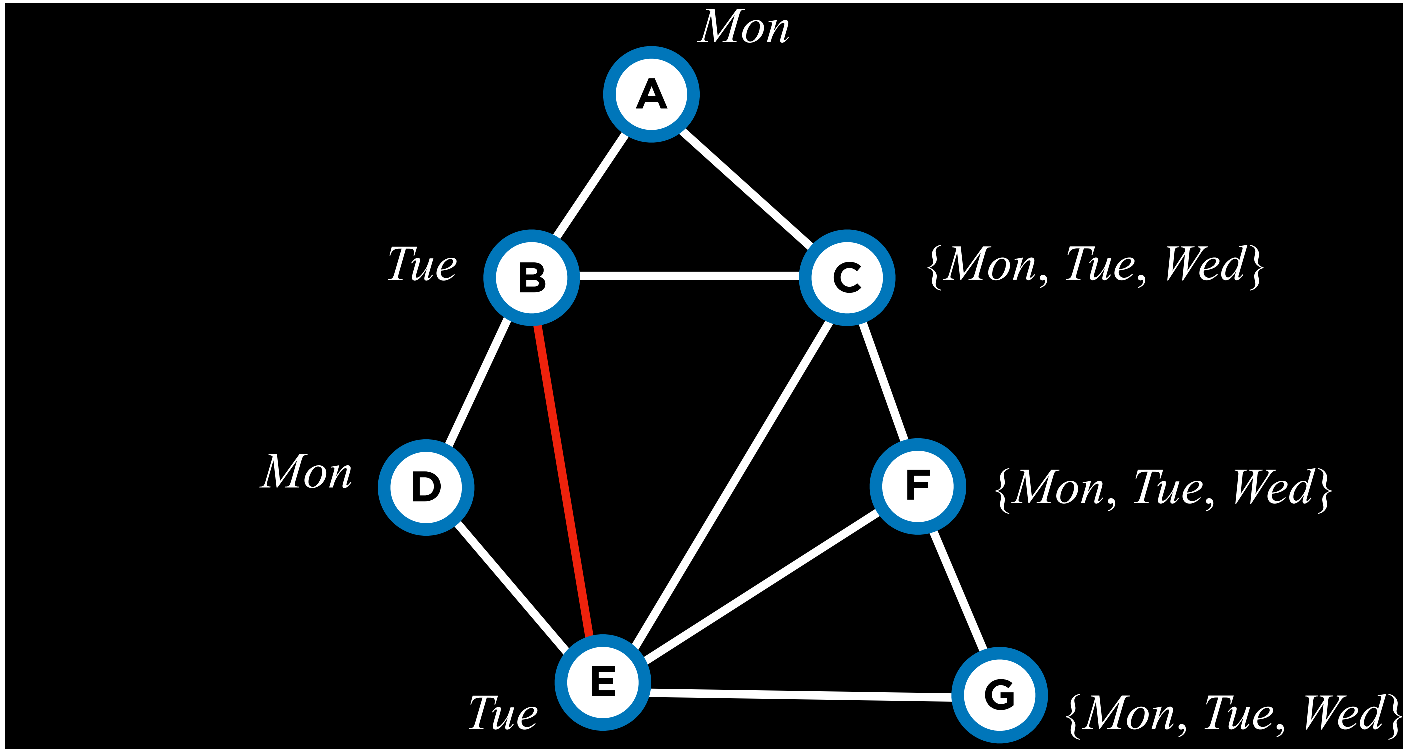

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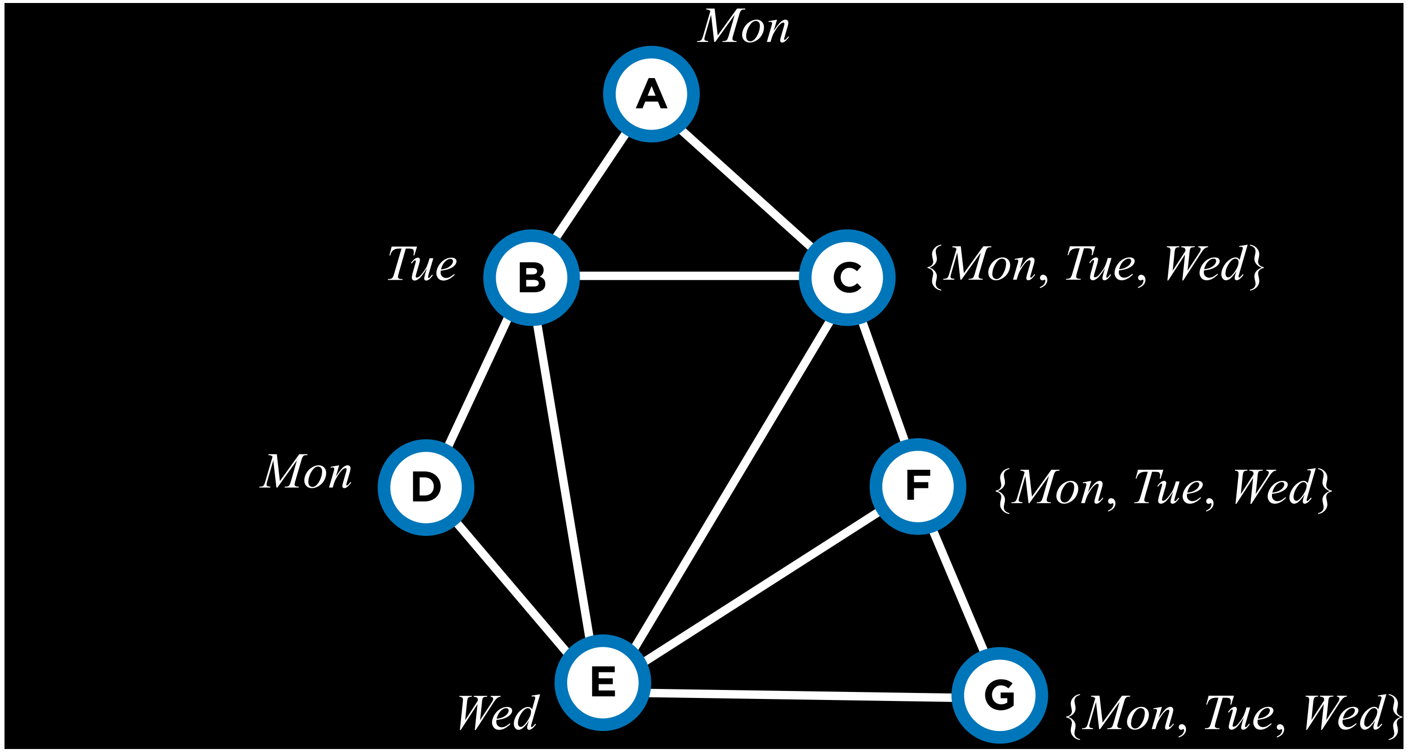

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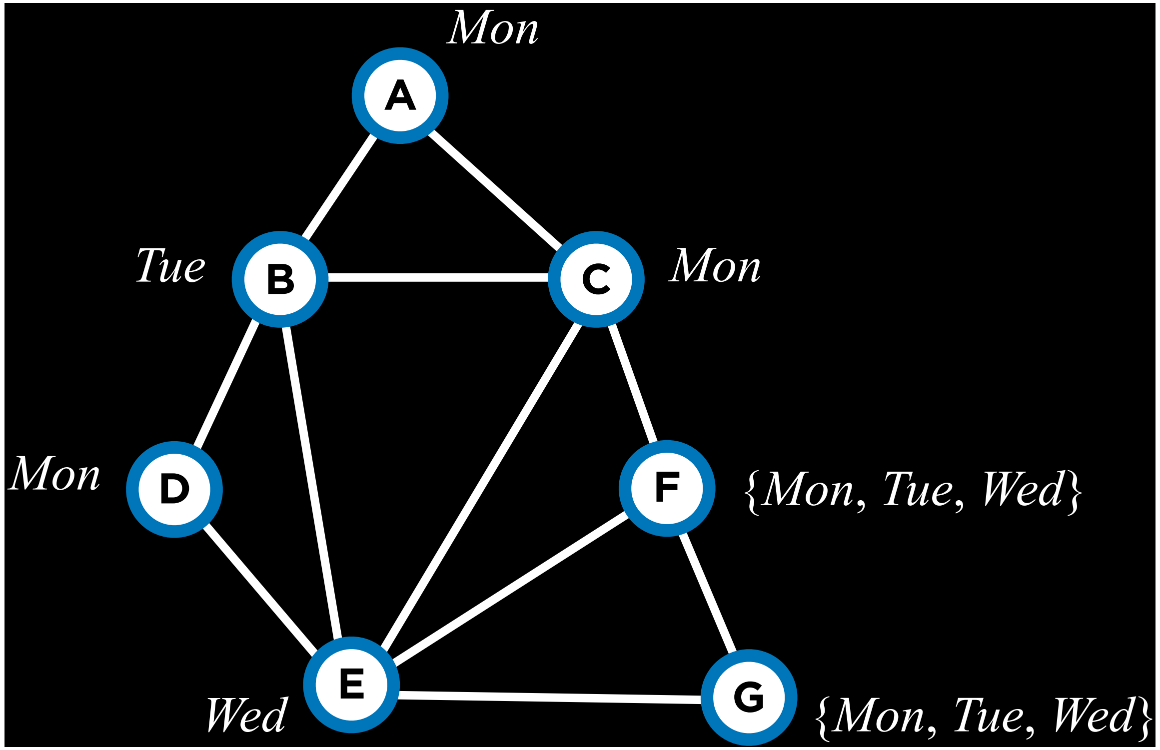

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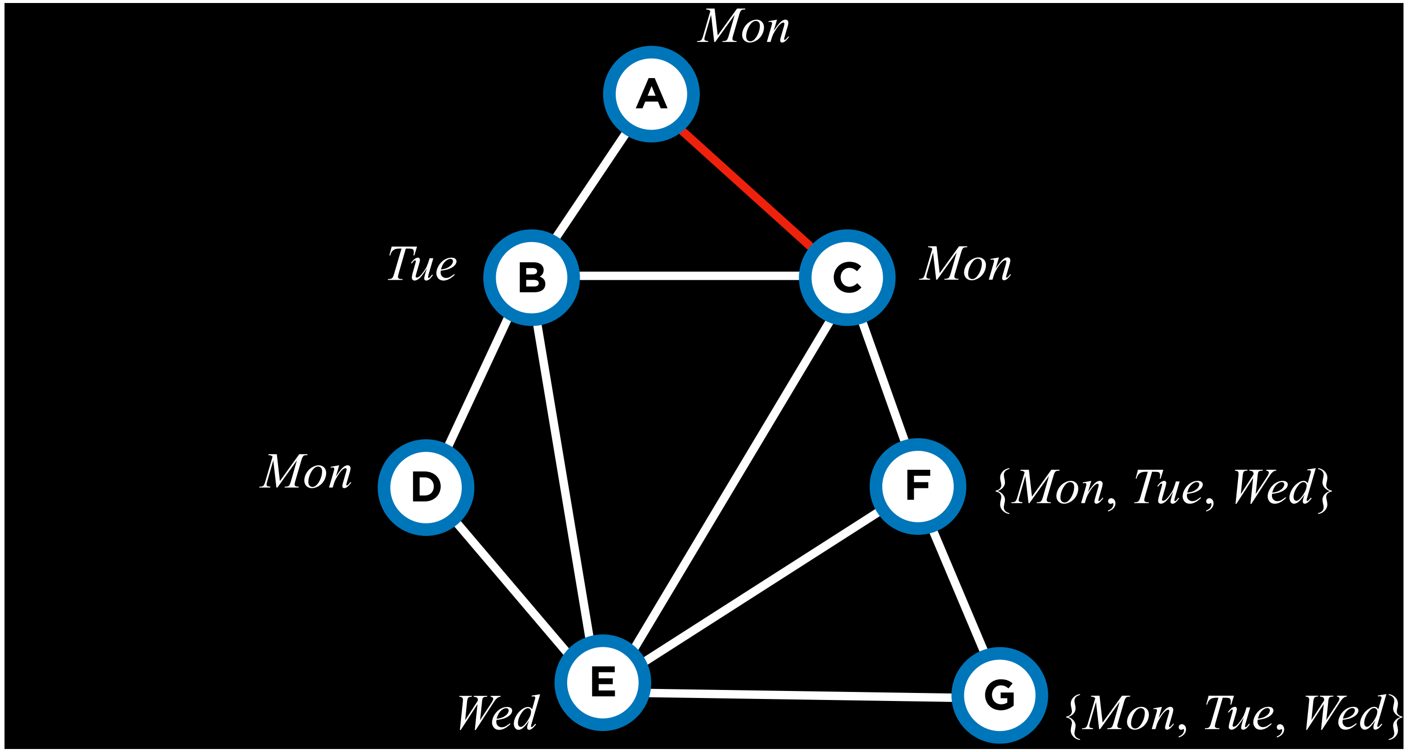

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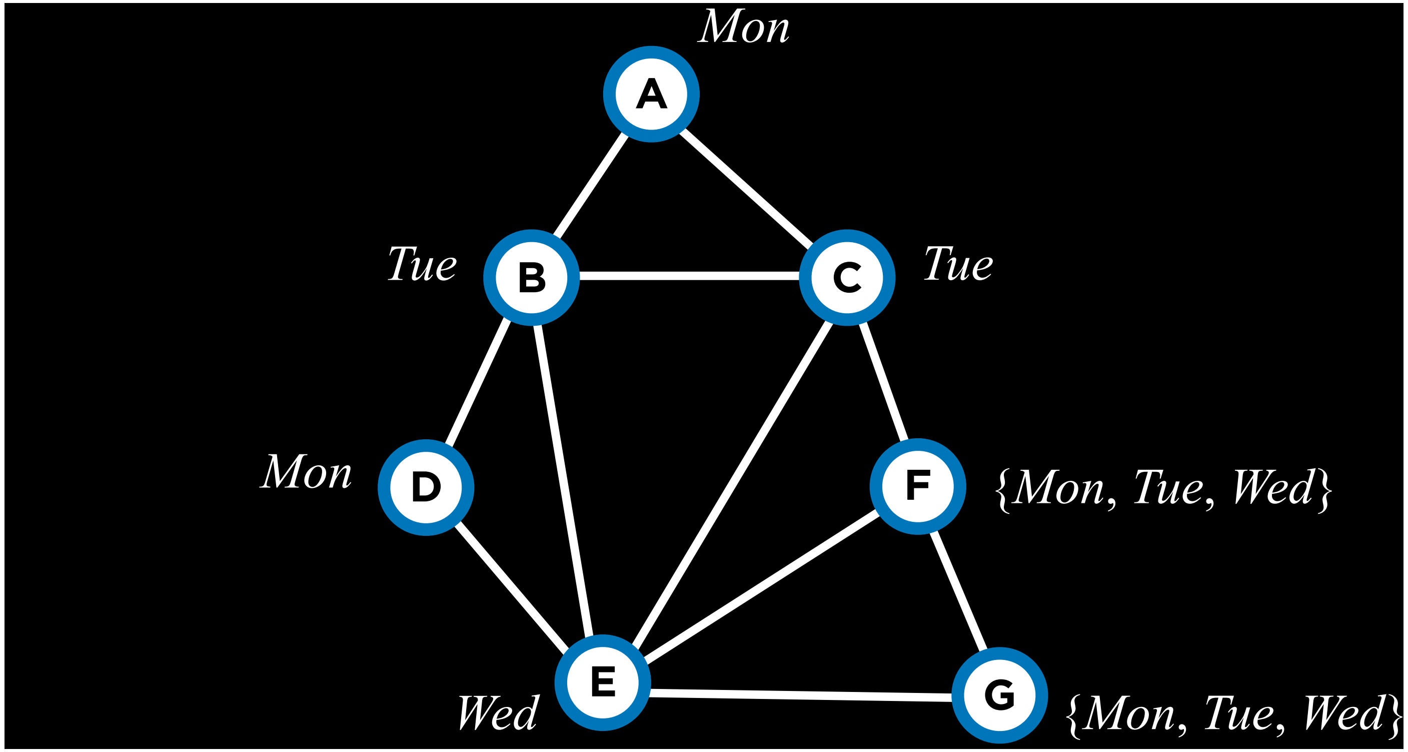

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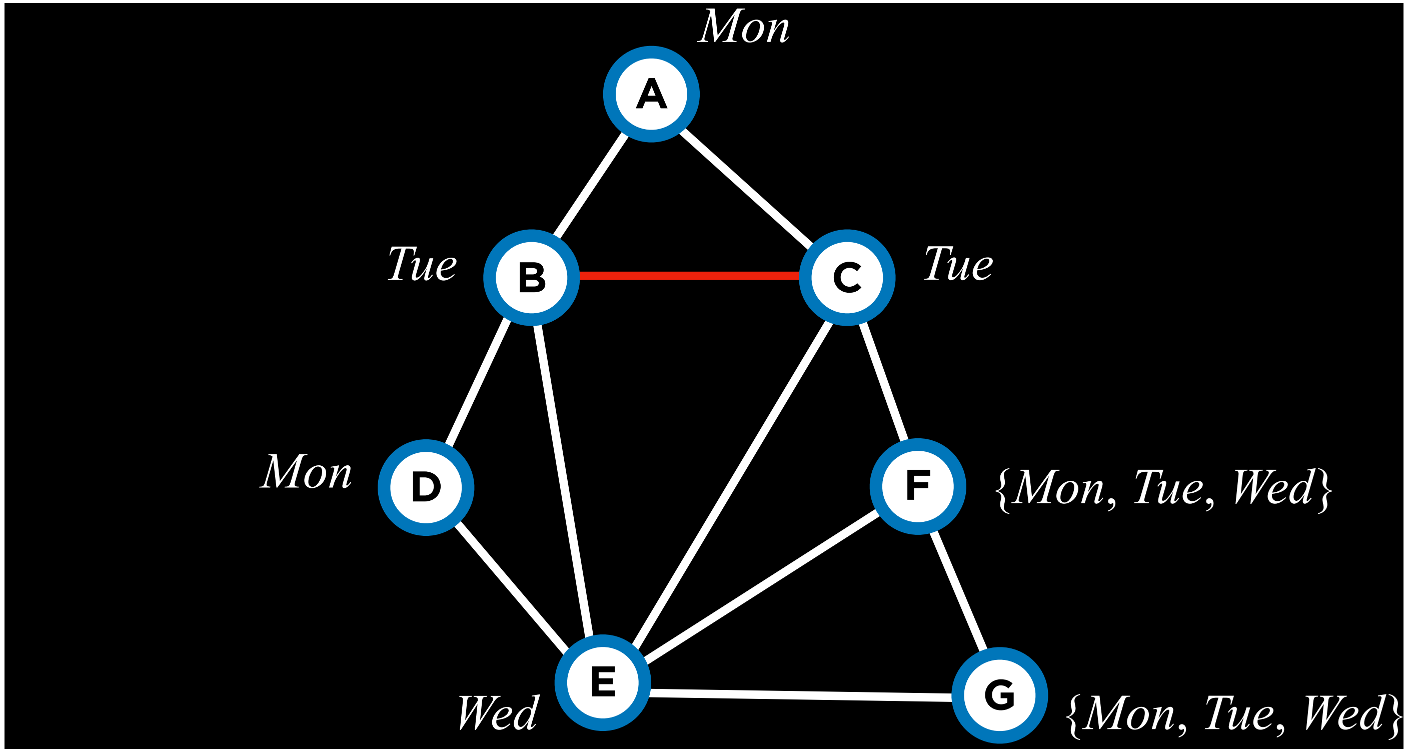

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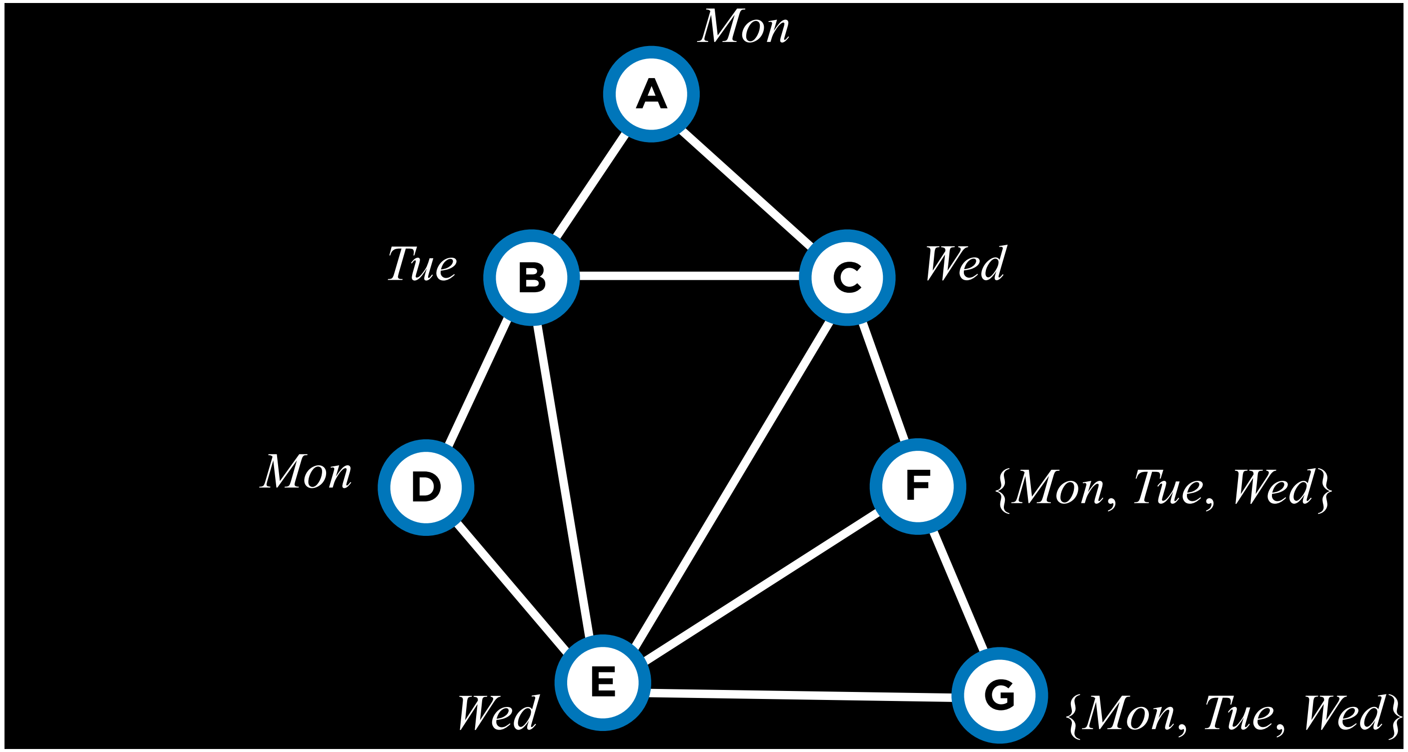

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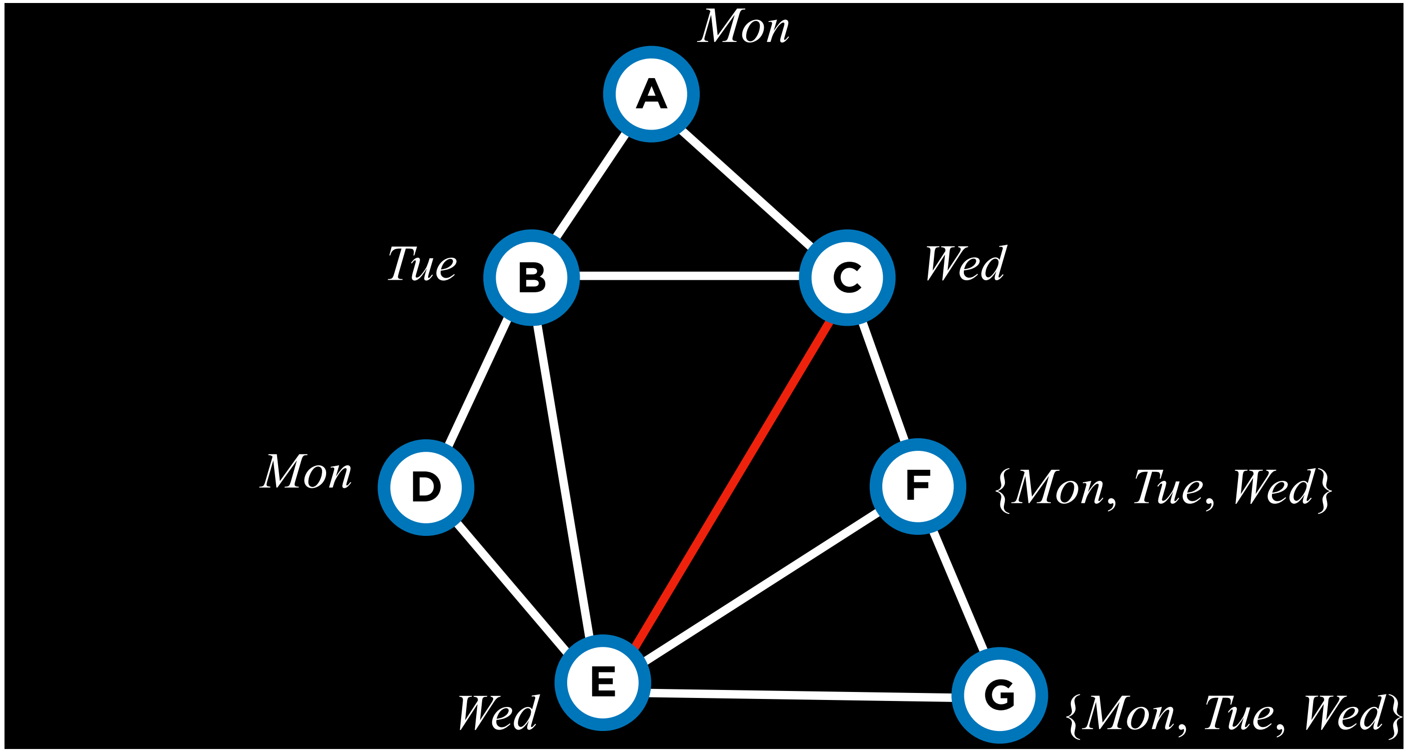

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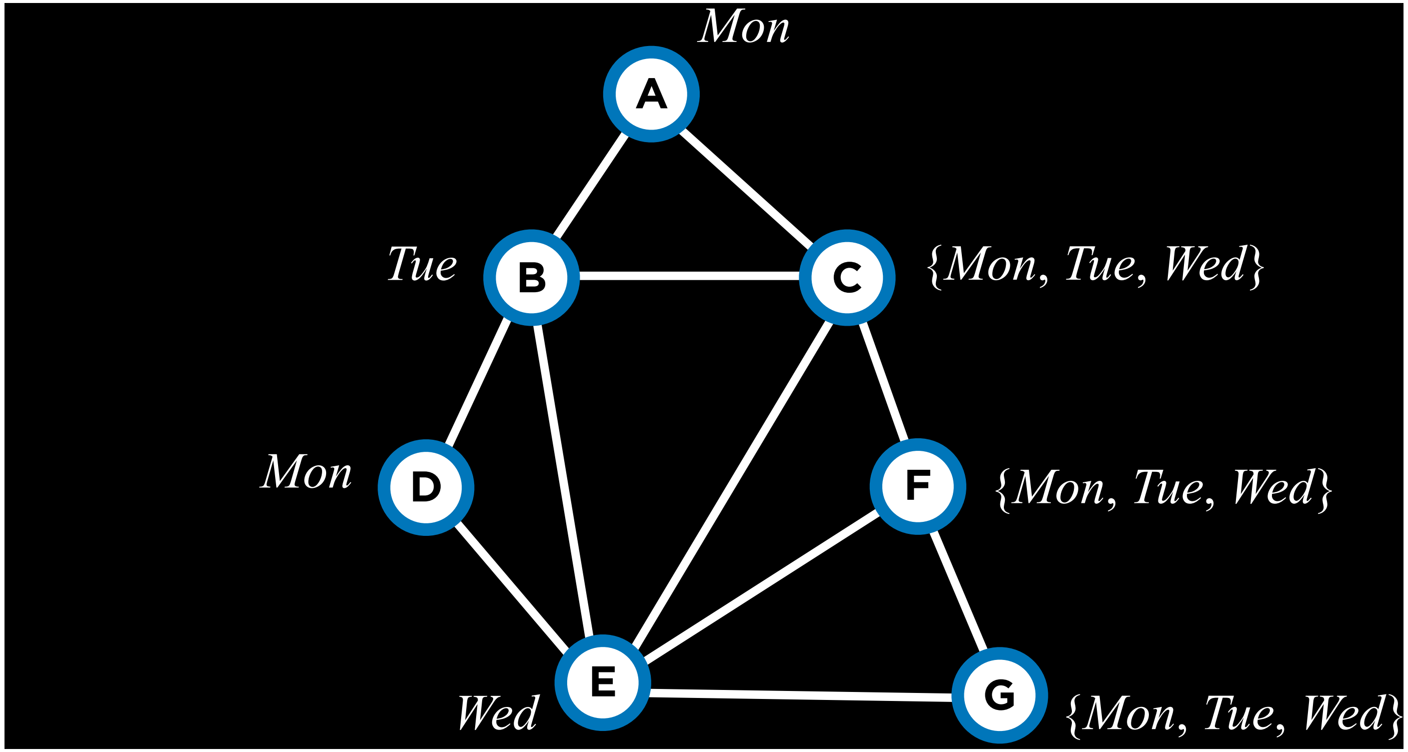

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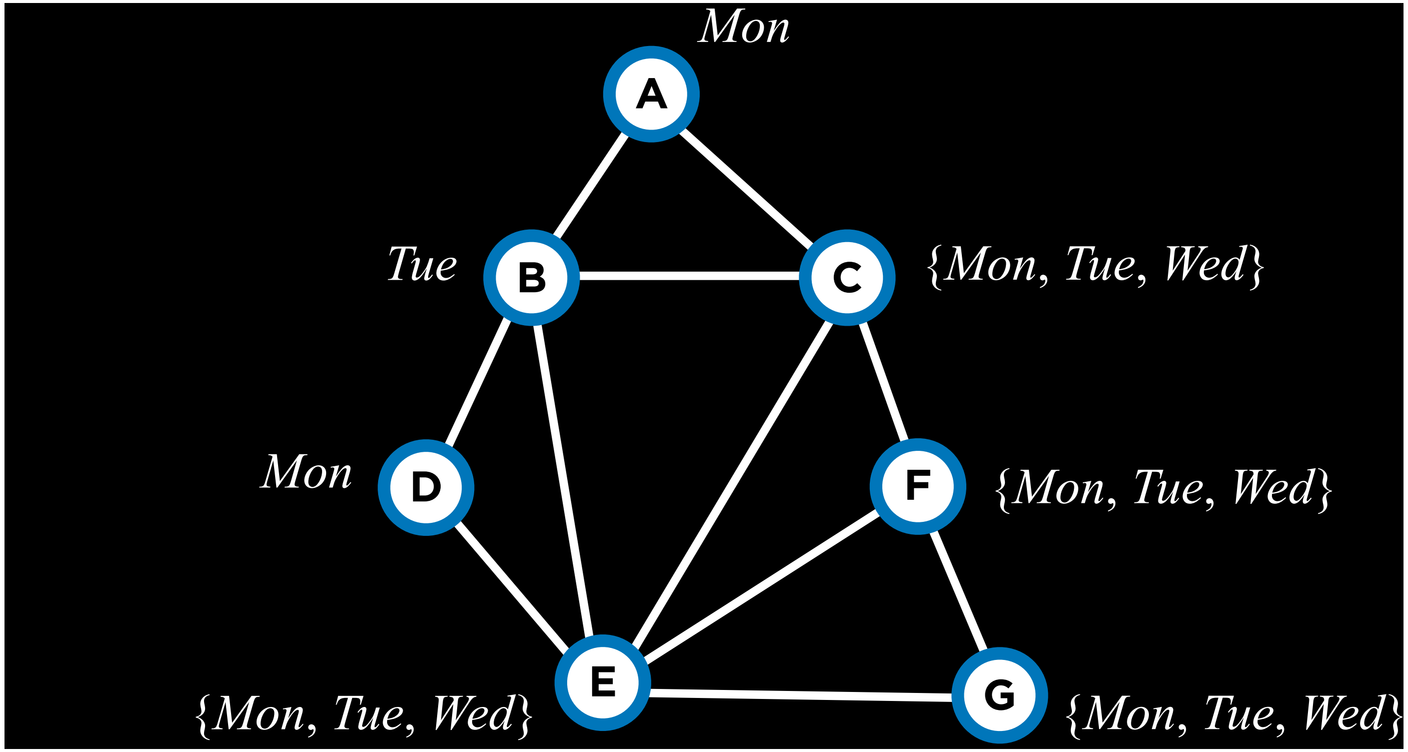

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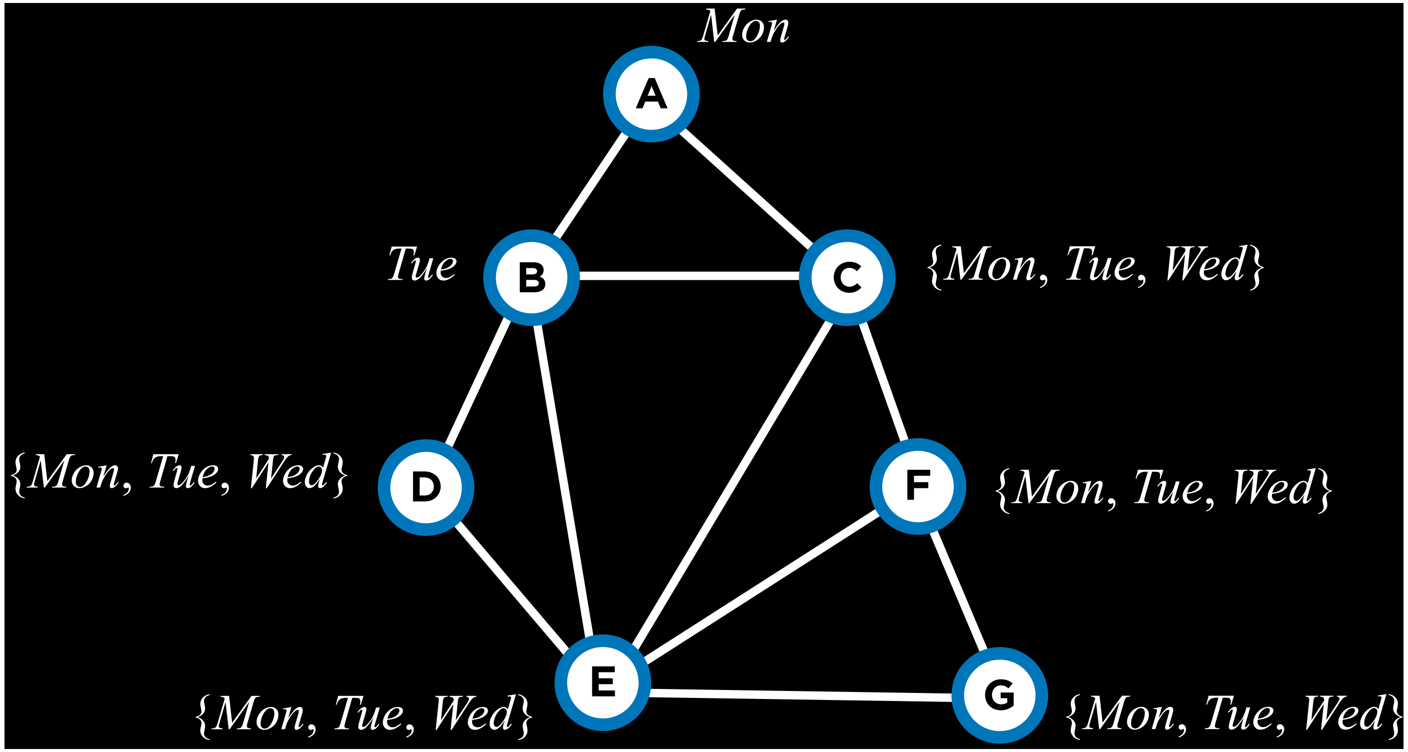

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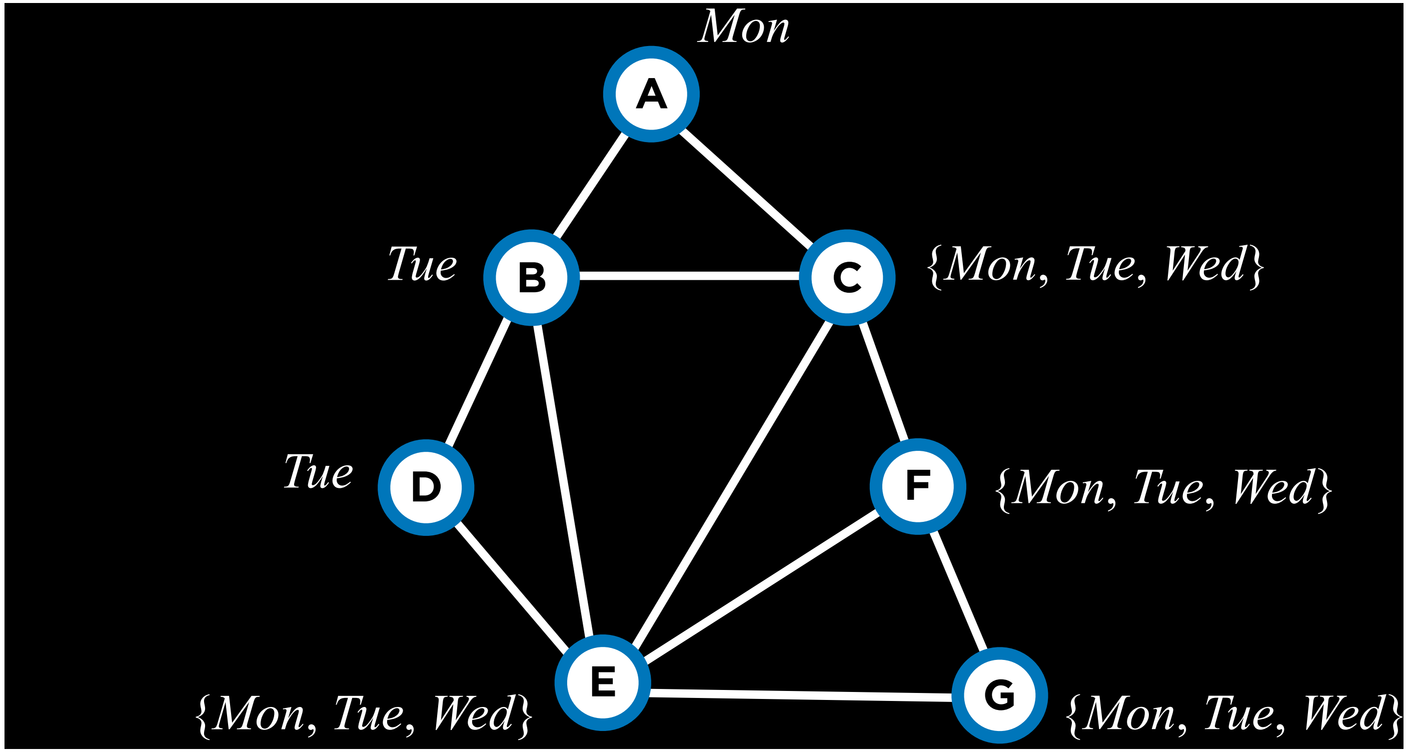

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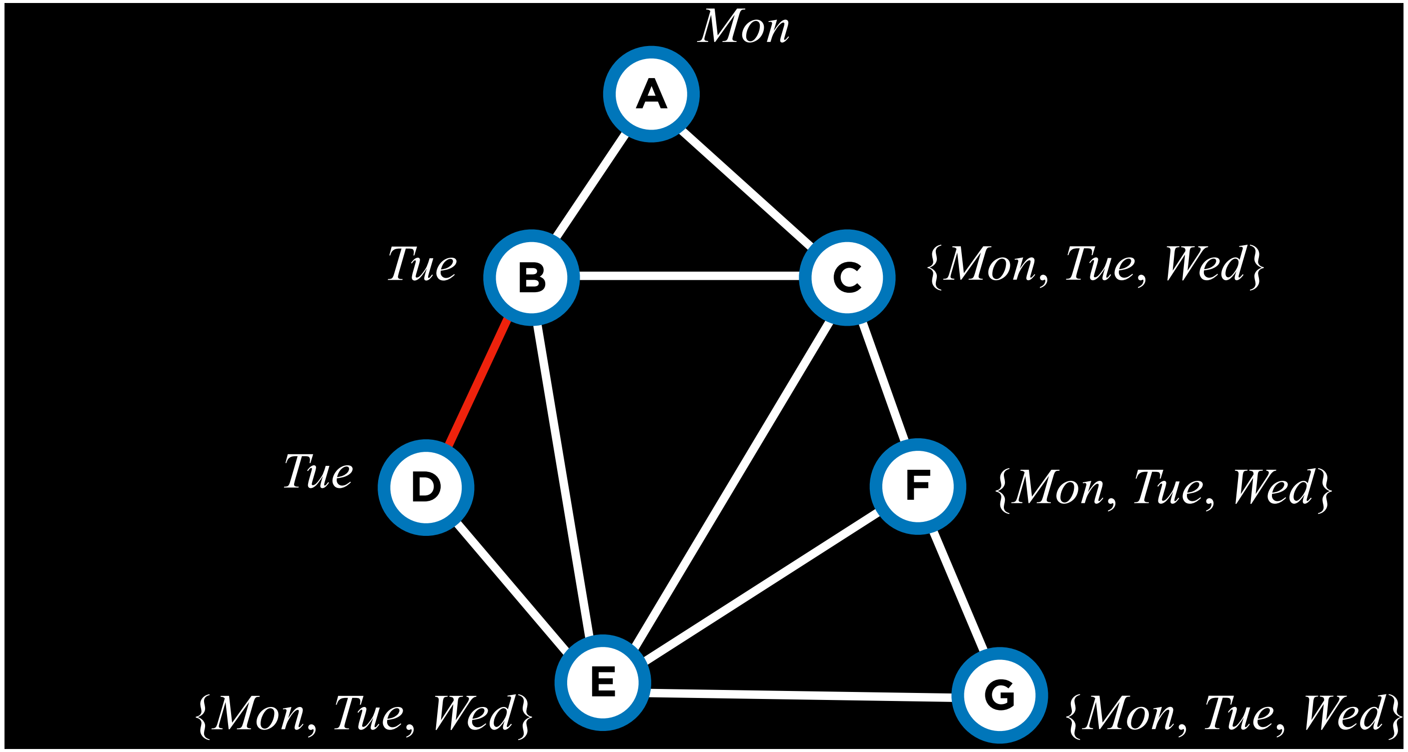

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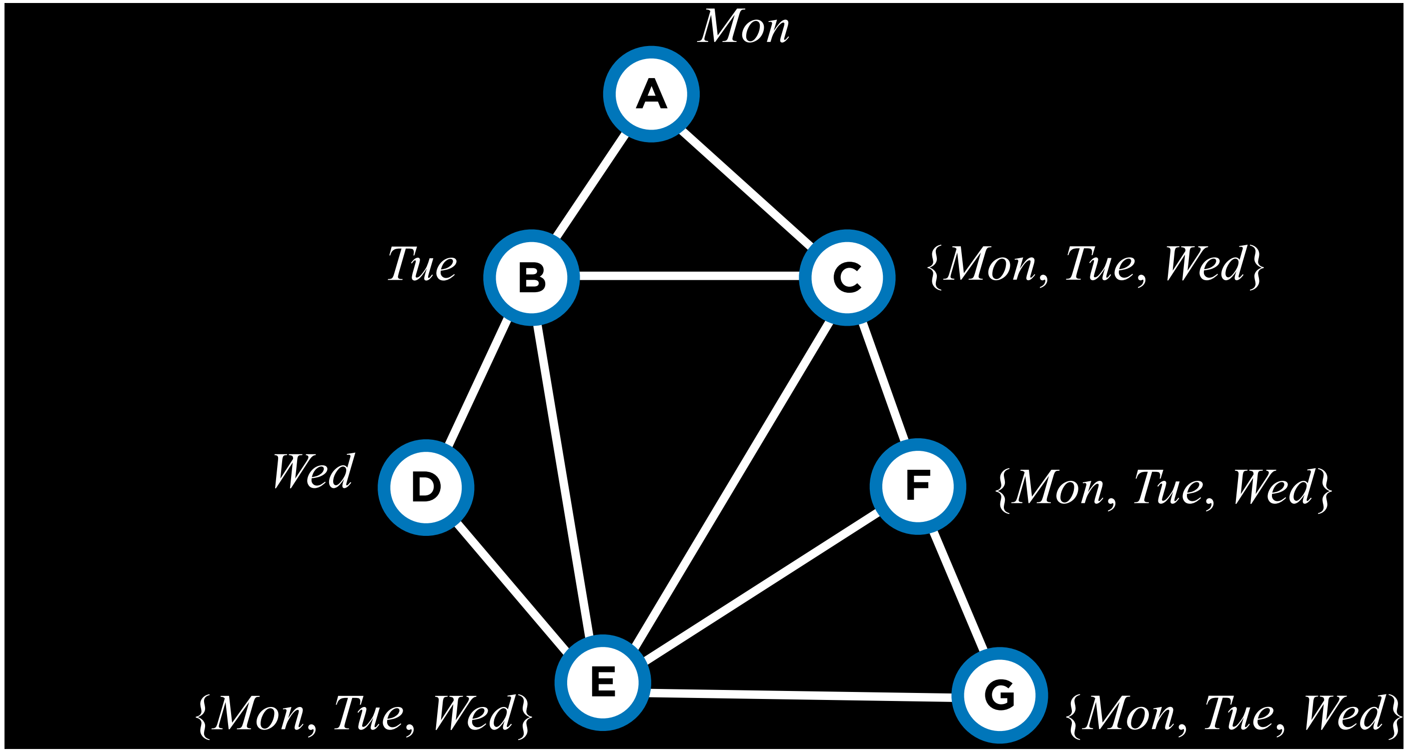
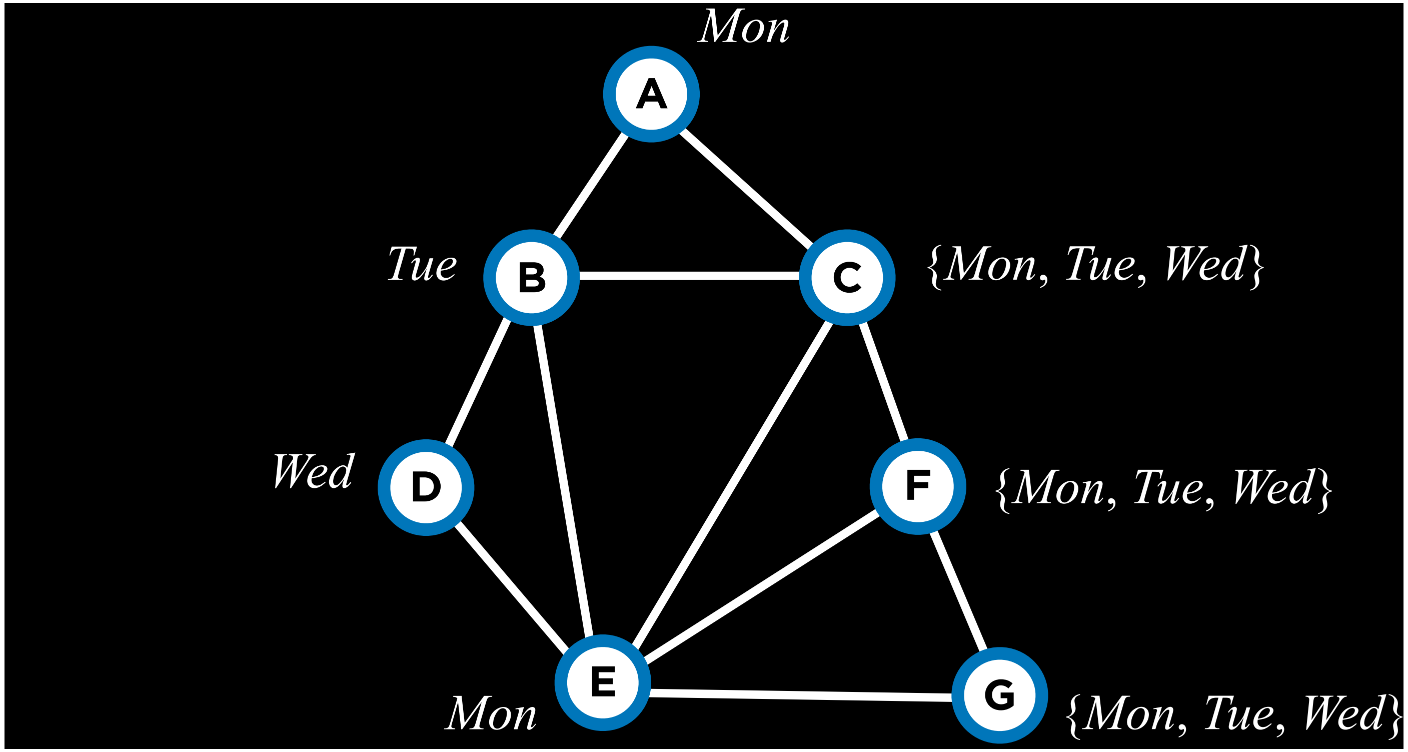
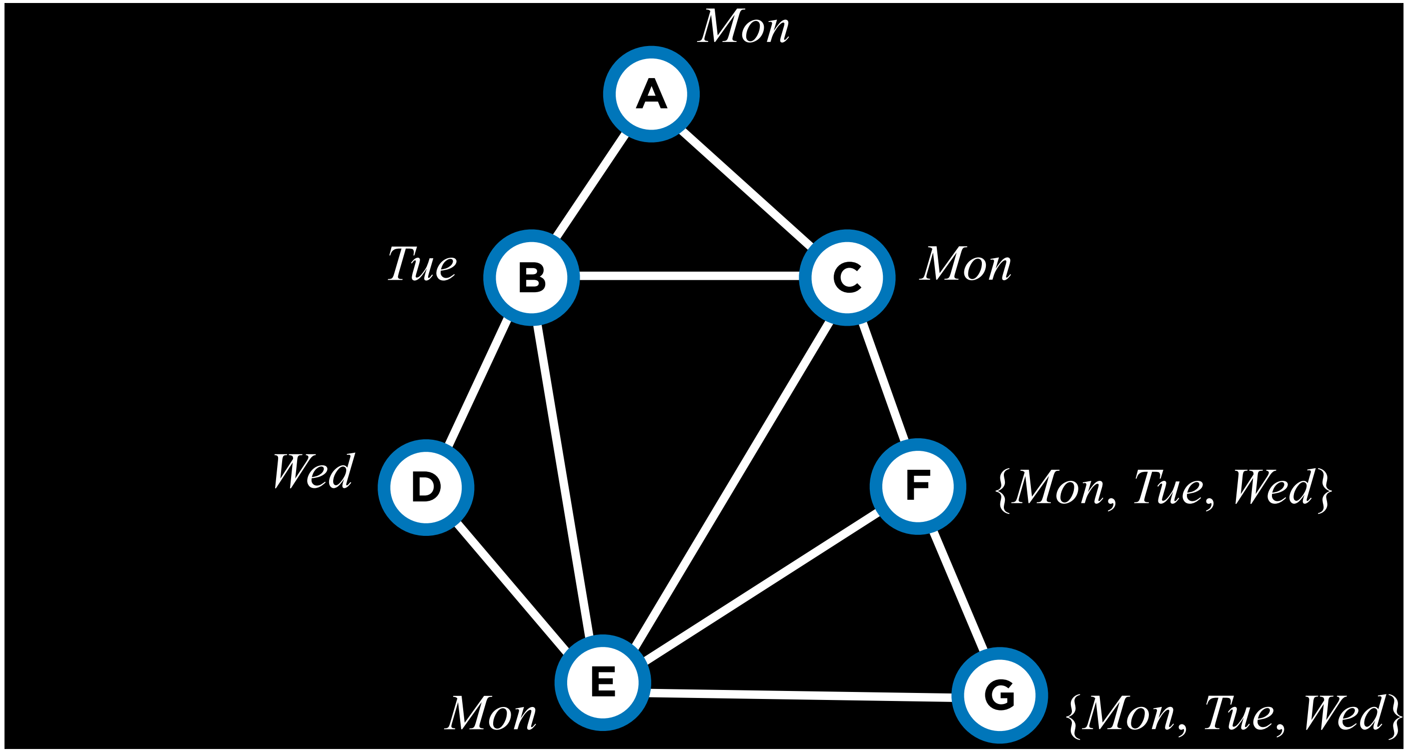


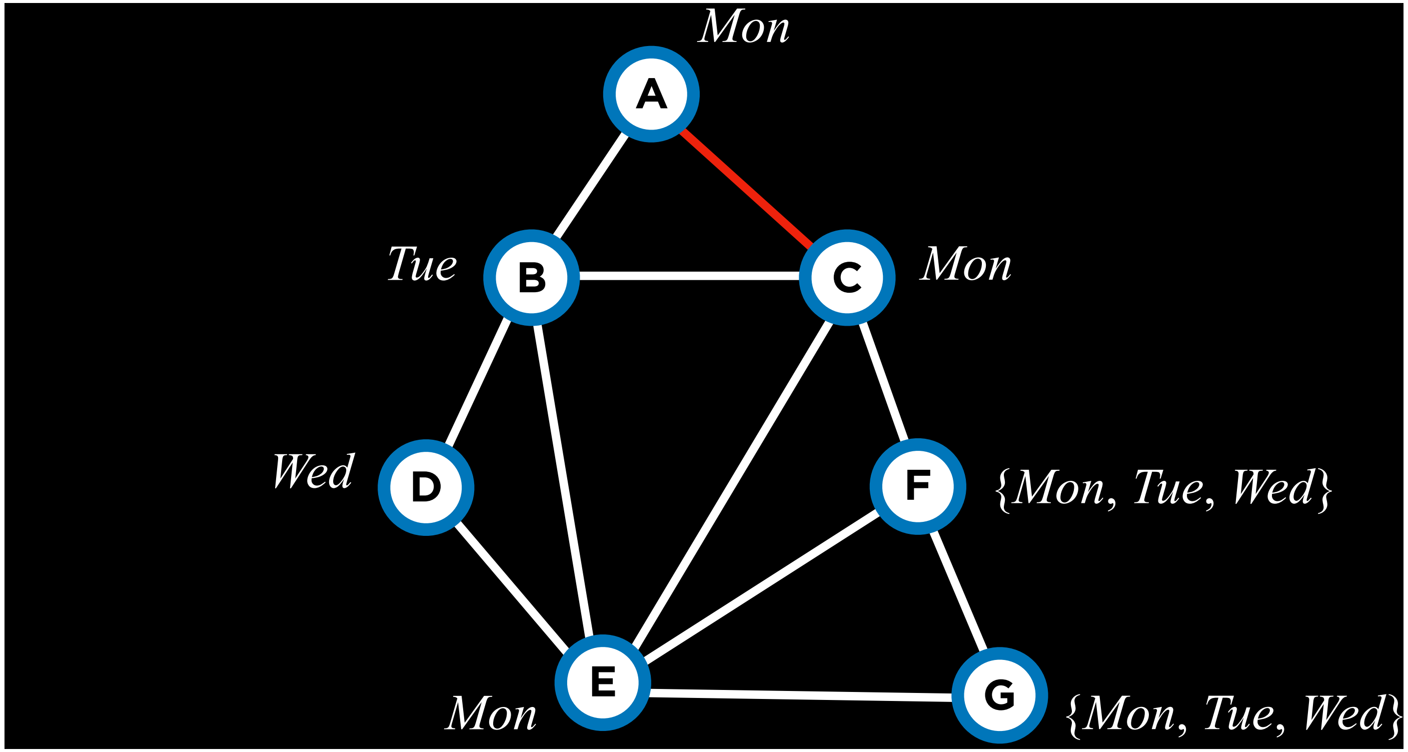
Backtracking in Practice



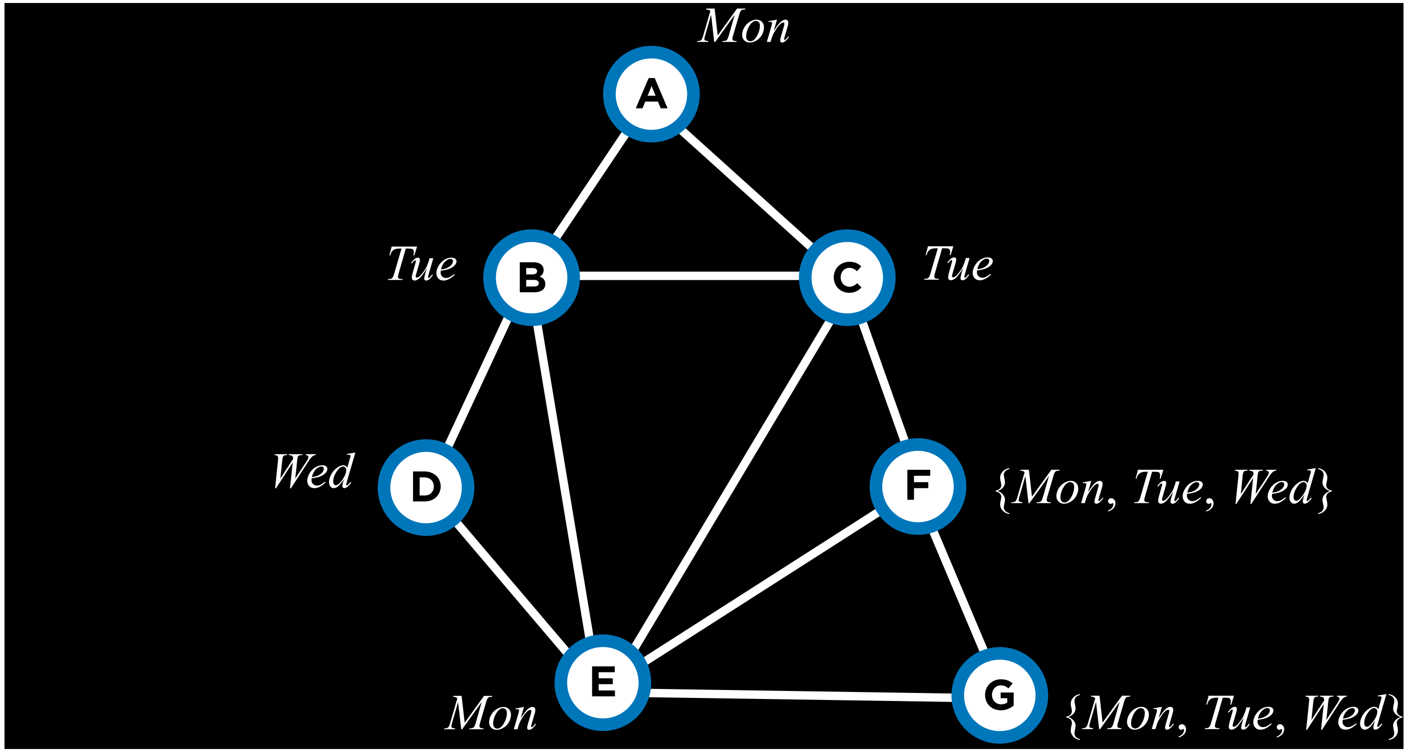
Backtracking in Practice



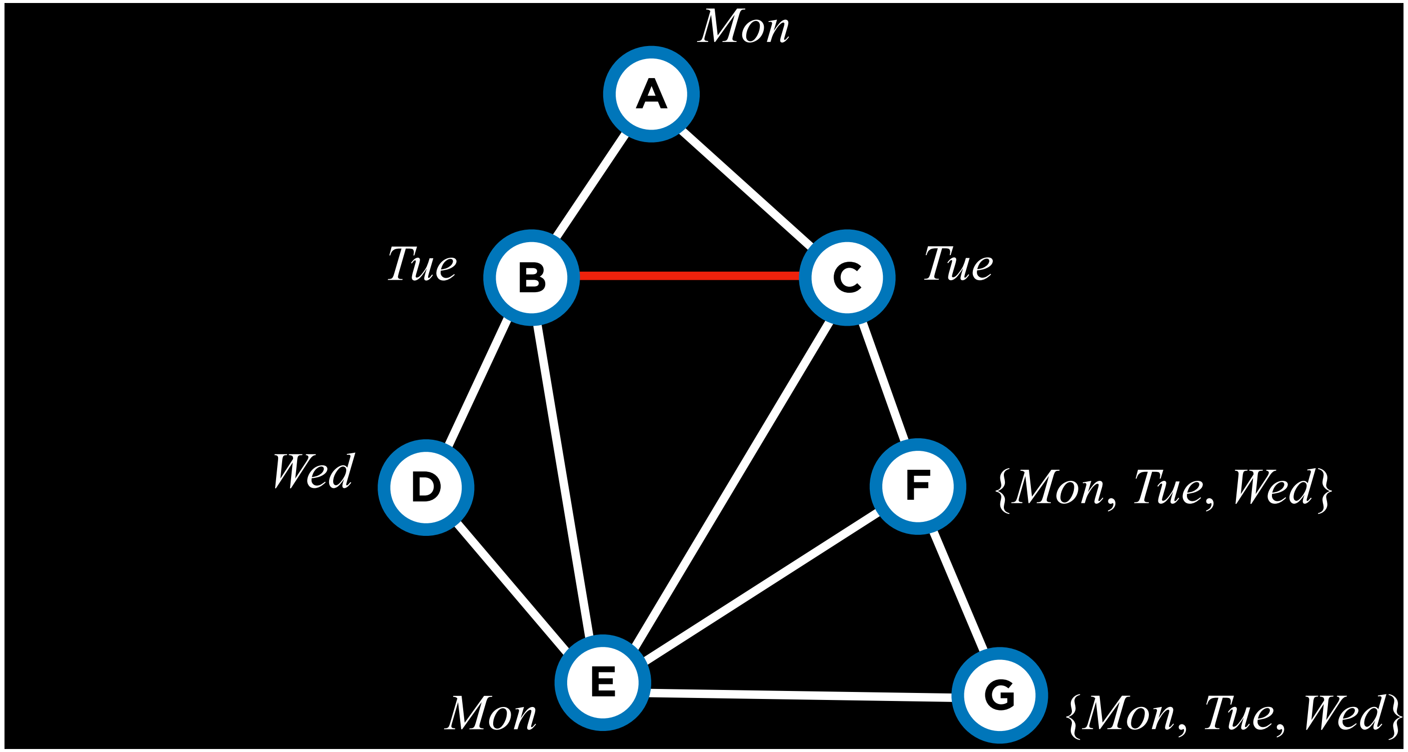
Backtracking in Practice



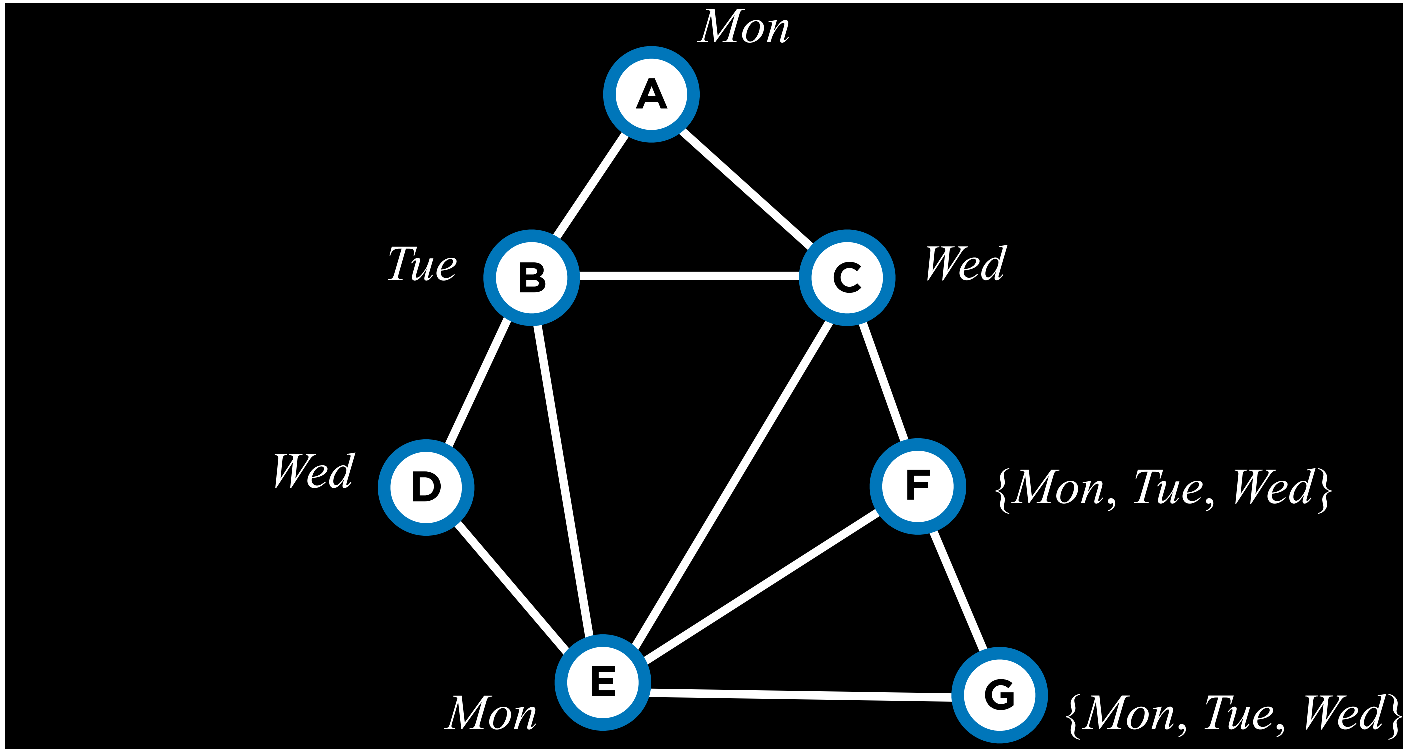
Backtracking in Practice



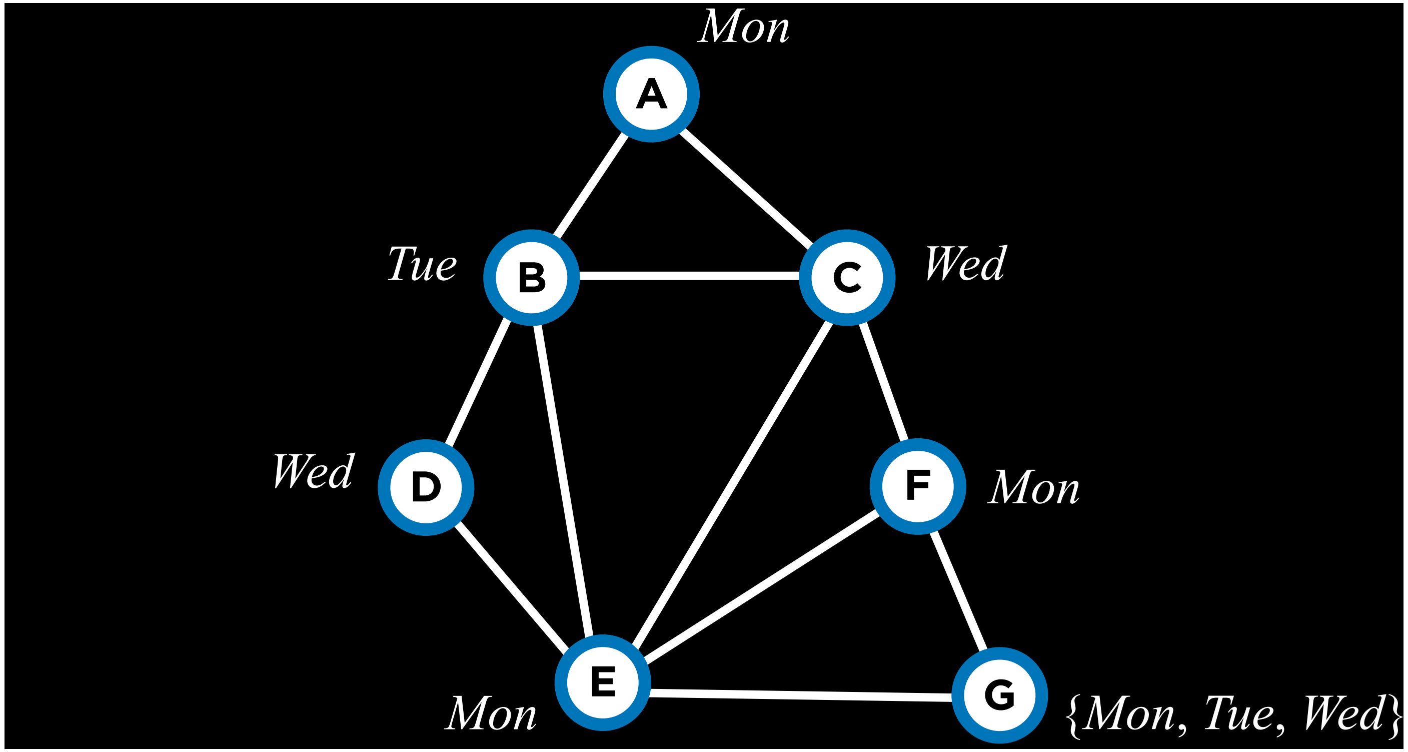
Backtracking in Practice



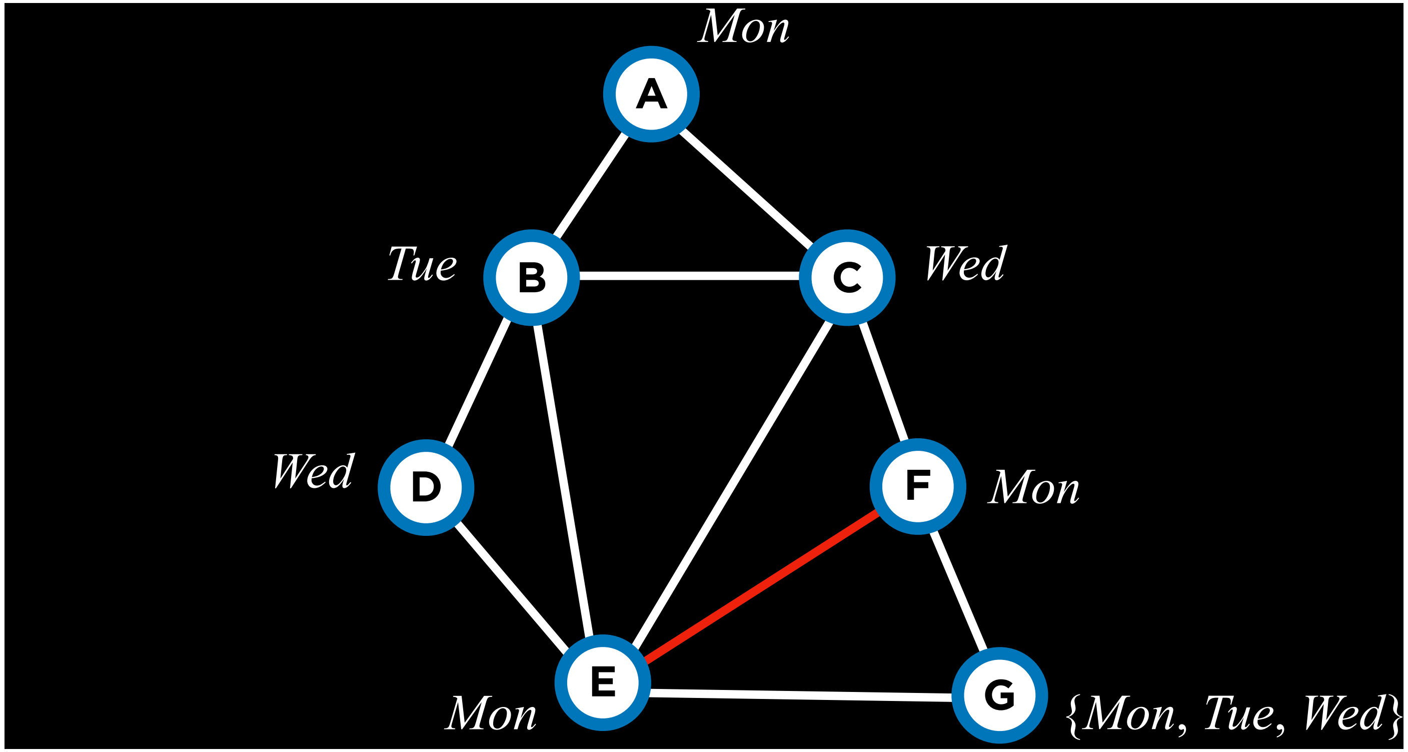
Backtracking in Practice



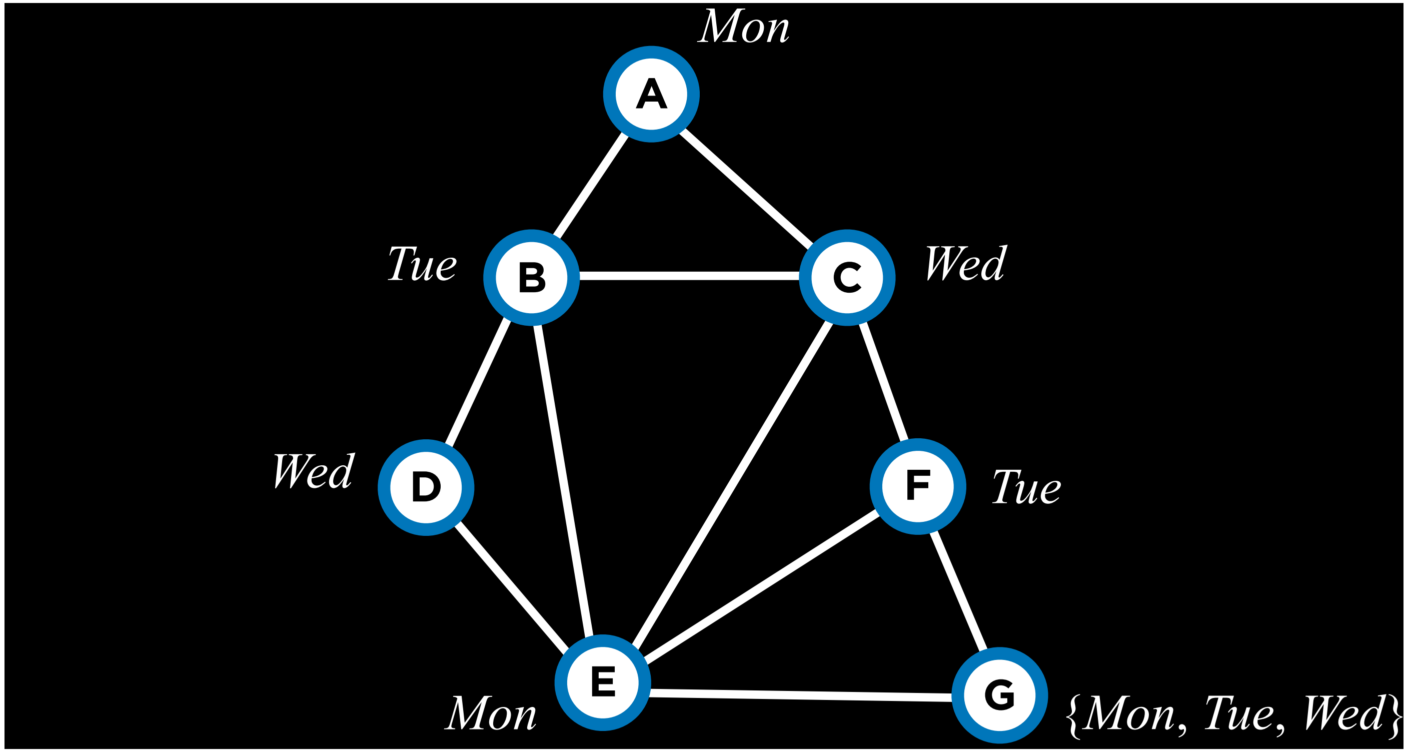
Backtracking in Practice



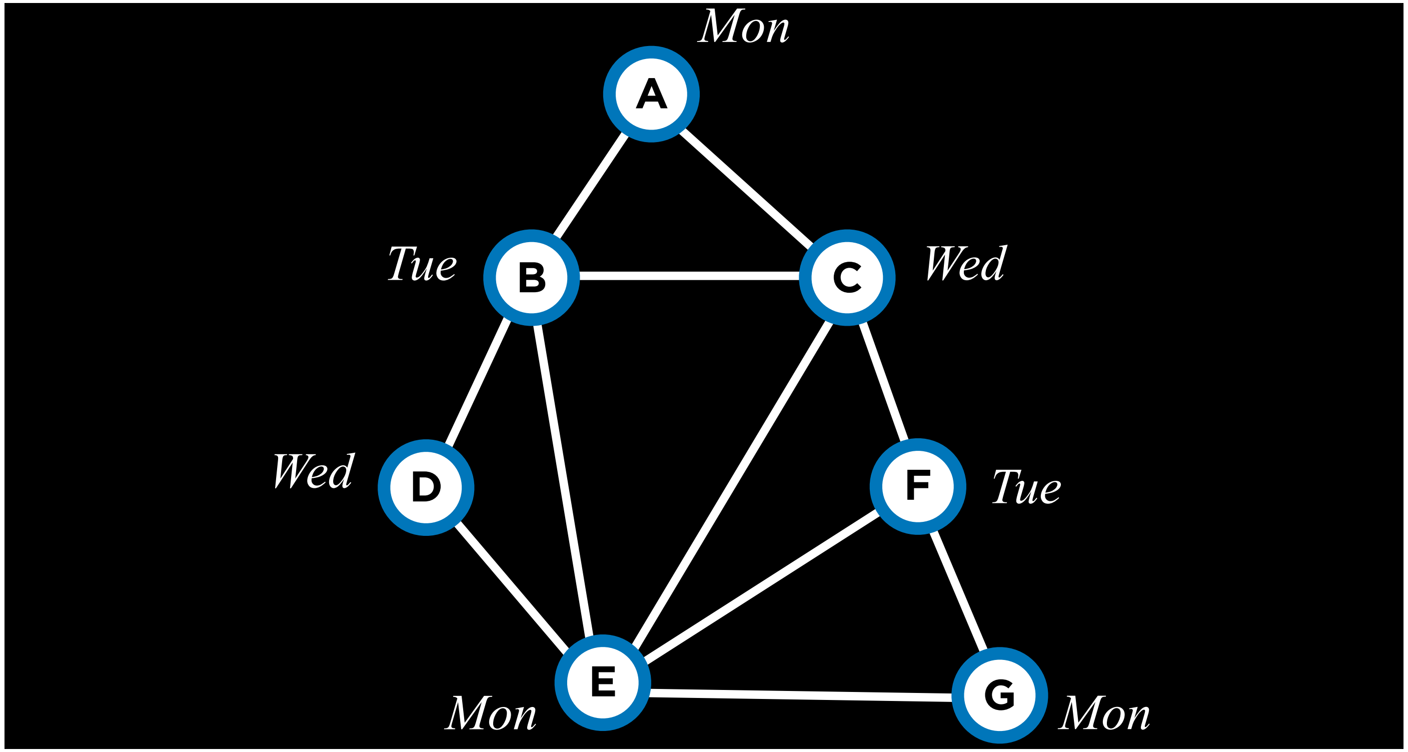
Backtracking in Practice



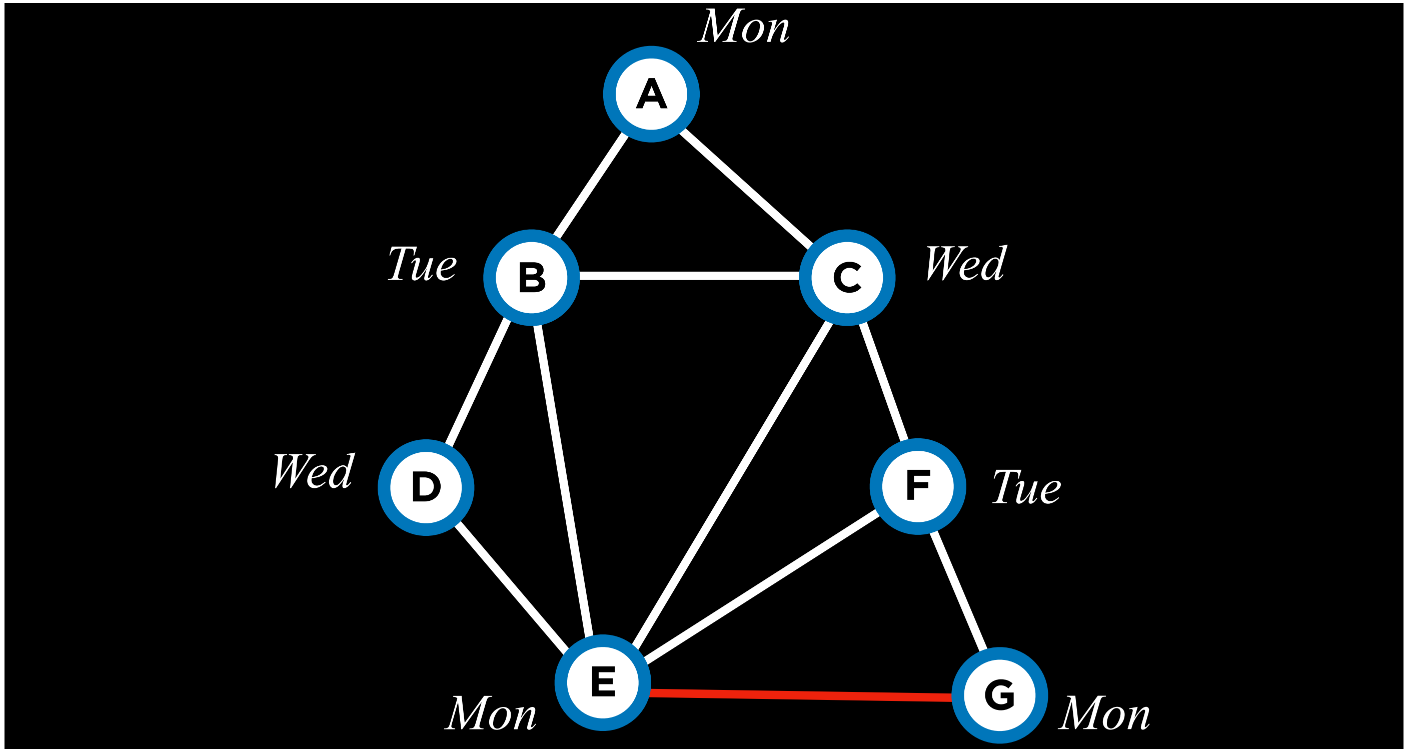
Backtracking in Practice



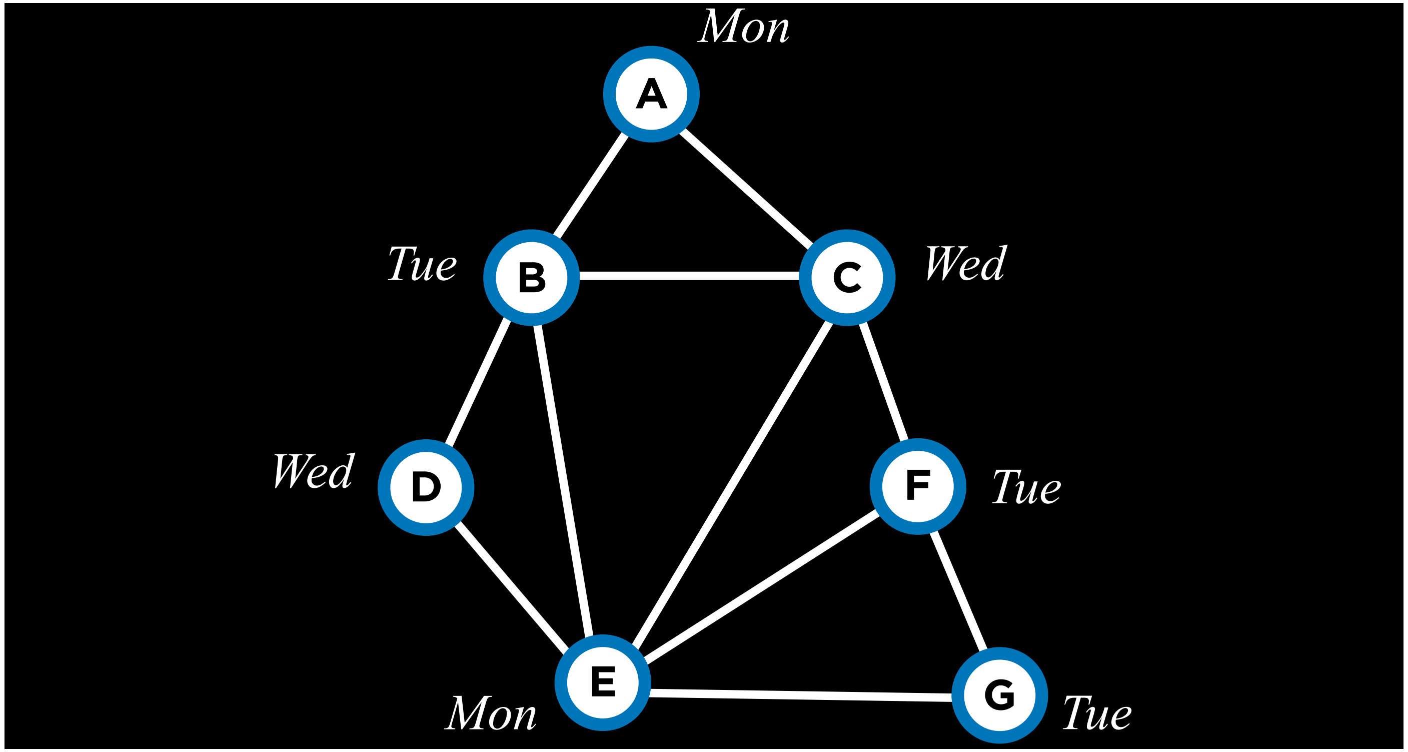
Backtracking in Practice



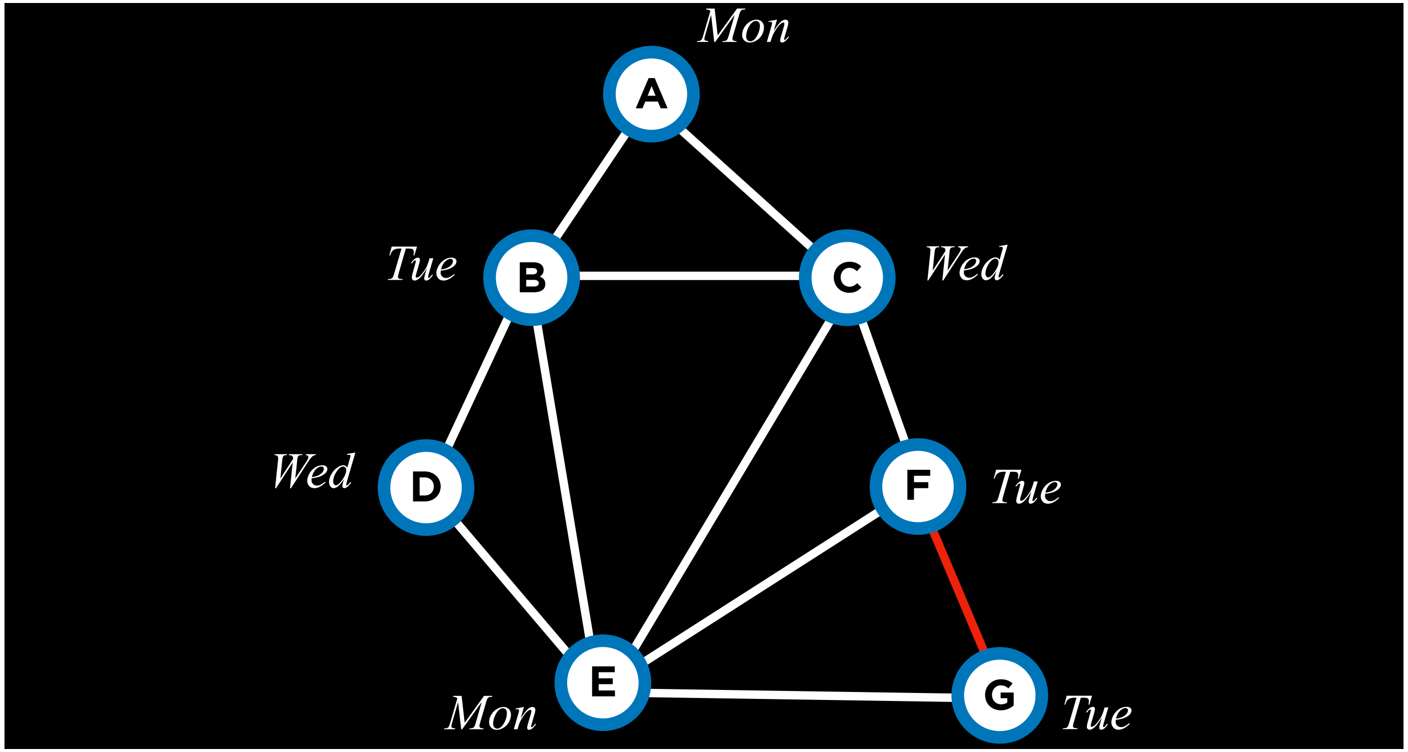
Backtracking in Practice



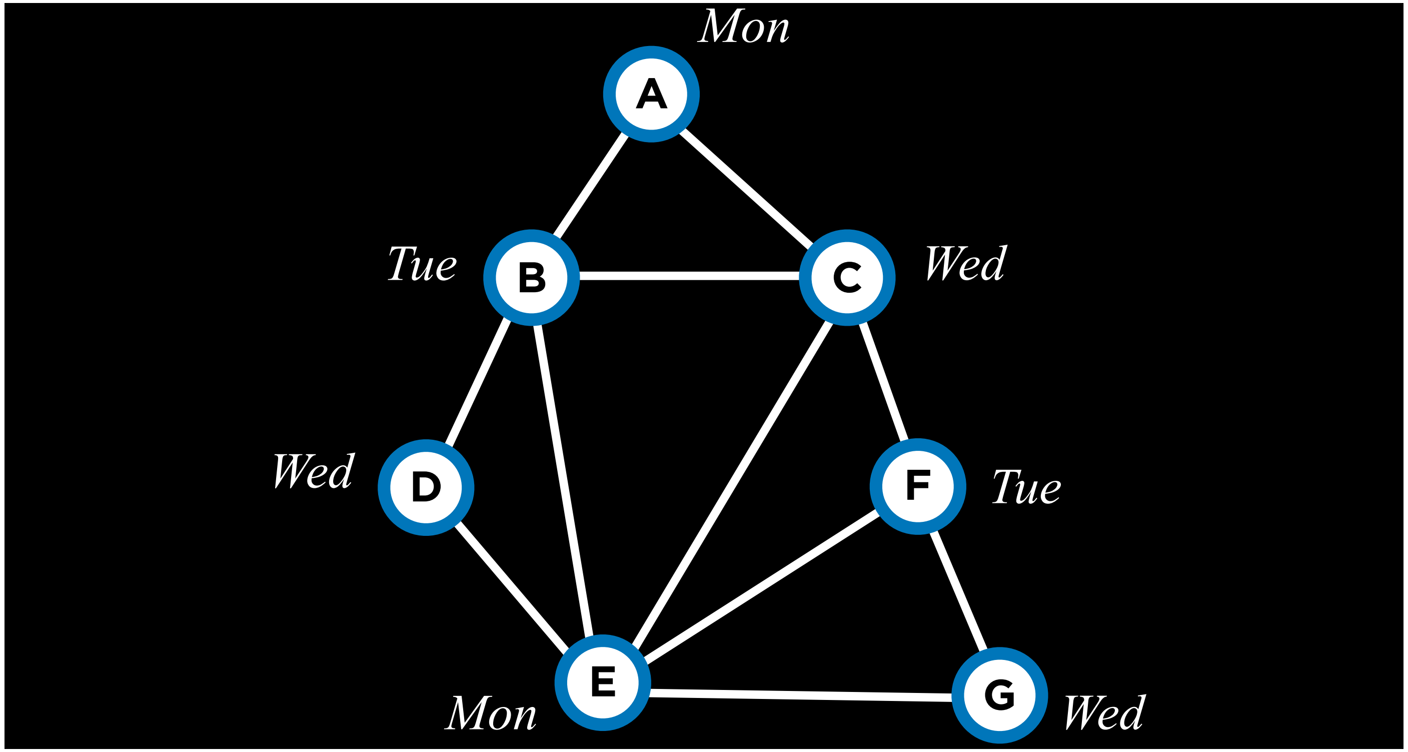
Backtracking in Practice



Backtracking in Practice

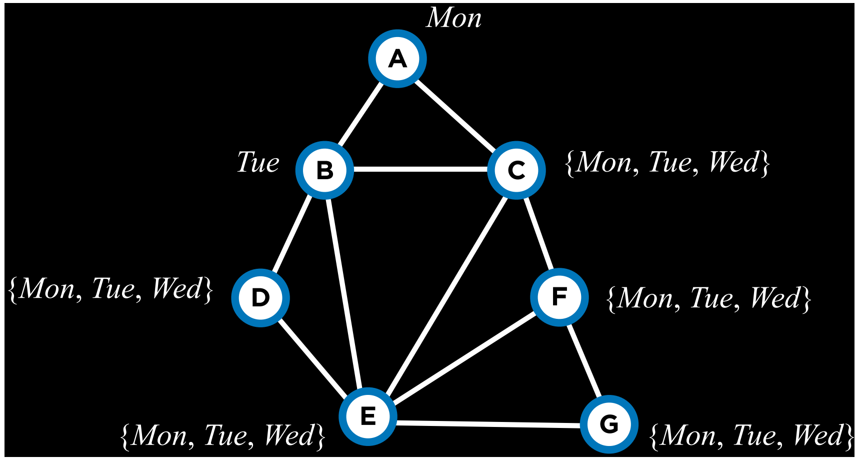


Backtracking in Practice

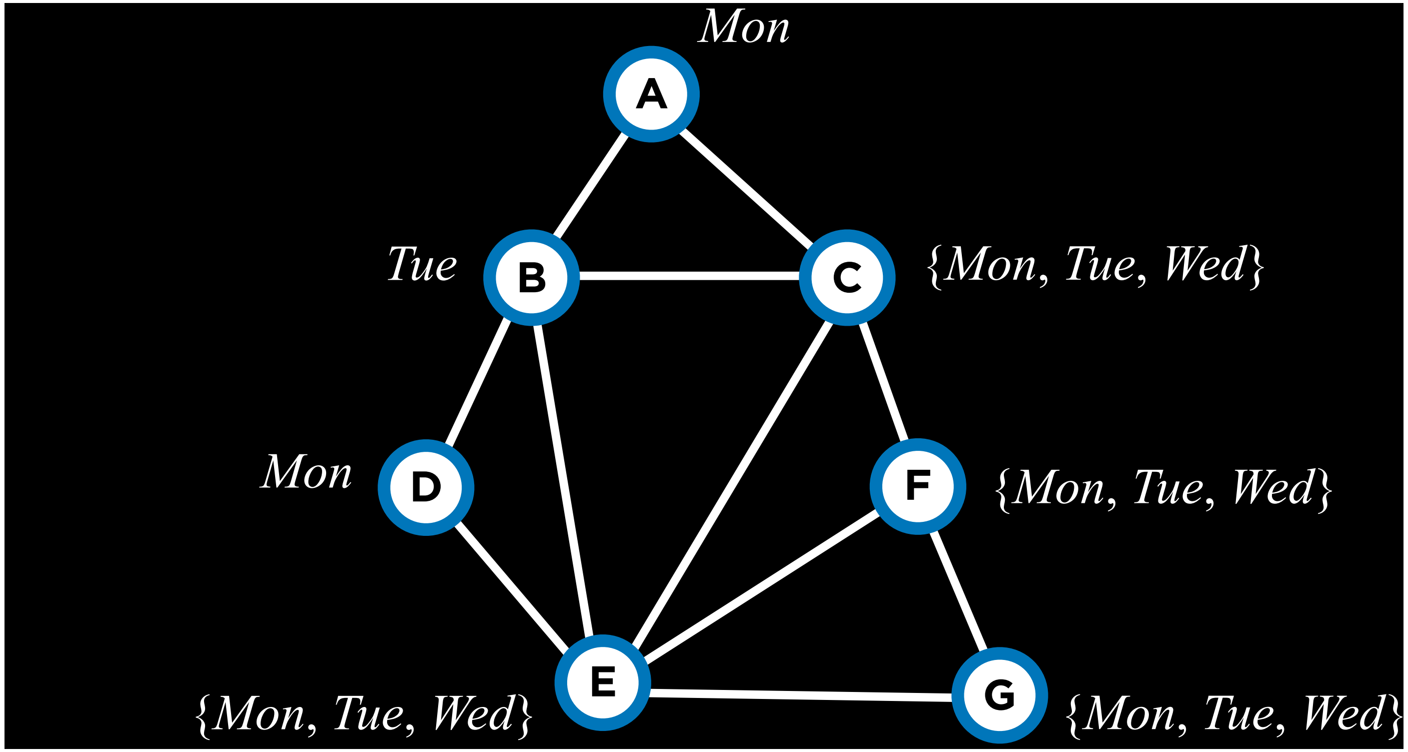


Inference

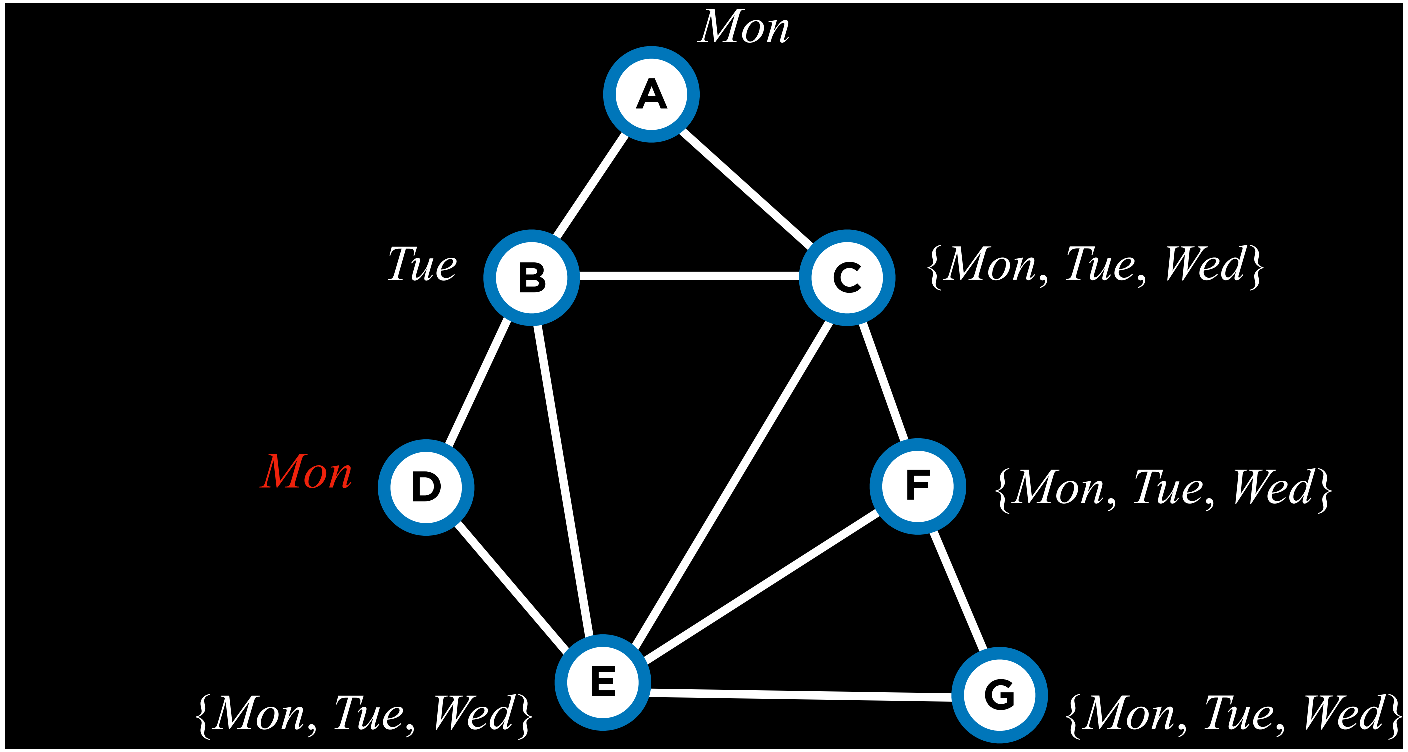
- We might be clever in order to improve the efficiency of how we solve these sorts of problems
- The idea is that of inference, using our problem knowledge to draw conclusions in order to make the rest of the problem-solving process easier
- Let's go back to where we got stuck the first time
  - We dealt with B and then we went on to D



Inference

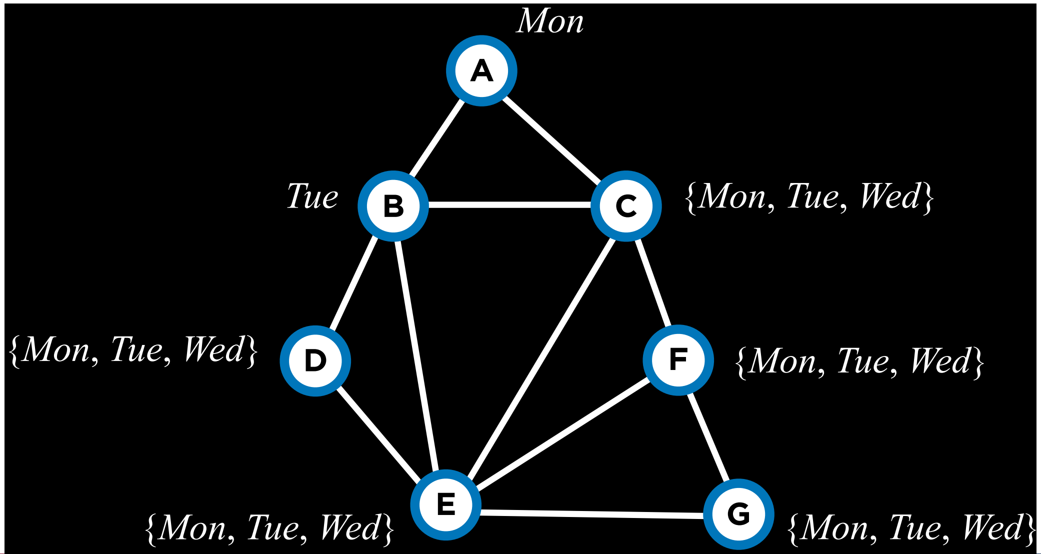

Mon
A
Tue
B
C
{Mon, Tue, Wed}
Mon
D
F
{Mon, Tue, Wed}
E
{Mon, Tue, Wed}
G
{Mon, Tue, Wed}

Inference


Mon
A
Tue
B
C
{Mon, Tue, Wed}
Mon
D
F
{Mon, Tue, Wed}
E
G
{Mon, Tue, Wed}
{Mon, Tue, Wed}

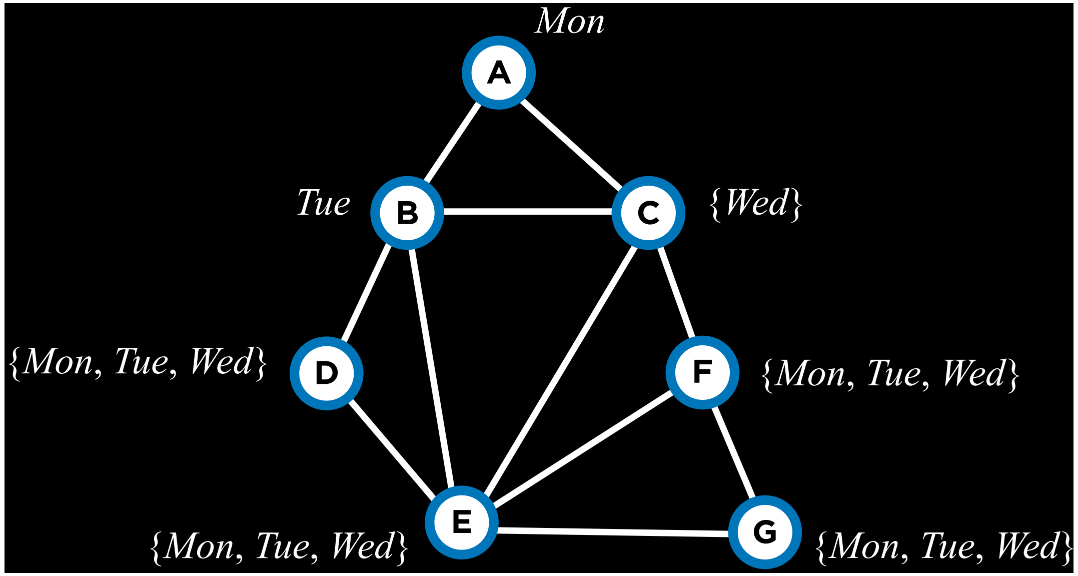
Inference

- We can look at the structure of this graph
  - For example, C's domain contains Monday and Tuesday making it not arc-consistent with A and B
  - Using that information by making C arc-consistent with A and B, we could remove Mon and Tue from C's domain and just leave C with Wed …

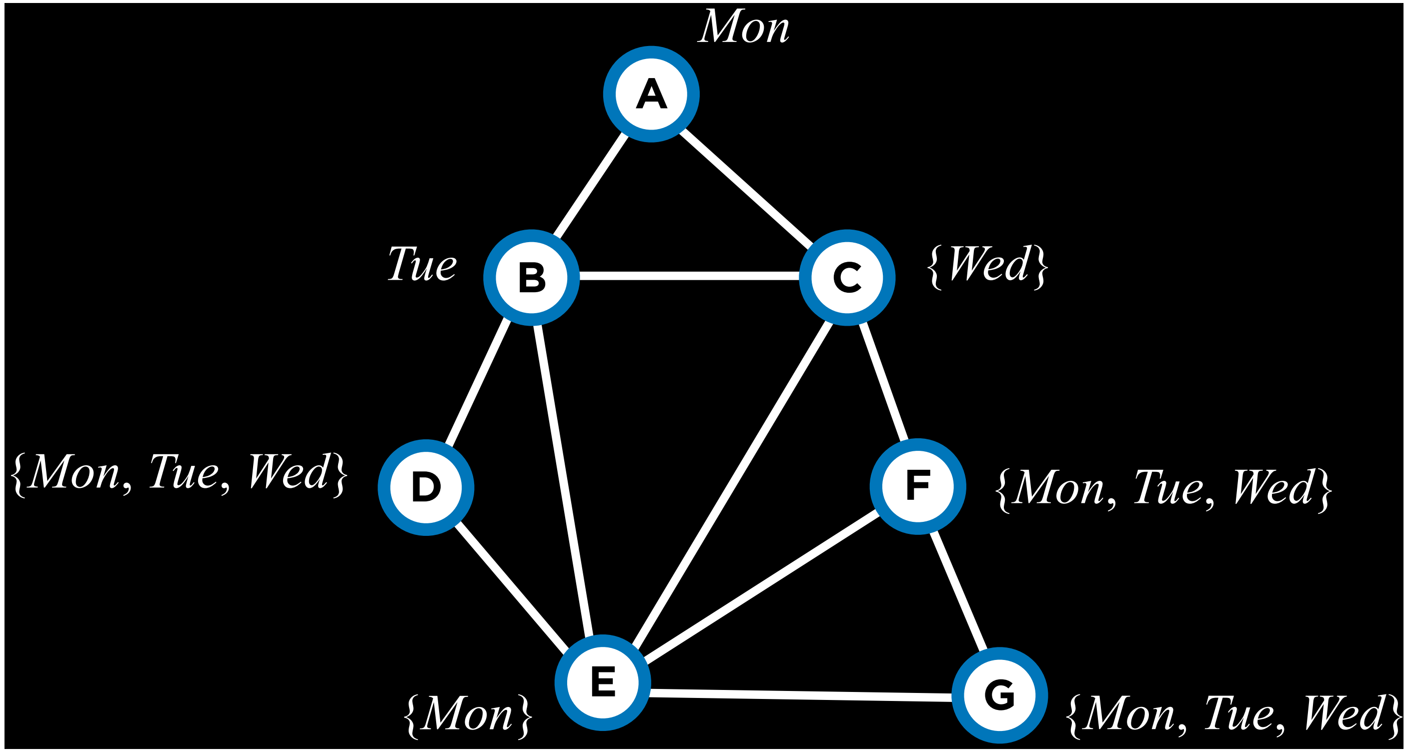


Inference

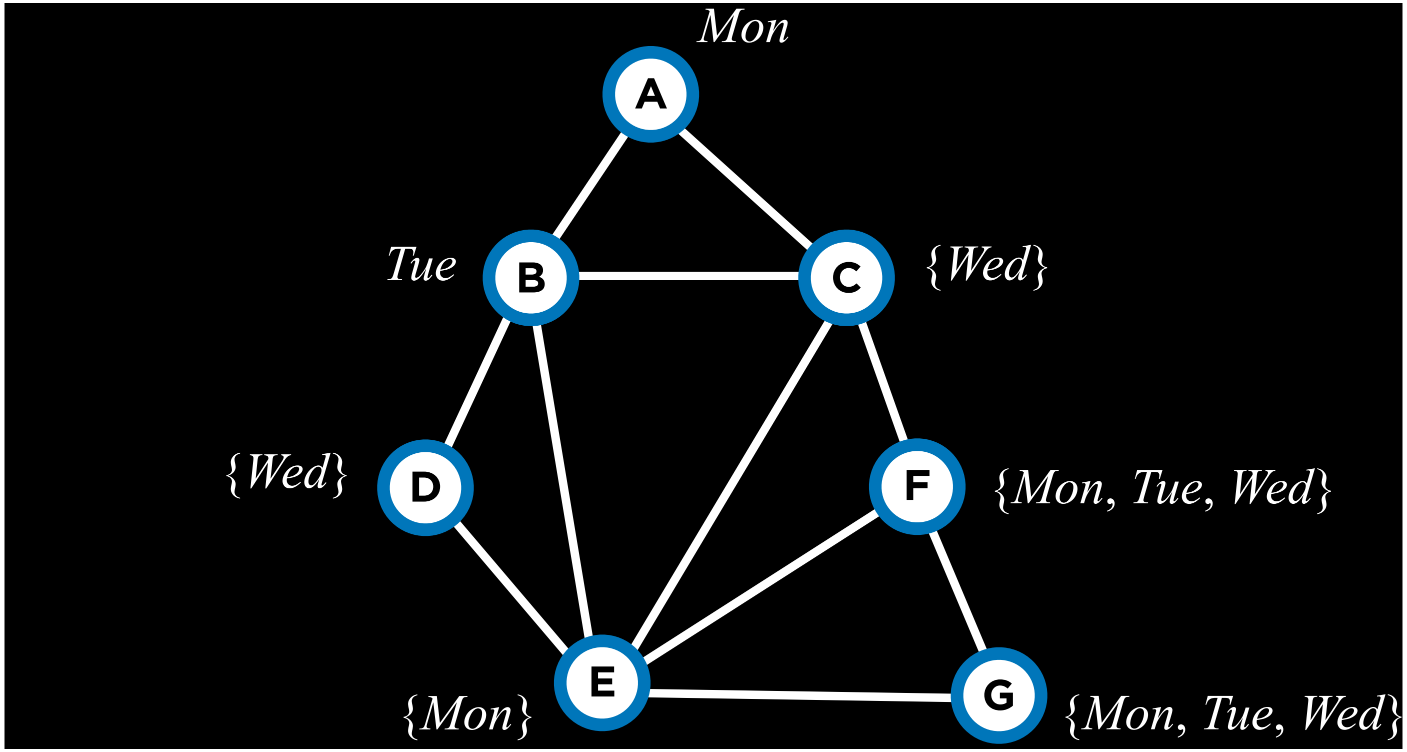
- Continuing to try and enforce arc consistency, there are some other conclusions we can draw
  - B's only option is Tue and C's only option is Wed
  - if we want to make E arc-consistent, E can't be Tue and Wed because that wouldn't be arc-consistent with B and C…



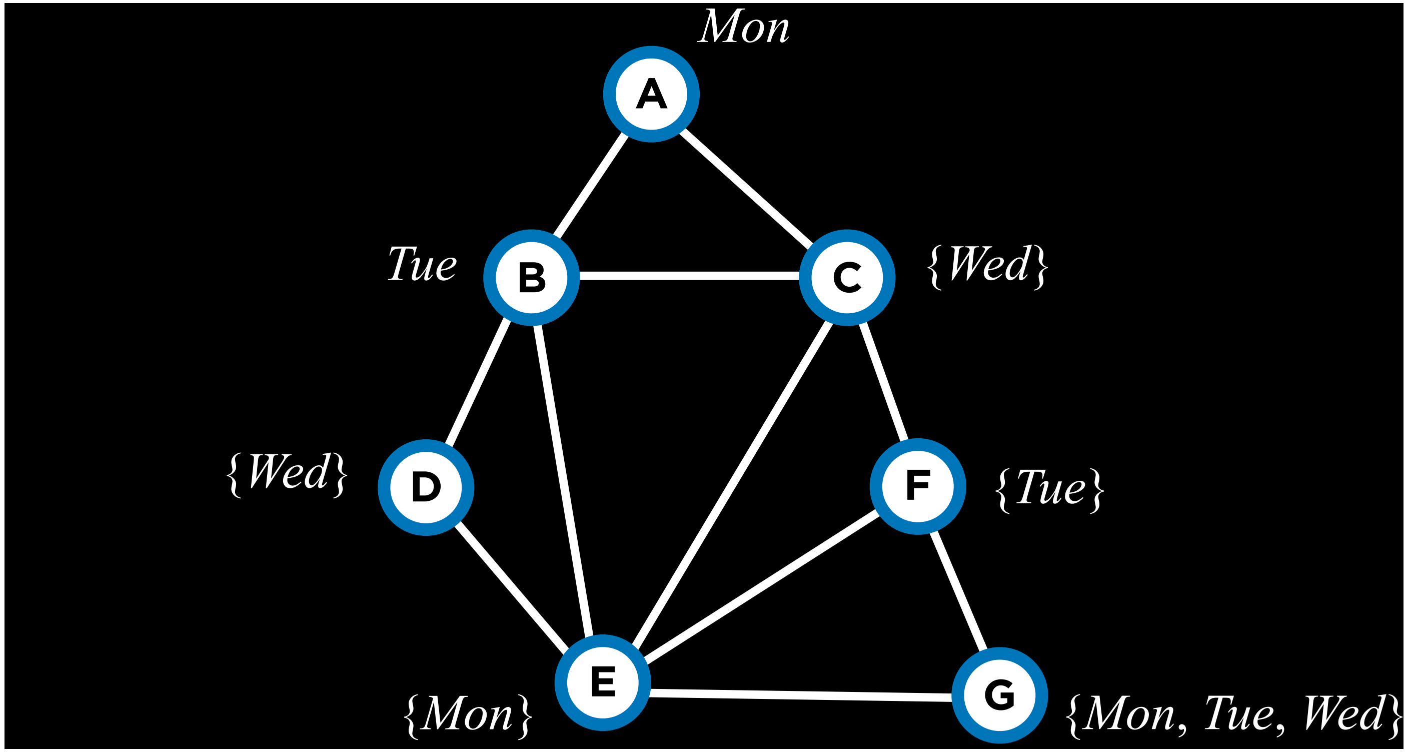
Inference



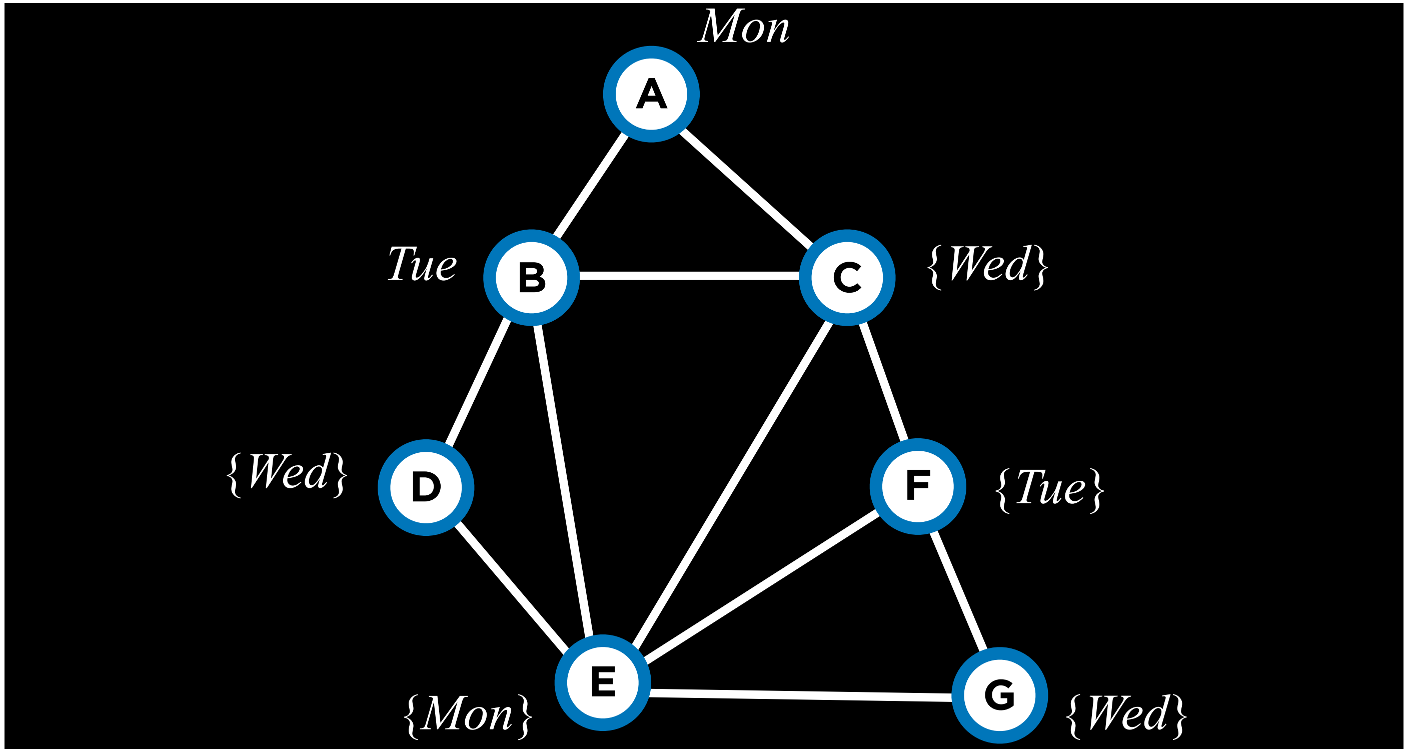
Inference



Inference

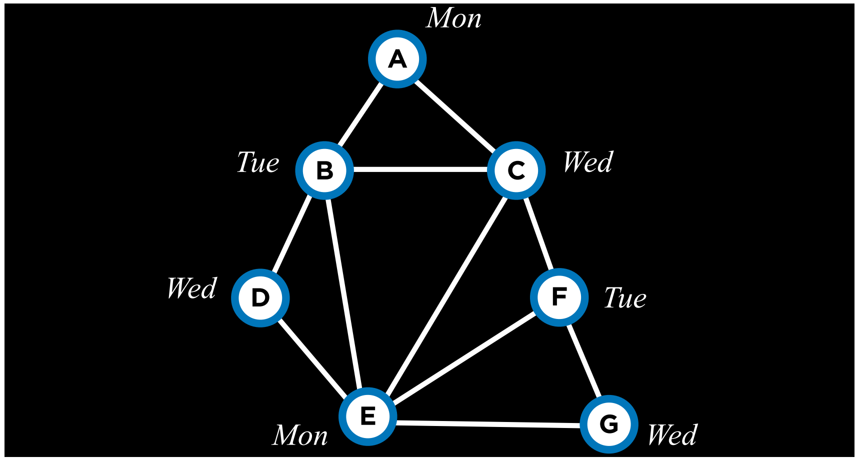


Inference



Inference

- It turns out that without having to do any additional search and backtrack, just by enforcing arc consistency, we were able to figure out what the assignment of all the variables should be
  - We interleave the search process and the inference step in trying to enforce arc consistency



Maintaining Arc-Consistency

- Algorithm for enforcing arc-consistency every time we make a new assignment
- When we make a new assignment to X, calls AC-3, starting with a queue of all arcs (Y,X) where Y is a neighbor of X
- Sometimes we can run the algorithm at the very beginning before we even begin searching to limit the domain of the variables making it easier to search

Maintaining Arc-Consistency

```
function BACKTRACK(assignment, csp):
  if assignment complete: return assignment
  var = SELECT-UNASSIGNED-VAR(assignment, csp)
  for value in DOMAIN-VALUES(var, assignment):
    if value consistent with assignment:
      add {var = value} to assignment
      inferences = INFERENCE(assignment, csp)
      if inferences ≠ failure: add inferences to assignment
      result = BACKTRACK(assignment, csp)
      if result ≠ failure: return result
    remove {var = value} and inferences from assignment
  return failure
```

Heuristics

- There are other heuristics that can be used to try to improve the efficiency of the search process
  - it concerns some of the functions employed in the revised backtracking algorithm
- To begin with, let's consider `SELECT-UNASSIGNED-VAR`
  - It selects some variable in the CSP that has not yet been assigned
  - So far, we have been selecting variables at random, but we can do better by using certain heuristics for choosing carefully which variable should be explored next

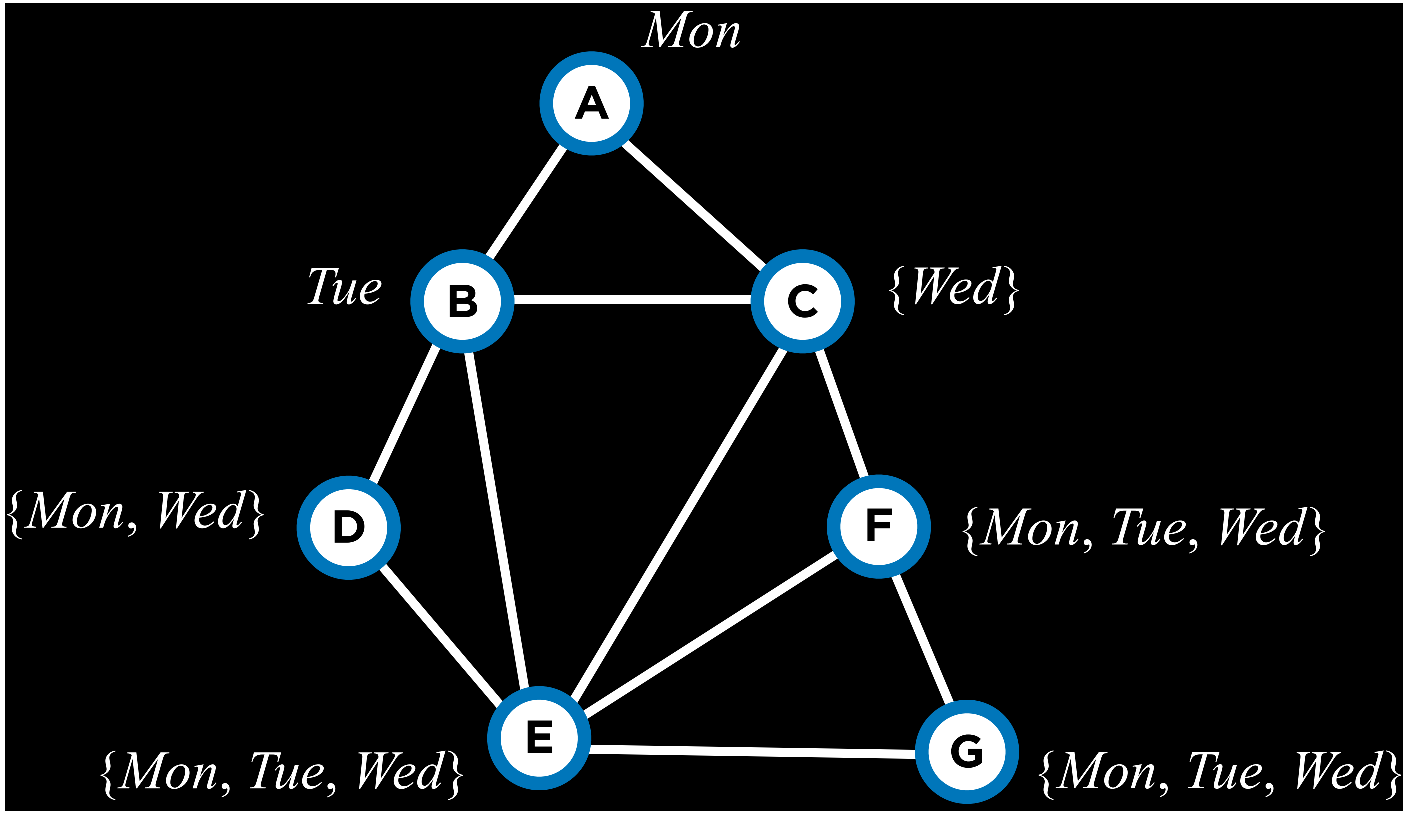
Using Heuristics Revisited

```
function BACKTRACK(assignment, csp):
   if assignment complete: return assignment
   var = SELECT-UNASSIGNED-VAR(assignment, csp)
   for value in DOMAIN-VALUES(var, assignment):
      if value consistent with assignment:
         add {var = value} to assignment
         inferences = INFERENCE(assignment, csp)
         if inferences ≠ failure: add inferences to assignment
         result = BACKTRACK(assignment, csp)
         if result ≠ failure: return result
      remove {var = value} and inferences from assignment
   return failure
```

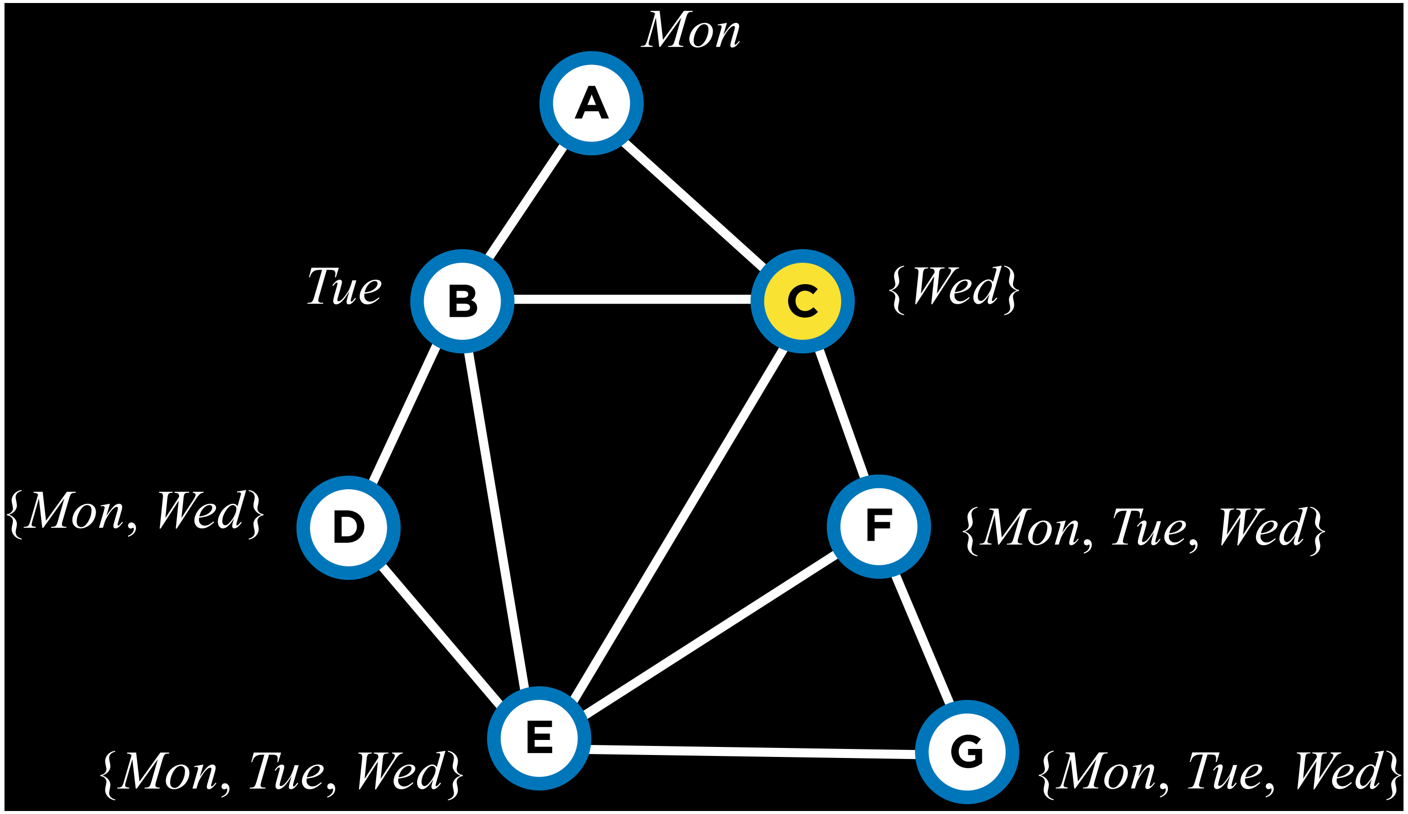
Select-Unassigned-Var

- Minimum remaining values (MRV) heuristic
  - Select the variable that has the smallest domain
  - The idea is if there are only two remaining values left, we can discard one of them very quickly to get to the other
    - one of those two has got to be the solution if a solution does exist
- Degree heuristic
  - Select the variable that has the highest degree
  - The idea is that by choosing a variable of high degree, one immediately constraints the rest of the variables more
    - and it's more likely to be able to eliminate large parts of the state-space that we don't need to search trough

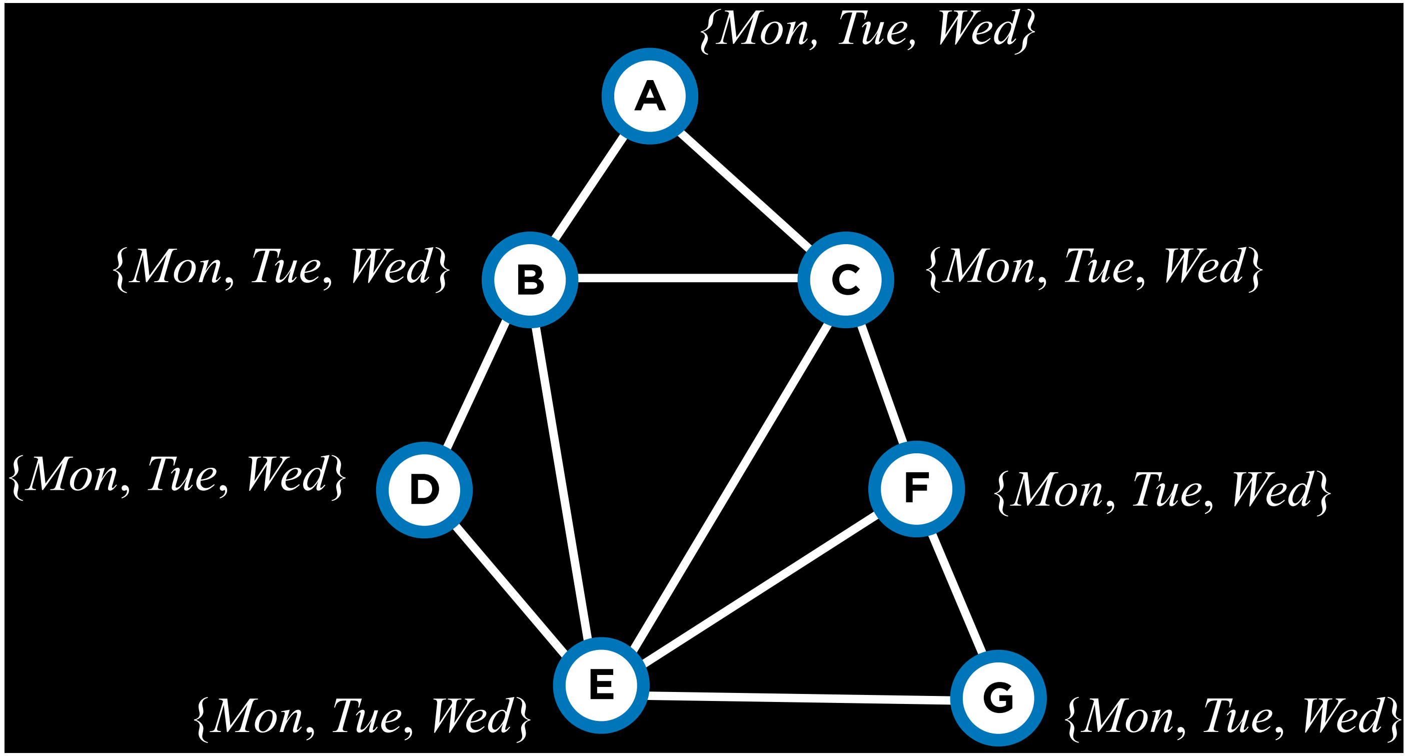
Minimum Remaining Values



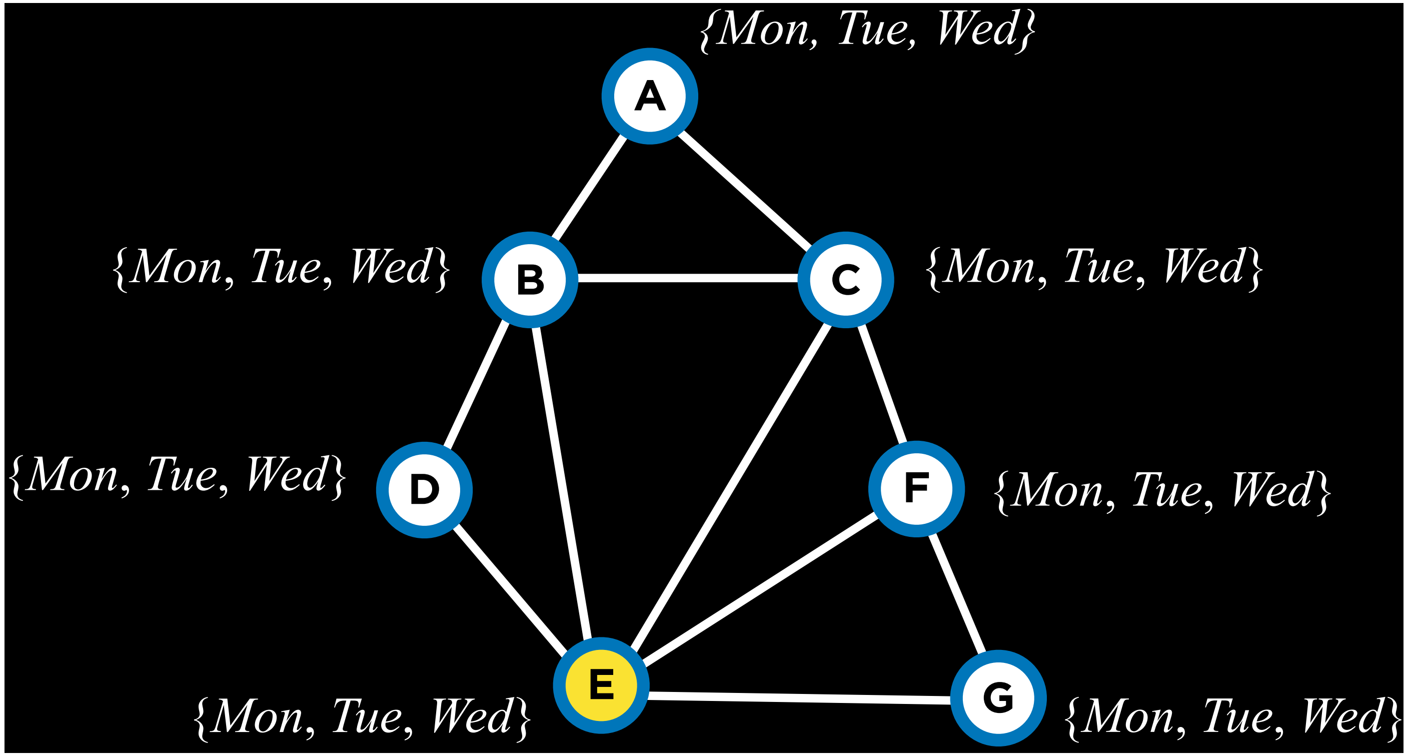
Minimum Remaining Values



Degree Heuristic



Degree Heuristic



Domain-Values Revisited

- Domain-values takes a domain for a variable and returns a sequence of all the values inside that variable's domain
  - We used a naïve approach where we just go in order Mon, Tue, Wed
- But this order might not be the most effective one to search in, it might be more effective to choose values that are likely to be solutions first and then go to other values
- How do we assess whether a value is likely to lead to a solution?
  - We can look at how many things get removed from domains by making this new assignment of a variable to a particular value

Domain-Values Revisited

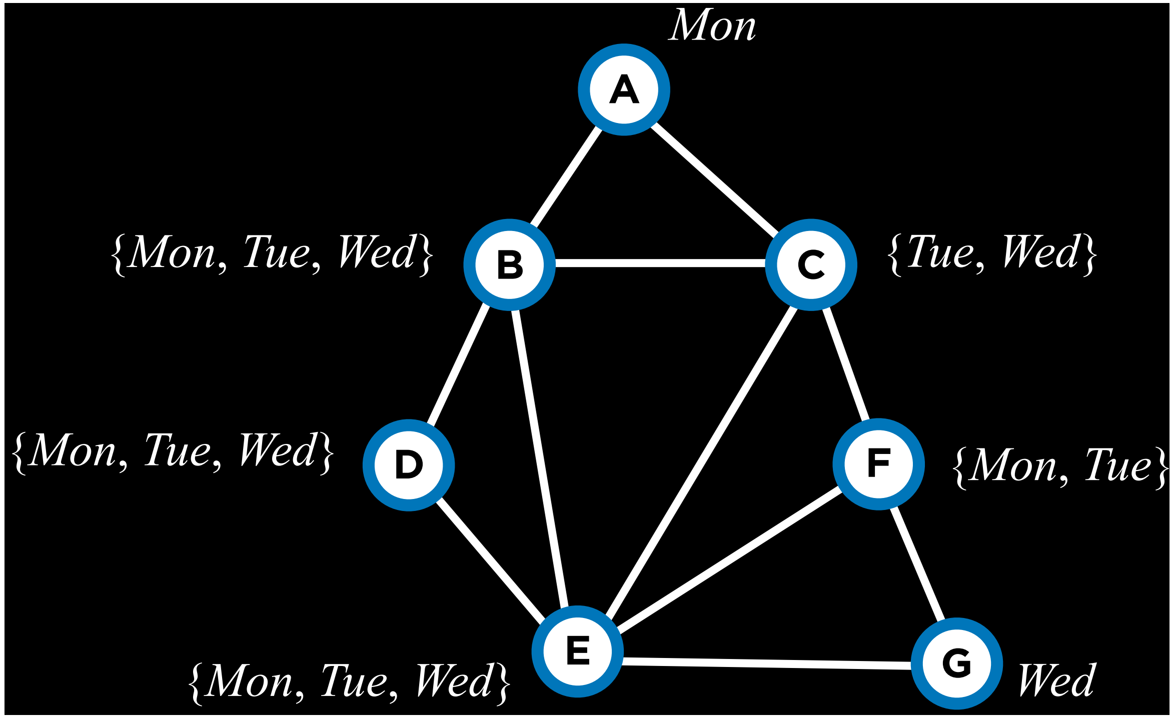
```
function BACKTRACK(assignment, csp):
   if assignment complete: return assignment
   var = SELECT-UNASSIGNED-VAR(assignment, csp)
   for value in DOMAIN-VALUES(var, assignment):
      if value consistent with assignment:
         add {var = value} to assignment
         inferences = INFERENCE(assignment, csp)
         if inferences ≠ failure: add inferences to assignment
         result = BACKTRACK(assignment, csp)
         if result ≠ failure: return result
      remove {var = value} and inferences from assignment
   return failure
```

Domain-Values

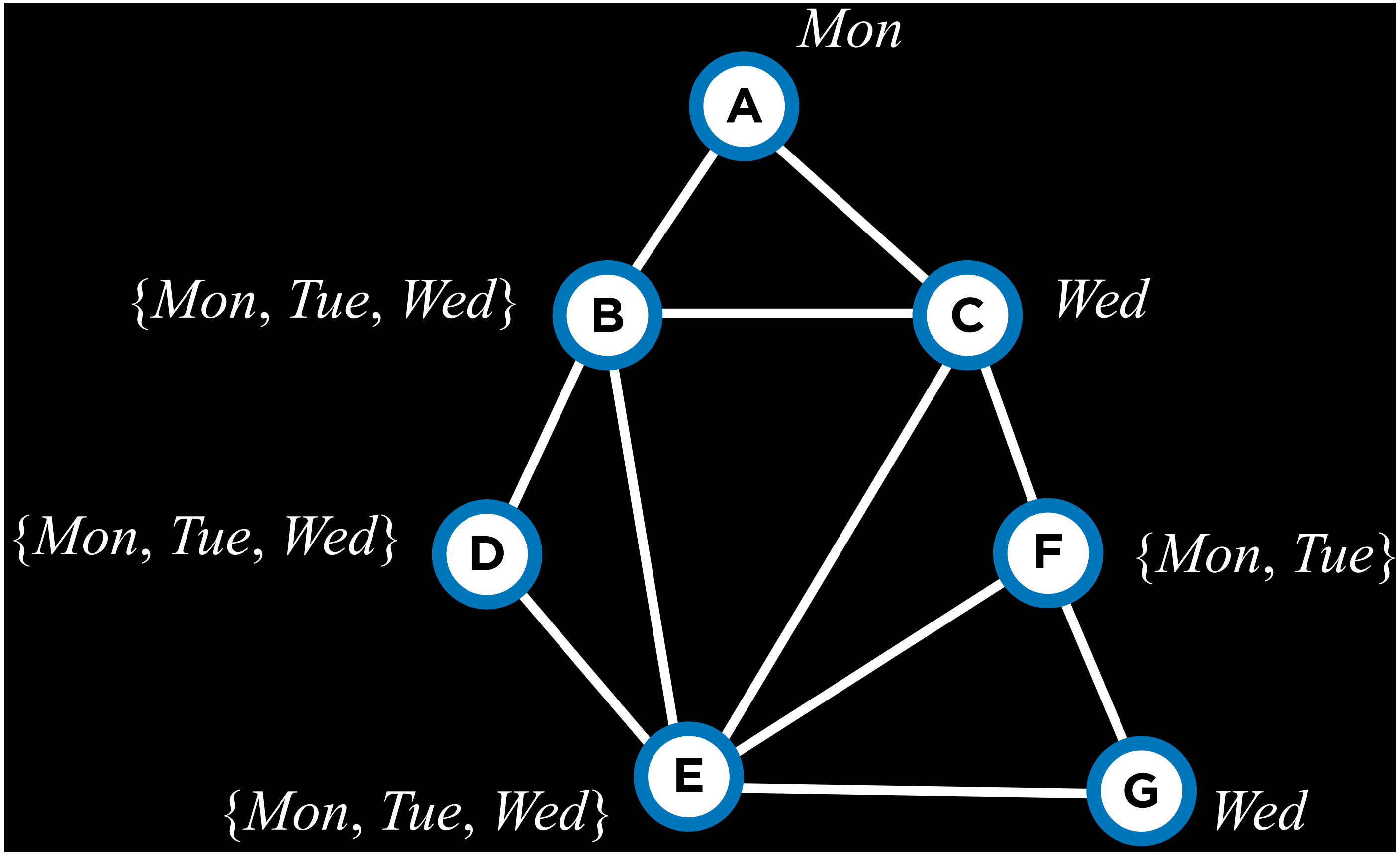
- Least-constraining value heuristic
  - Returns values in order by the number of choices that are ruled out for neighboring variables
    - Try least-constraining values first
  - The idea is that if one starts with a value that rules out a lot of other choices, we're ruling out a lot of possibilities likely to lead to a solution

Least-constraining Value Heuristic

- Considering C, what should I choose first, Tue or Wed?



Least-constraining Value Heuristic



Least-constraining Value Heuristic

- By continuing this process, we will find a solution
  - an assignment of variables to values where each of these classes has an exam date with no conflict