Course of "Automatic Control Systems"
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Asymptotic Bode diagrams

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⚔ Let us consider a transfer function $W(s)$ an LTI system

$$W(s) = K \frac{s^{\nu} \prod_i (1 + \sigma_i s)^{m_i} \prod_q \left(1 + \frac{2\xi_q}{\omega_{nq}} s + \frac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_j (1 + \tau_j s)^{n_j} \prod_p \left(1 + \frac{2\zeta_p}{\omega_{np}} s + \frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}}$$

⚔ The aim of the Bode diagrams is to represent the magnitude and the phase of transfer function $W(s)|_{s=j\omega}$ as function of ω.

⚔ The function $W(s)|_{s=j\omega}$ corresponds to the harmonic response function only if the LTI system is asymptotically stable.

⚔ In the following we will generally refer to Bode diagrams of a transfer function assuming implicitly that $W(s)$ is evaluated for $s = j\omega$
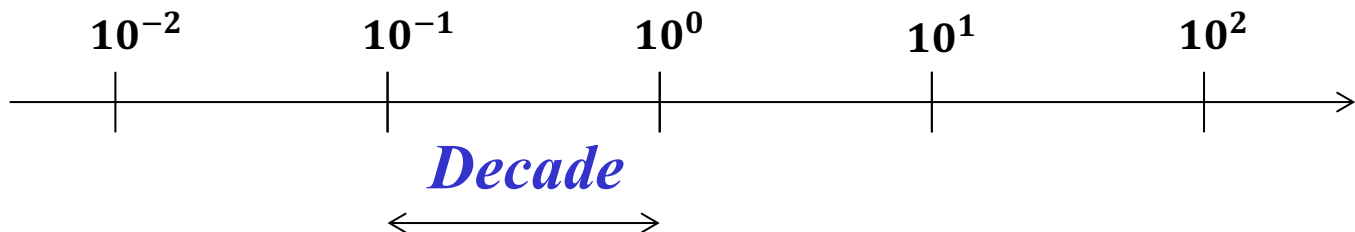
⚐ In the Bode diagrams magnitude and phase of $W(j\omega)$ are represented on two different Cartesian planes.

⚐ ***The x-axis of both magnitude and phase Bode diagrams are in a logarithmic scale ($log_{10}\omega$)***

On a logarithmic scale, the distance between two frequencies ω_1 and ω_2 depends on the difference of the logarithms and hence on the ratio on the frequencies

$$\log(\omega_2) - \log(\omega_1) = \log\left(\frac{\omega_2}{\omega_1}\right)$$

A decade is defined as the distance between two frequencies whose ratio is 10.

$$10^{-2} \qquad 10^{-1} \qquad 10^{0} \qquad 10^{1} \qquad 10^{2}$$

Decade

⚔ The y-axis of the magnitude and phase Bode diagrams indicate respectively
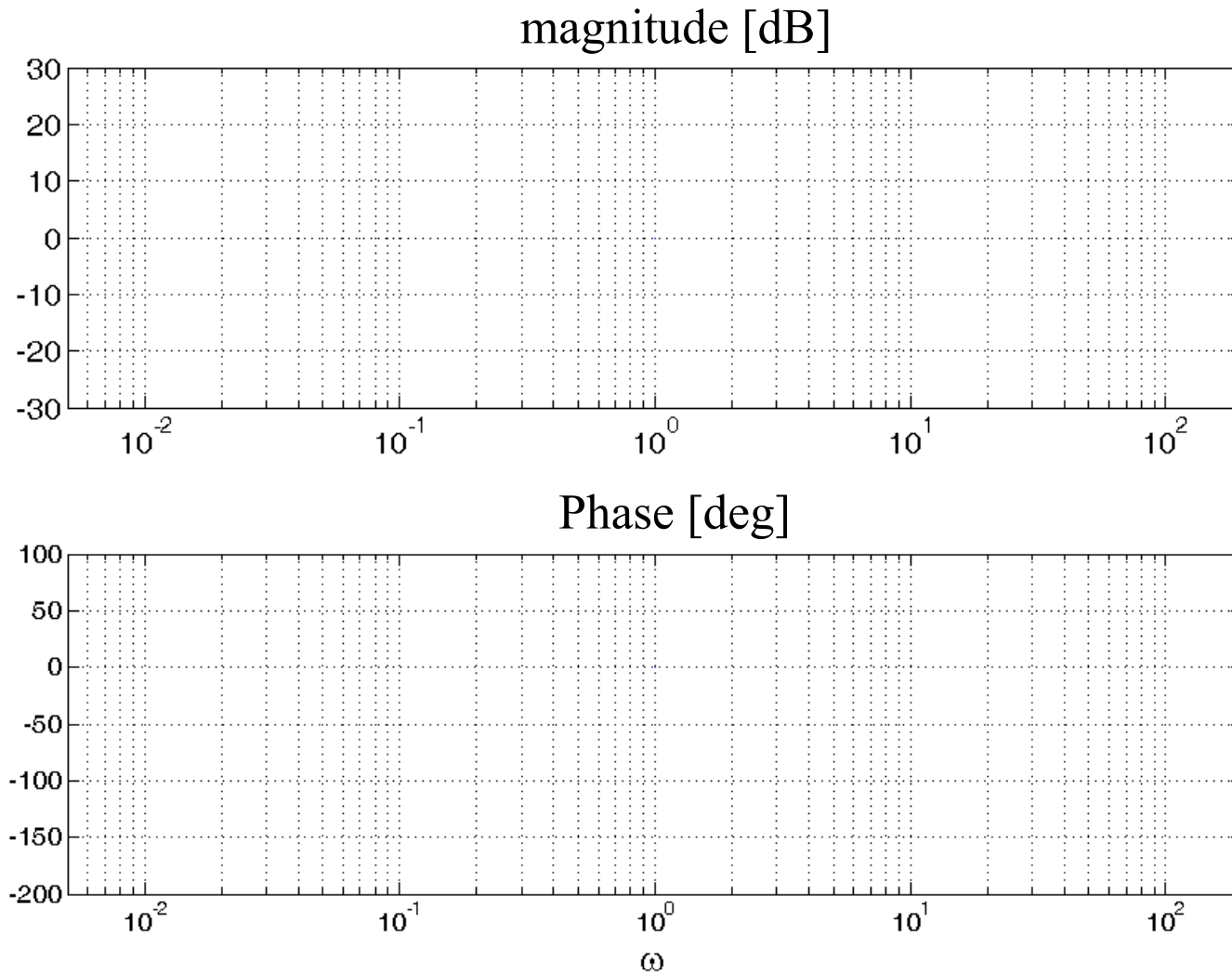
✦ *the magnitude of the transfer function in dB (decibel)*

$$|W(j\omega)|_{db} = 20 \log_{10} |W(j\omega)|$$

✦ *the phase of the transfer function in degrees or radians*

$$\angle W(j\omega)$$

magnitude [dB]

Phase [deg]

⚜ The magnitude of the Bode diagrams is expressed in decibel firstly because the logarithmic scale allows to consider *large magnitude intervals with limited space* (ex: $|10|_{db} = 20$, $|100|_{db} = 40$, $|1000|_{db} = 60$)

⚜ Moreover, the magnitude of $W(s)|_{s=j\omega}$ in decibel can be written has

$$|W(j\omega)|_{db} = 20\log_{10}\left(K \frac{s^{\nu}\prod_{i}(1+\sigma_i s)^{m_i}\prod_{q}\left(1+\dfrac{2\xi_q}{\omega_{nq}}s+\dfrac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_{j}(1+\tau_j s)^{n_j}\prod_{p}\left(1+\dfrac{2\zeta_p}{\omega_{np}}s+\dfrac{s^2}{\omega_{np}^2}\right)^{\kappa_p}}\Bigg|_{s=j\omega}\right) =$$

and using the main properties of the logarithm….

$$\left|W(j\omega)\right|_{db} = 20\log_{10} K +$$

Constant term

$$+ 20\log_{10} s^{\nu} +$$

Monomial term

$$+ \sum_{i} 20\log_{10}\left(1+\sigma_i s\right)^{m_i} - \sum_{j} 20\log_{10}\left(1+\tau_j s\right)^{n_j}$$

Binomial terms

Trinomial terms

$$+ \sum_{q} 20\log_{10}\left(1+\frac{2\xi_q}{\omega_{nq}}s+\frac{s^2}{\omega_{nq}^2}\right)^{\eta_q} - \sum_{p} 20\log_{10}\left(1+\frac{2\zeta_p}{\omega_{np}}s+\frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}$$

♠ The magnitude of $W(s)|_{s=j\omega}$ in decibel is given by the sum of four terms: ***constant, monomial, binomial and trinomial terms***

♠ The phase function has the some product property of the logarithm. Hence in the following we will construct the magnitude and phase Bode diagrams considering these four terms separately.

⚜ *Constant term:* **K**

⚜ *Monomial term:* Zero/Pole in the origin of multiplicity ν: $\quad \mathbf{20 \log_{10} s^{\nu}}$

⚜ *Binomial term* : Real zero/pole of multiplicity ν: $\quad \mathbf{20 \log_{10}(1 + \tau s)^{\pm\nu}}$

⚜ *Trinomial term* : Complex conjugate zero/pole of multiplicity ν:

$$20 \log_{10}\left(1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}\right)^{\pm\nu}$$
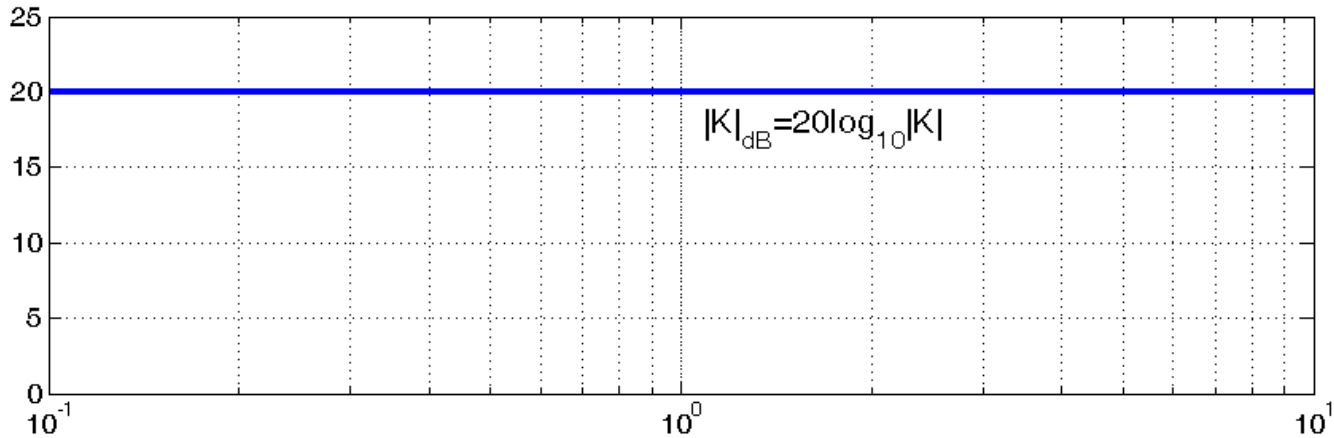
magnitude [dB]
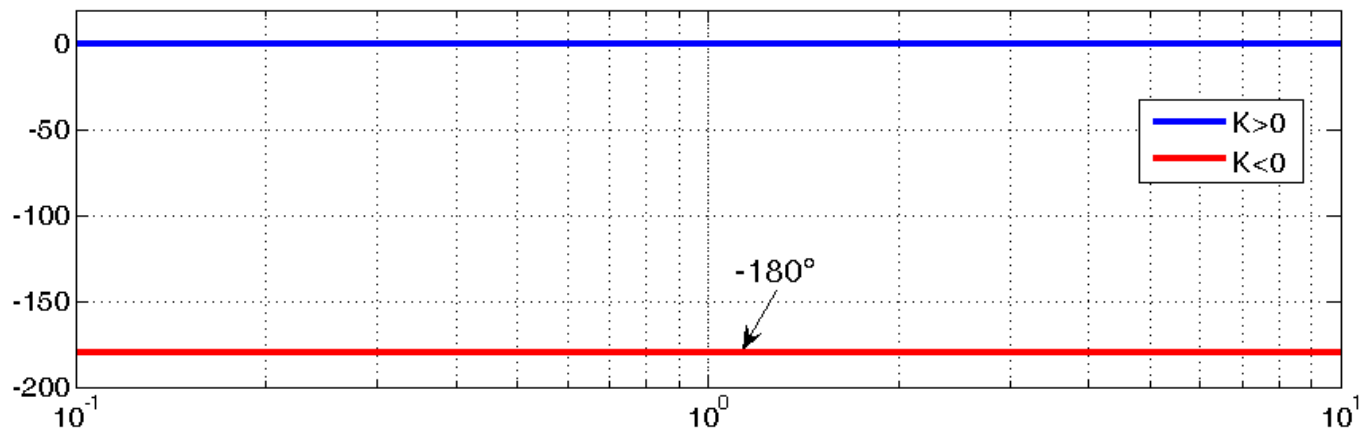
$|K|_{dB}=20\log_{10}|K|$

$|K|$	$|K|_{dB}$
0.01	-40
0.1	-20
1	0
2	6
3	10
5	14
10	20
100	40
1000	60

Phase [deg]

-180°

K>0
K<0

$$|\mathbf{W}(\mathbf{j}\boldsymbol{\omega})|_{\mathbf{dB}} = \mathbf{20} \cdot \boldsymbol{log_{10}} \left(|(\boldsymbol{j\omega})^{\boldsymbol{\nu}}|\right)$$

magnitude [dB]

$40 \, dB/dec$

$20 \, dB/dec$

$-20 \, dB/dec$

$-40 \, dB/dec$

Phase [deg]

$$\angle\mathbf{W}(\mathbf{j}\boldsymbol{\omega}) = \angle\,(\boldsymbol{j\omega})^{\boldsymbol{\nu}}$$
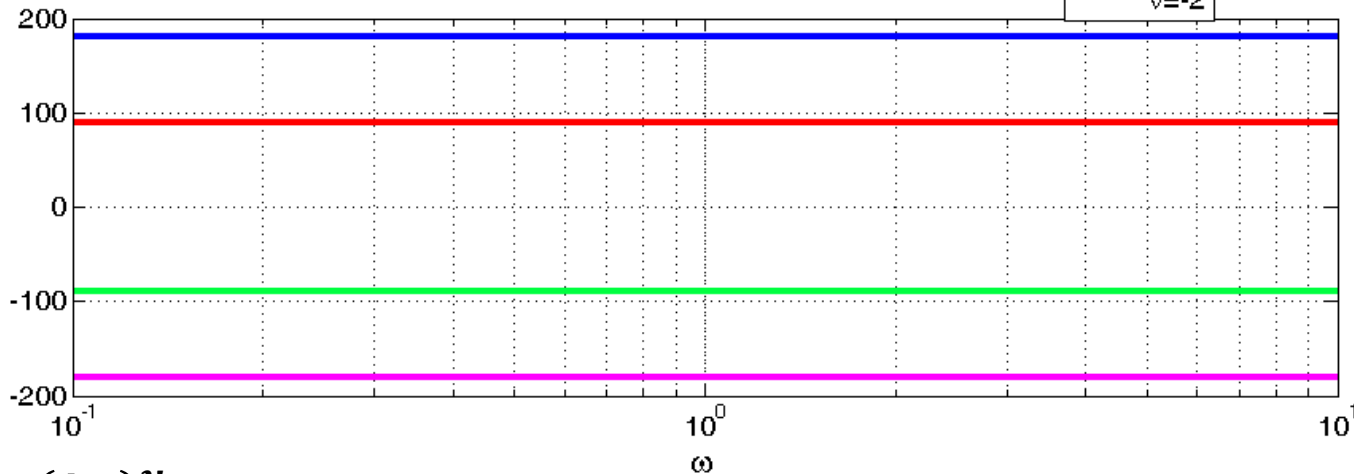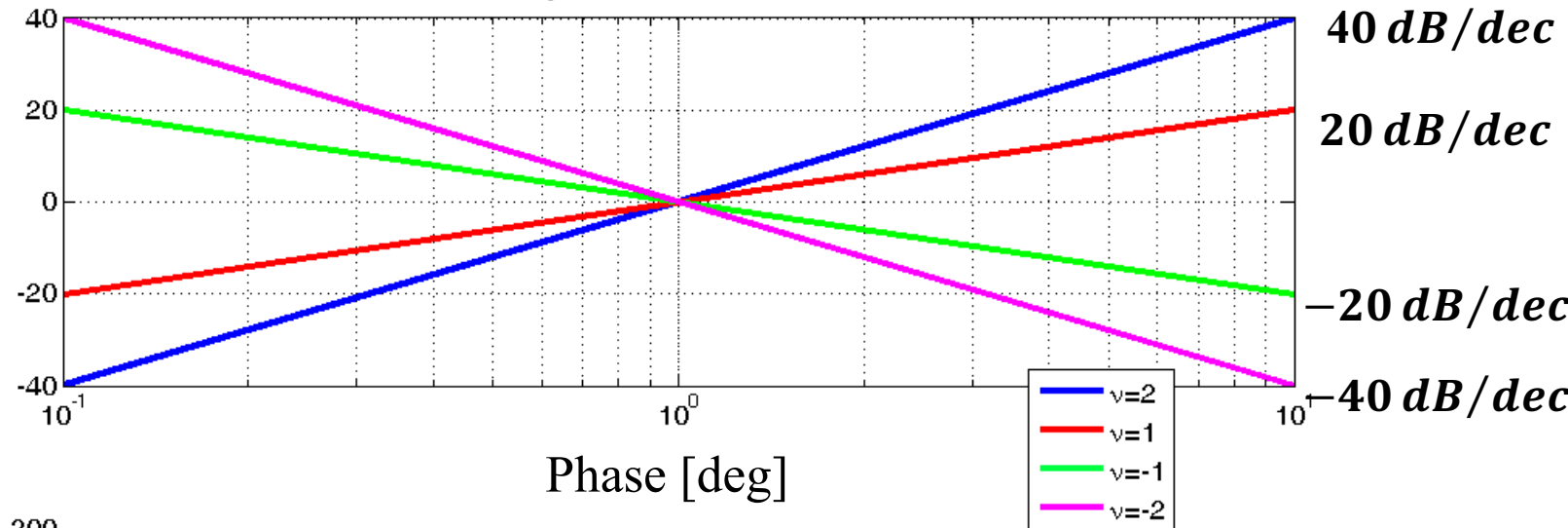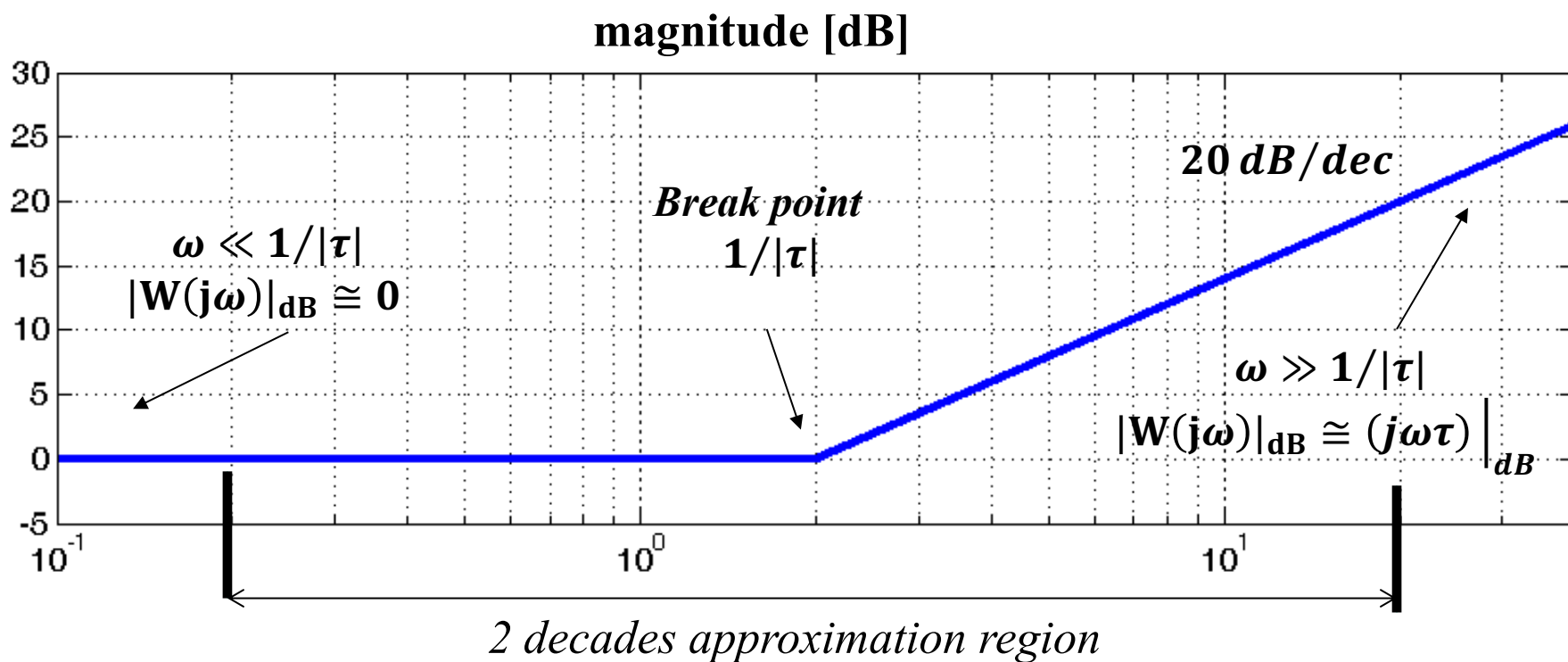
⚜ *The Bode diagrams definition of binomial and trinomial terms is more demanding*

⚜ For these cases we will first consider the *asymptotic Bode diagrams* that give a correct information of magnitude and phase of the considered terms for $\omega \rightarrow 0$ and $\omega \rightarrow +\infty$ (*or at least 1 decade before and after the break point of the binomial term*)

⚜ In the interval *from a decade before to a decade after the break point* the magnitude and phase asymptotic Bode diagrams are linked with linear connections

Case 1: real zero of multiplicity one $(1 + s\tau)$

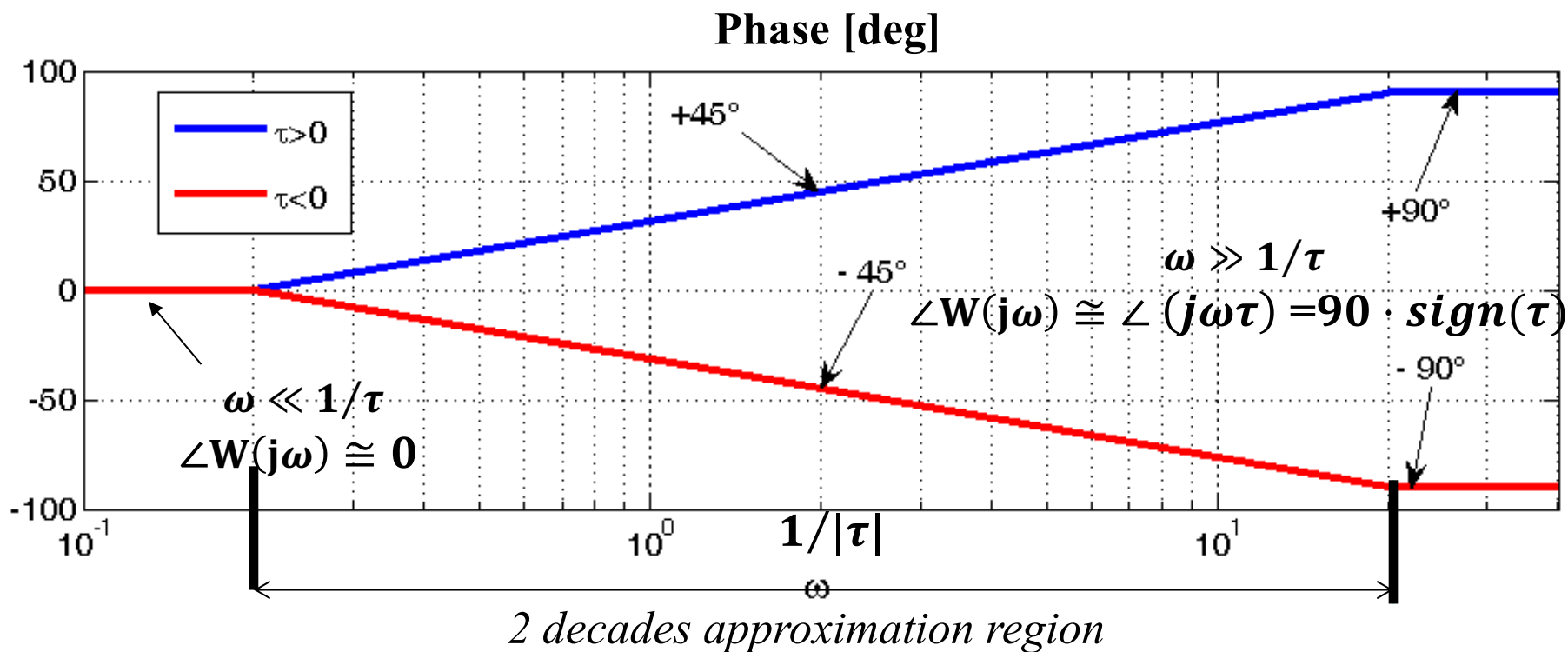magnitude in decibel $|W(j\omega)|_{dB} = 20 \cdot log_{10}(|1 + j\omega\tau|) = 20 \cdot log_{10}\sqrt{1 + (\omega\tau)^2}$

magnitude [dB]

$\omega \ll 1/|\tau|$
$|W(j\omega)|_{dB} \cong 0$

Break point
$1/|\tau|$

$20\, dB/dec$

$\omega \gg 1/|\tau|$
$|W(j\omega)|_{dB} \cong (j\omega\tau)\Big|_{dB}$

2 decades approximation region

The magnitude diagram is independent of the sign of τ

Case 1: real zero of multiplicity one $(1 + s\tau)$

Phase in degree $\angle W(j\omega) = \angle (1 + j\omega\tau) = \tan^{-1}(\omega\tau)$

Phase [deg]

+45°

+90°

$\omega \gg 1/\tau$

$\angle W(j\omega) \cong \angle (j\omega\tau) = 90 \cdot sign(\tau)$

- 45°

- 90°

$\omega \ll 1/\tau$

$\angle W(j\omega) \cong 0$

$1/|\tau|$

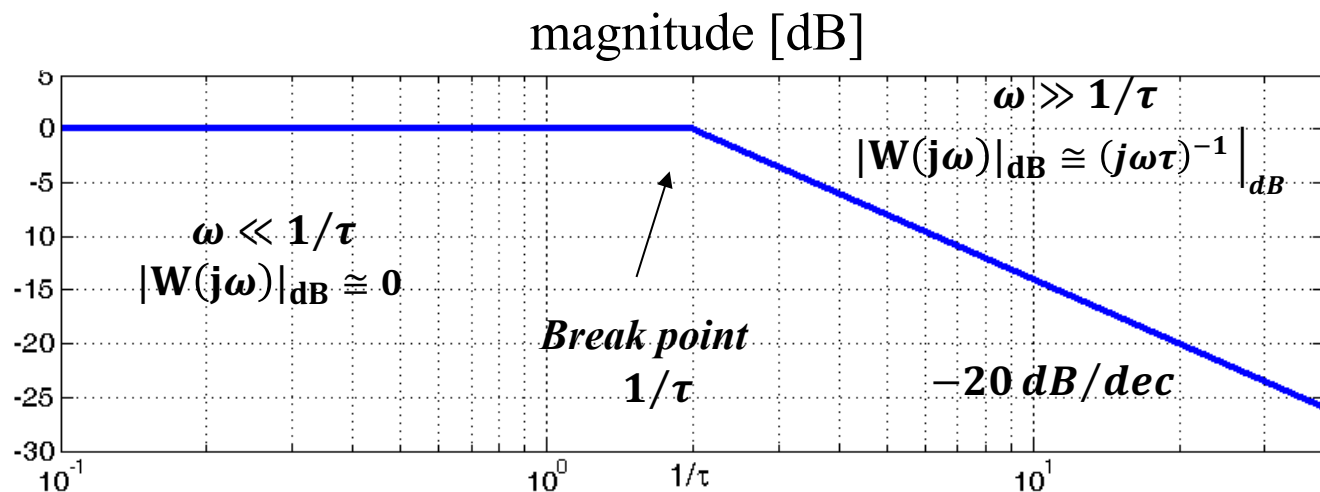ω

τ>0

τ<0

2 decades approximation region

The phase diagram depends on the sign of τ

Case 2: real pole of multiplicity one $(1 + s\tau)^{-1}$

magnitude in decibel
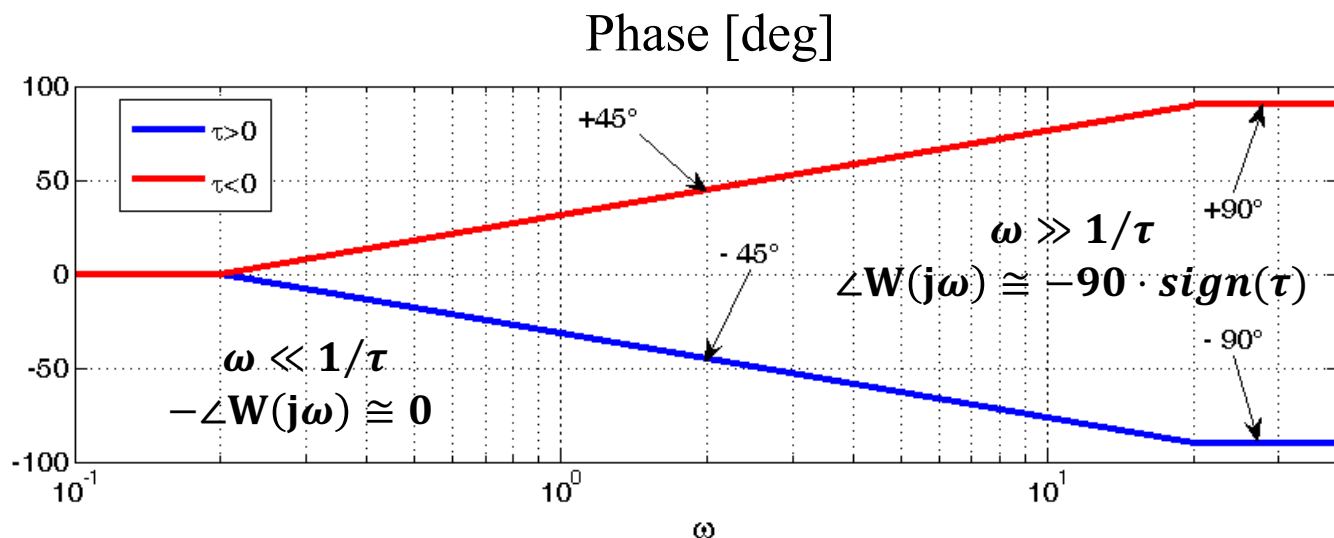
$$-20 \cdot log_{10}\,(1 + j\omega\tau)$$

The magnitude diagram is independent of the sign of τ

magnitude [dB]

$\omega \gg 1/\tau$

$|W(j\omega)|_{dB} \cong (j\omega\tau)^{-1}\big|_{dB}$

$\omega \ll 1/\tau$

$|W(j\omega)|_{dB} \cong 0$

Break point
$1/\tau$

$-20\,dB/dec$

Phase in degree

$$-\angle\,(1 + j\omega\tau)$$

The phase diagram depends on the sign of τ

Phase [deg]

$\tau > 0$

$\tau < 0$

+45°

- 45°

$\omega \gg 1/\tau$

$\angle W(j\omega) \cong -90 \cdot sign(\tau)$

+90°

- 90°

$\omega \ll 1/\tau$

$-\angle W(j\omega) \cong 0$

ω

Case 1: complex conjugate zeros of multiplicity one $\left(1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}\right)$

magnitude in decibel $\quad |W(j\omega)|_{dB} = 20 \cdot log_{10}\left(1 + j\frac{2\zeta\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right) = 20 \cdot log_{10}\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta}{\omega_n}\right)^2}$
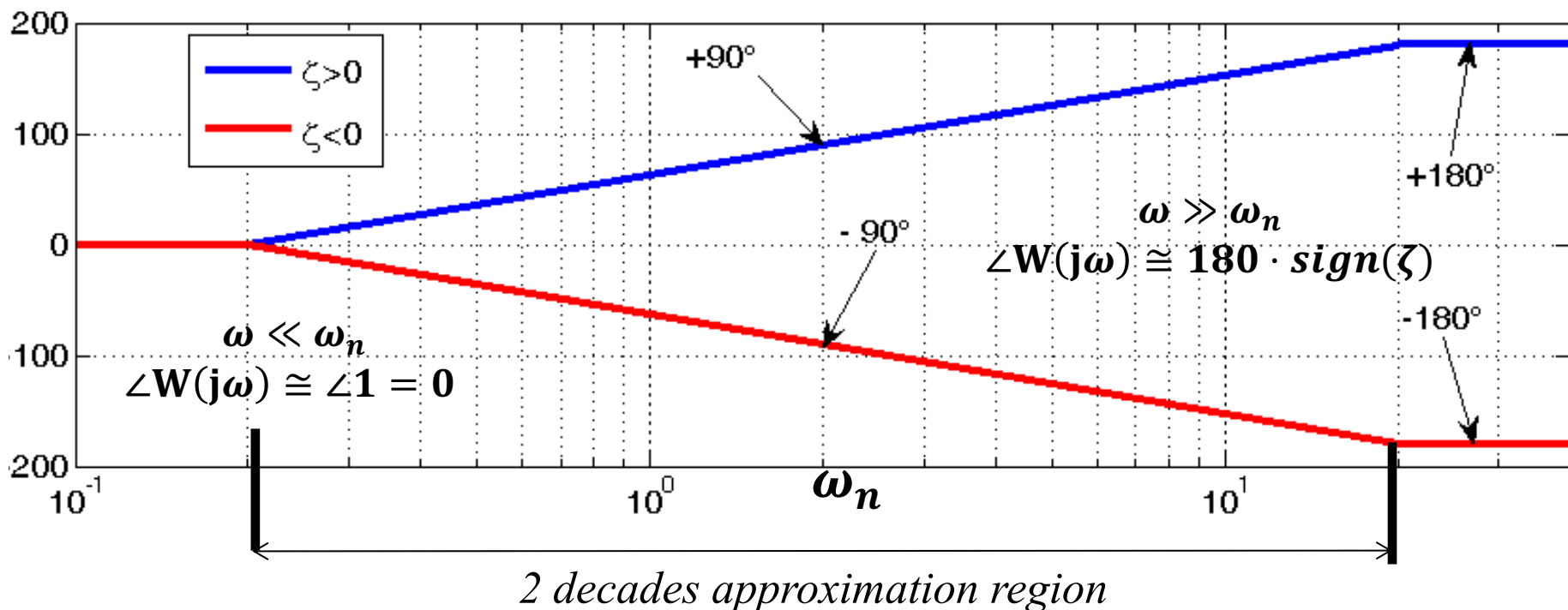
magnitude [dB]
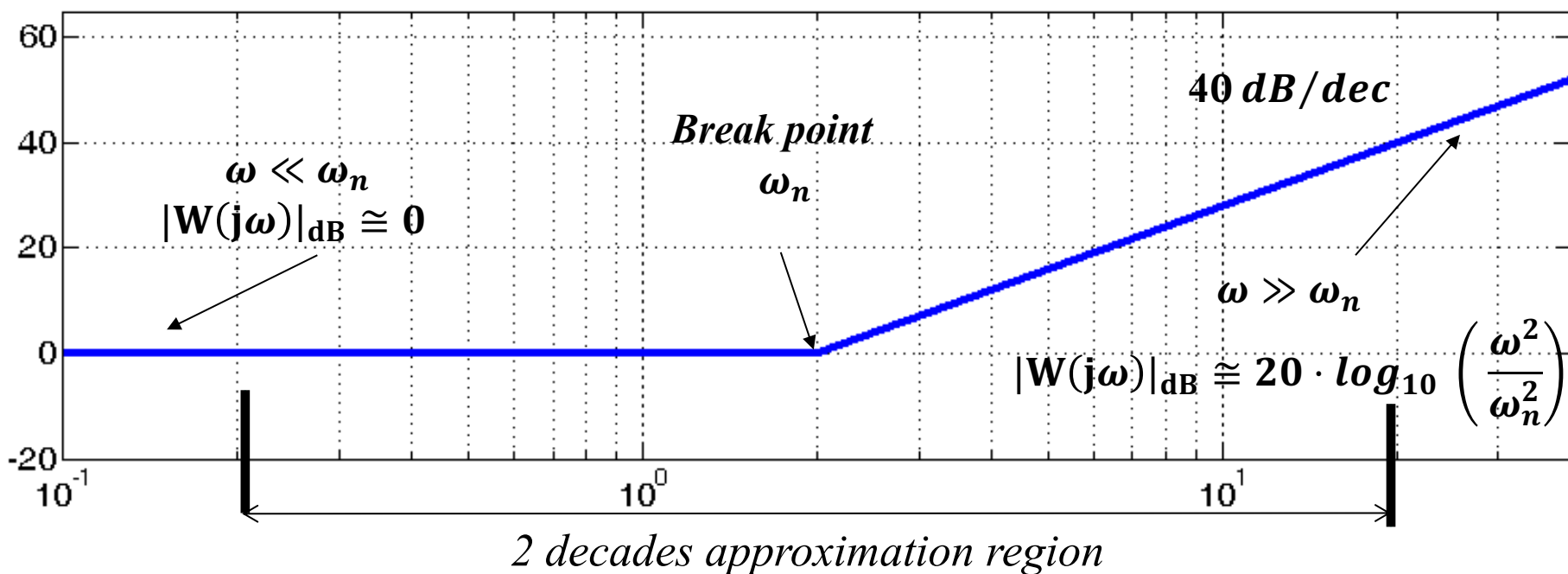
2 decades approximation region

The magnitude diagram is independent of the sign of ζ.

Case 1: complex conjugate zeros of multiplicity one $\left(1 + \dfrac{2\zeta s}{\omega_n} + \dfrac{s^2}{\omega_n^2}\right)$

Phase in degree $\angle W(j\omega) = \angle\left(1 + j\dfrac{2\zeta\omega}{\omega_n} - \dfrac{\omega^2}{\omega_n^2}\right) = \tan^{-1}\dfrac{\frac{2\zeta}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$
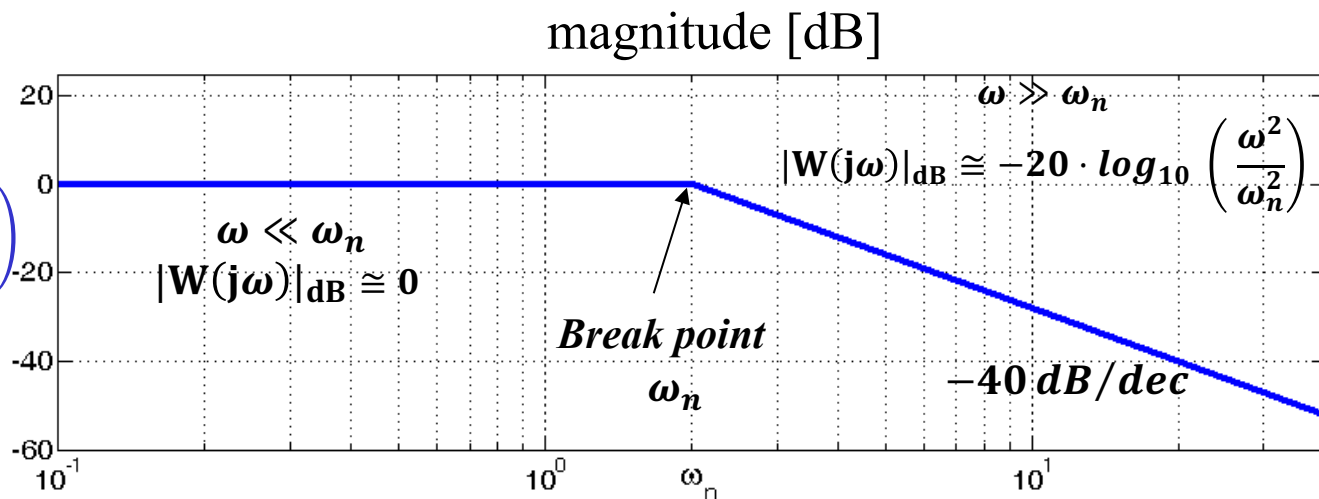
Phase [deg]

2 decades approximation region

The phase diagram depends on the sign of ζ.

Case 2: complex conjugate poles of multiplicity one $\left(1 + \dfrac{2\zeta s}{\omega_n} + \dfrac{s^2}{\omega_n^2}\right)^{-1}$

magnitude [dB]

magnitude in decibel

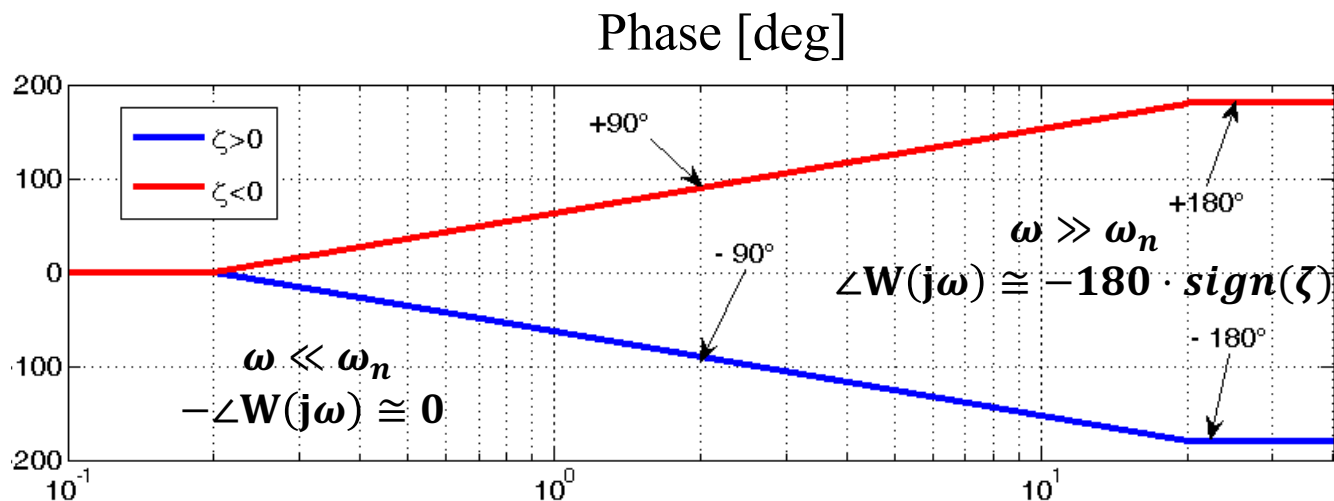$-20 \cdot log_{10}\left(1 + j\dfrac{2\zeta\omega}{\omega_n} - \dfrac{\omega^2}{\omega_n^2}\right)$

The magnitude diagram is independent of the sign of ζ

$\omega \gg \omega_n$

$|W(j\omega)|_{dB} \cong -20 \cdot log_{10}\left(\dfrac{\omega^2}{\omega_n^2}\right)$

$\omega \ll \omega_n$
$|W(j\omega)|_{dB} \cong 0$

Break point
ω_n

$-40\, dB/dec$

Phase [deg]

Phase in degree

$-\angle\left(1 + j\dfrac{2\zeta\omega}{\omega_n} - \dfrac{\omega^2}{\omega_n^2}\right)$

The phase diagram depends on the sign of ζ

+90°
- 90°

$\omega \ll \omega_n$
$-\angle W(j\omega) \cong 0$

$\omega \gg \omega_n$
$\angle W(j\omega) \cong -180 \cdot sign(\zeta)$

+180°
- 180°

ζ>0
ζ<0

⚔ Let us consider the transfer function of an LTI system

$$W(s) = \frac{10(s+5)}{(s+1)(s+10)}$$

⚔ The LTI system is asymptotically stable and the harmonic response is

$$W(j\omega) = 5\frac{(1+j0.2\omega)}{(1+j\omega)(1+j0.1\omega)}$$

⚔ The harmonic response is composed by a constant term $k = 5$, and three binomial terms $(1+j0.2\omega), (1+j\omega)^{-1}, (1+j0.1\omega)^{-1}$

⚔ Summing magnitude and phase of the four contributions we can evaluate the Bode diagrams of the system

magnitude [dB]

Phase [deg]