Course of "Automatic Control Systems"
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# Harmonic response function

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⚕ Let us consider an ***asymptotically stable LTI system.***

⚕ Given an input signal $u(t)$ and an initial condition $x(0)$, we define

    ✦ ***steady-state response $y_{ss}(t)$***, the regular behavior of the total response $y(t)$ *(if exist)* after an infinite time from the application of the input.

    ✦ ***transient response $y_t(t)$***, the difference between the total response of the system and the steady-state response $y_t(t) = y(t) - y_{ss}(t).$

⚔ ***The steady-state response*** of asymptotically stable system is independent from the initial condition.

⚔ It depends on the particular input applied to the system

polynomial inputs $\longrightarrow$ polynomial steady state

sinusoidal inputs $\longrightarrow$ sinusoidal steady state

⚜ The step response is characterized by "decaying" exponential functions related to the system evolution modes and a constant value
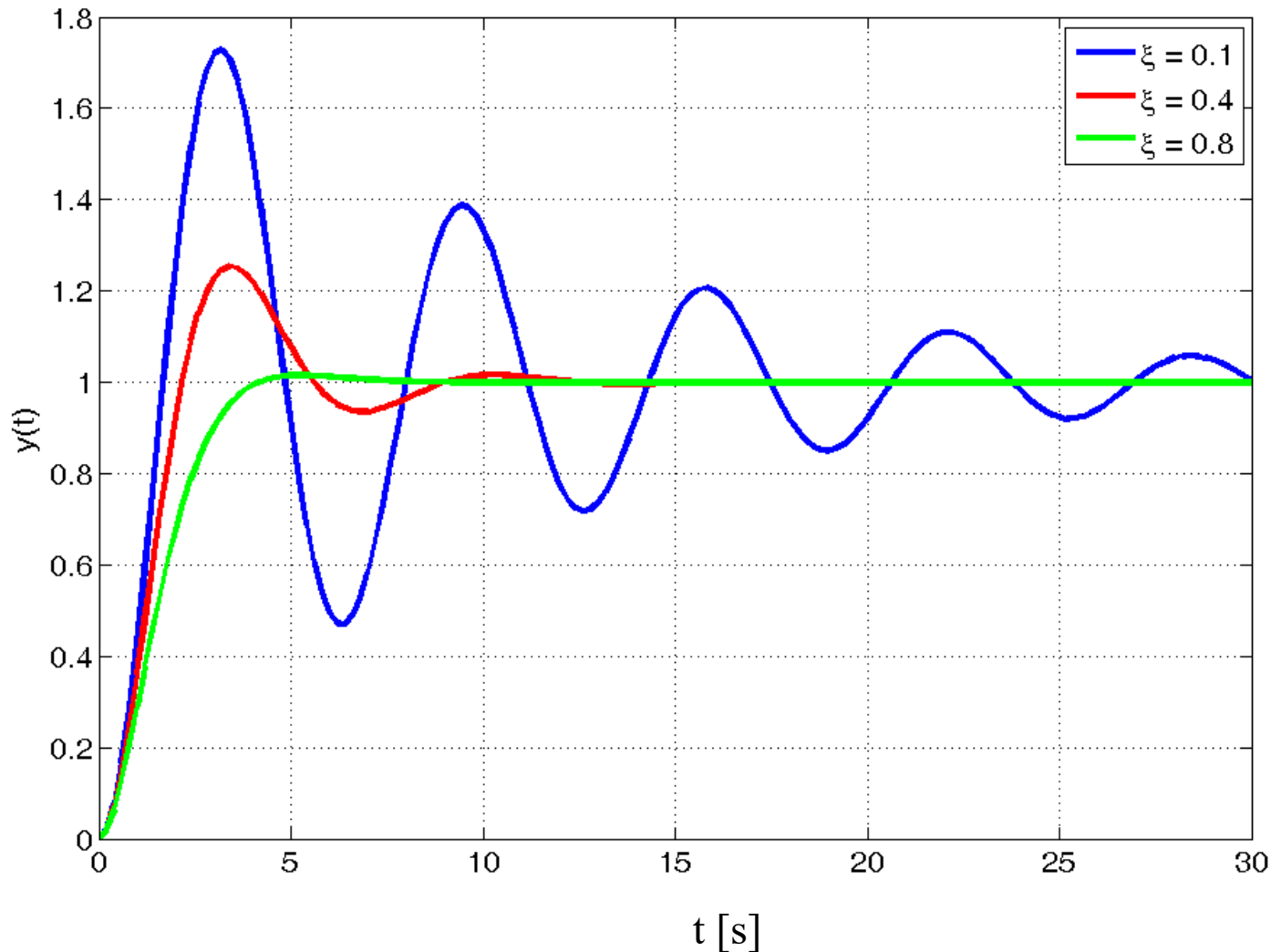
$$x_0 = 0$$

$u(t)=1(t)$
$W(s)$
$y_s(t)$
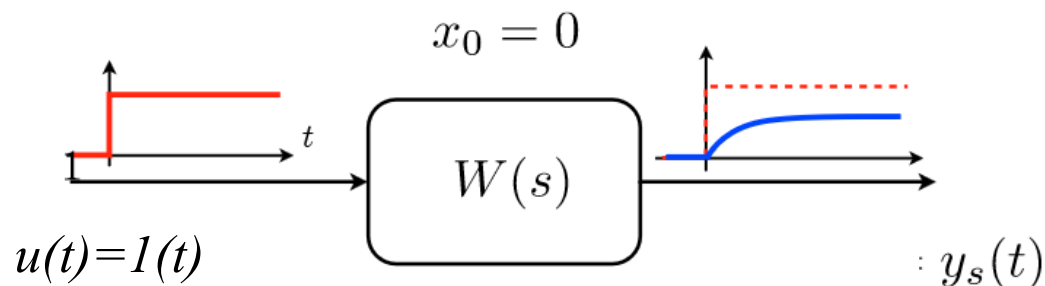
⚜ The "decaying" exponential functions determine **the transient** part of the response while the constant term is the **steady-state** value.

$y_s(t)$

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transient

steady-state   (asymptotic behavior)

constant value

$R_0$

conceptual
frontier

$t$
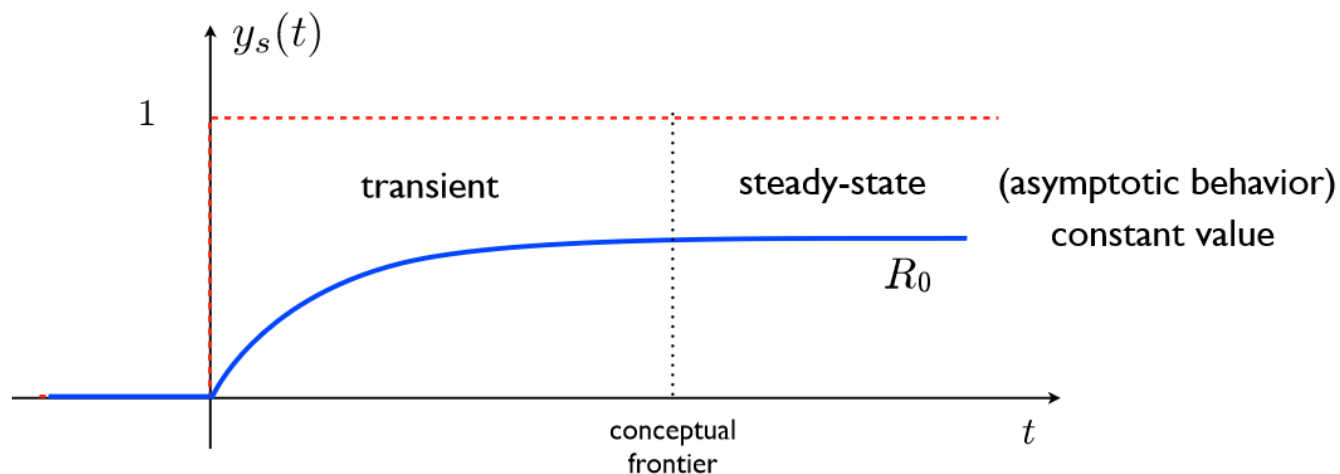
⚔ Different  evolution modes determine different transient responses.

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Let us consider input signal belonging to the class of complex exponential functions:

- $u(t) = e^{st}, s = \alpha + j\omega$
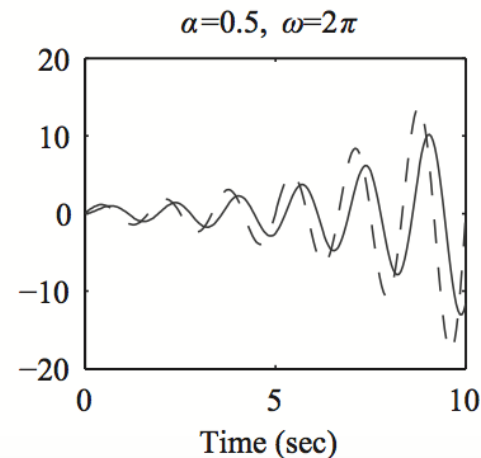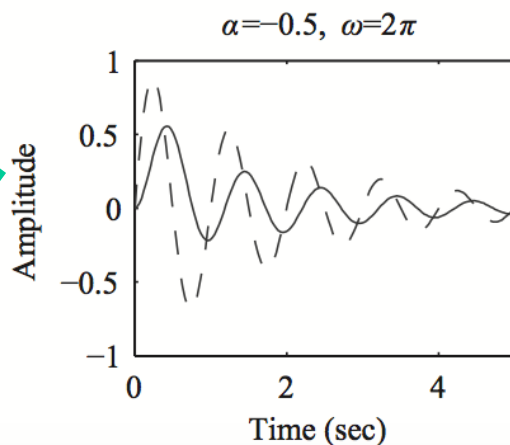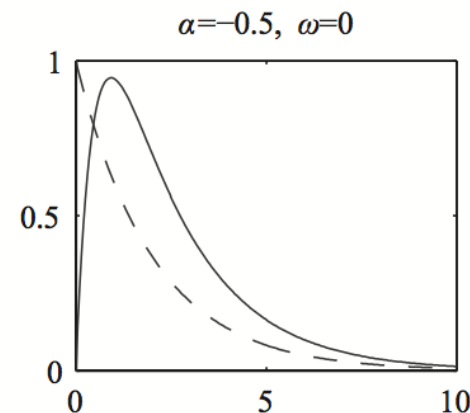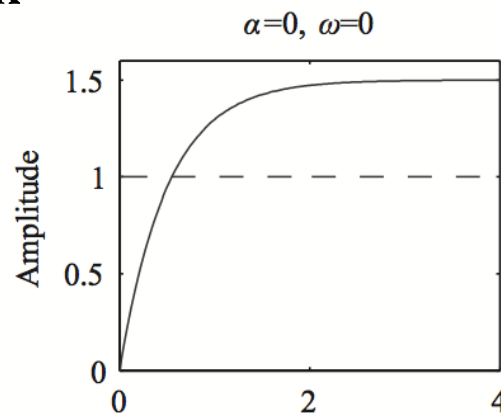
Recall that

- $e^{st} = e^{(\alpha + j\omega)t}$

$$= e^{\alpha t}(\cos(\omega t) + j\sin(\omega t))$$

Many signals may be written as a linear combination of complex exponential functions.

After an initial transient, the LTI response is proportional to the input (i.e. exhibits the same form of the input).

**LTI:** $\dot{y}(t) + 2\,y(t) = 3u(t)$



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- **A SISO system of *n-th* order, $t_0=0$, $x(0)=x_0$:**

$$\dot{x}(t) = A\,x(t) + Bu(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

$\mathcal{L}$

$$sX(s) - x_0 = A\,X(s) + BU(s)$$
$$Y(s) = C\,X(s) + DU(s)$$

$$X(s) = (sI - A)^{-1}\,x_0 + (sI - A)^{-1}B\,U(s)$$

$$W(s) = C(sI - A)^{-1}B + D$$

$$Y(s) = \underbrace{C(sI - A)^{-1}x_0}_{Y_l} + \underbrace{(C(sI - A)^{-1}B + D)\,U(s)}_{Y_f}$$

- **For exponential input:**

$$u(t) = e^{\lambda t}, t \geq 0$$

$\mathcal{L}$

$$Y_f(s) = \boldsymbol{W}(s)U(s) = W(s)\frac{1}{s - \lambda} = \cdots + \frac{k}{s - \lambda}$$

$\mathcal{L}^{-1}$

$$\cdots + ke^{\lambda t}$$

- *Initial condition $x(0)$* that nullifies the evolution modes

$$\Longrightarrow \quad x(t) = x(0)e^{\lambda t}$$

By using the state equation

$$\dot{x}(t) = A\,x(t) + Bu(t) \quad \Longrightarrow \quad \lambda x(0)e^{\lambda t} = Ax(0)e^{\lambda t} + Be^{\lambda t}$$

$$\Longrightarrow \quad (\lambda I - A)x(0) = B \quad \Longrightarrow \quad x(0) = (\lambda I - A)^{-1}B, \text{ if } \lambda \text{ is not an eigenvalue of } A$$

Then,

$$x(t) = x(0)e^{\lambda t} = (\lambda I - A)^{-1}Be^{\lambda t}, t > 0$$

$$y(t) = C\,x(t) + D\,u(t) = C(\lambda I - A)^{-1}Be^{\lambda t} + De^{\lambda t} = (C(\lambda I - A)^{-1}B + D)e^{\lambda t} = \boldsymbol{W}(\boldsymbol{\lambda})e^{\lambda t}.$$

If the system is asymptotically stable, $\quad x(t) = (\lambda I - A)^{-1}Be^{\lambda t}, \quad y(t) = \boldsymbol{W}(\boldsymbol{\lambda})e^{\lambda t},$

these functions represent the asymptotic movements of the state and the output of the system, for any initial condition $x(0)$.

⚲ Let us consider an asymptotically stable LTI system with a transfer function $W(s)$ subject to a sinusoidal input signal

$$u(t) = U_0 \sin(\omega_0 t + \varphi)$$

$$\boxed{W(s)}$$ $$y(t)$$
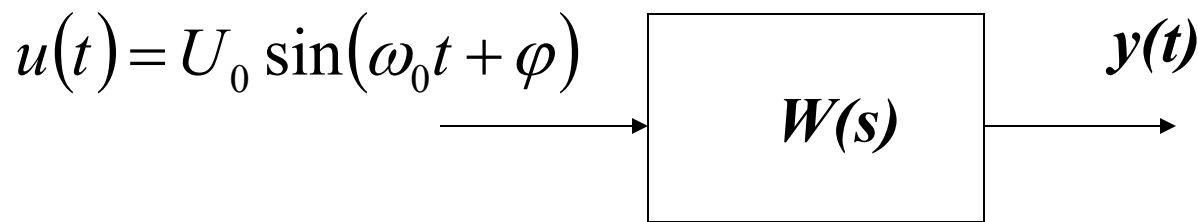
⚲ The evaluation of the steady state response of LTI system to sinusoidal inputs is very interest taking into account that *any signal can be decomposed in the sum of a finite (periodic signal) and infinite number (aperiodic signal) of sinusoids by means of the Fourier series.*

✦ It is possible to prove that the steady state response of an LTI system with transfer function $W(s)$ to a sinusoidal inputs $u(t) = U_0 \sin(\omega_0 t + \phi)$ can be written in the time domain as

$$y_{ss}(t) = U_0 |W(s)|_{s=j\omega_0} \sin\left(\omega_0 t + \varphi + \angle W(s)_{s=j\omega_0}\right)$$

where

✦ $|W(s)|_{s=j\omega_0}$ is the magnitude of the Laplace transform of $W(s)$ evaluated in $s = j\omega_0$.

✦ $\angle W(s)|_{s=j\omega_0}$ is the phase of the Laplace transform of $W(s)$ evaluated in $s = j\omega_0$.

**For a sinusoidal input,**

$$u(t) = \sin(\omega t), t \geq 0, \omega = \frac{2\pi}{T},$$

**we exploit the results achieved for an exponential input.**

**Indeed,** $\sin(\omega t) = \text{Im}(e^{j\omega t})$. **Recall that** $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$.

**For** $\tilde{u}(t) = e^{j\omega t}$, **if A without eigenvalues in** $\pm j\omega$, **then there is an initial state**

$x(0) = (j\omega I - A)^{-1}B$, **such that the movements of the state and the output:**

- $\tilde{x}(t) = x(0)e^{j\omega t} = (j\omega I - A)^{-1}Be^{j\omega t}, t > 0$

- $\tilde{y}(t) = C\,x(t) + D\,u(t) = C(j\omega I - A)^{-1}Be^{j\omega t} + De^{j\omega t}$
$$= (C(j\omega I - A)^{-1}B + D)e^{j\omega t} = W(j\omega)e^{j\omega t}$$
$$= |W(j\omega)|e^{j\arg(\boldsymbol{W}(j\omega))}e^{j\omega t} = |\boldsymbol{W}(j\omega)|e^{j\left(\omega t + \arg(\boldsymbol{W}(j\omega))\right)}$$

**Recall that** $z = a + ib = r(\cos\theta + i\sin\theta) = re^{j\theta}$ **with** $\theta = \arg(z) = \tan^{-1}\frac{b}{a} + 2k\pi$

$$\tilde{x}(t) = (j\omega I - A)^{-1}Be^{j\omega t}, \quad \tilde{y}(t) = |W(j\omega)|e^{j\left(\omega t + \arg(W(j\omega))\right)}, t > 0$$

These functions represent the asymptotic movements, for LTI asymptotically stable with $\tilde{u}(t) = e^{j\omega t}$.

For $u(t) = \sin(\omega t) = \text{Im}\left(e^{j\omega t}\right)$, then

$$x(t) = \text{Im}\left(\tilde{x}(t)\right), \ y(t) = \text{Im}\left(\tilde{y}(t)\right) = |W(j\omega)|\sin\left(\omega t + \arg(W(j\omega))\right), t > 0$$

In general for $u(t) = U\sin(\omega_0 t + \varphi), t > 0$,

there is an initial state such that the output is a sinusoidal signal:
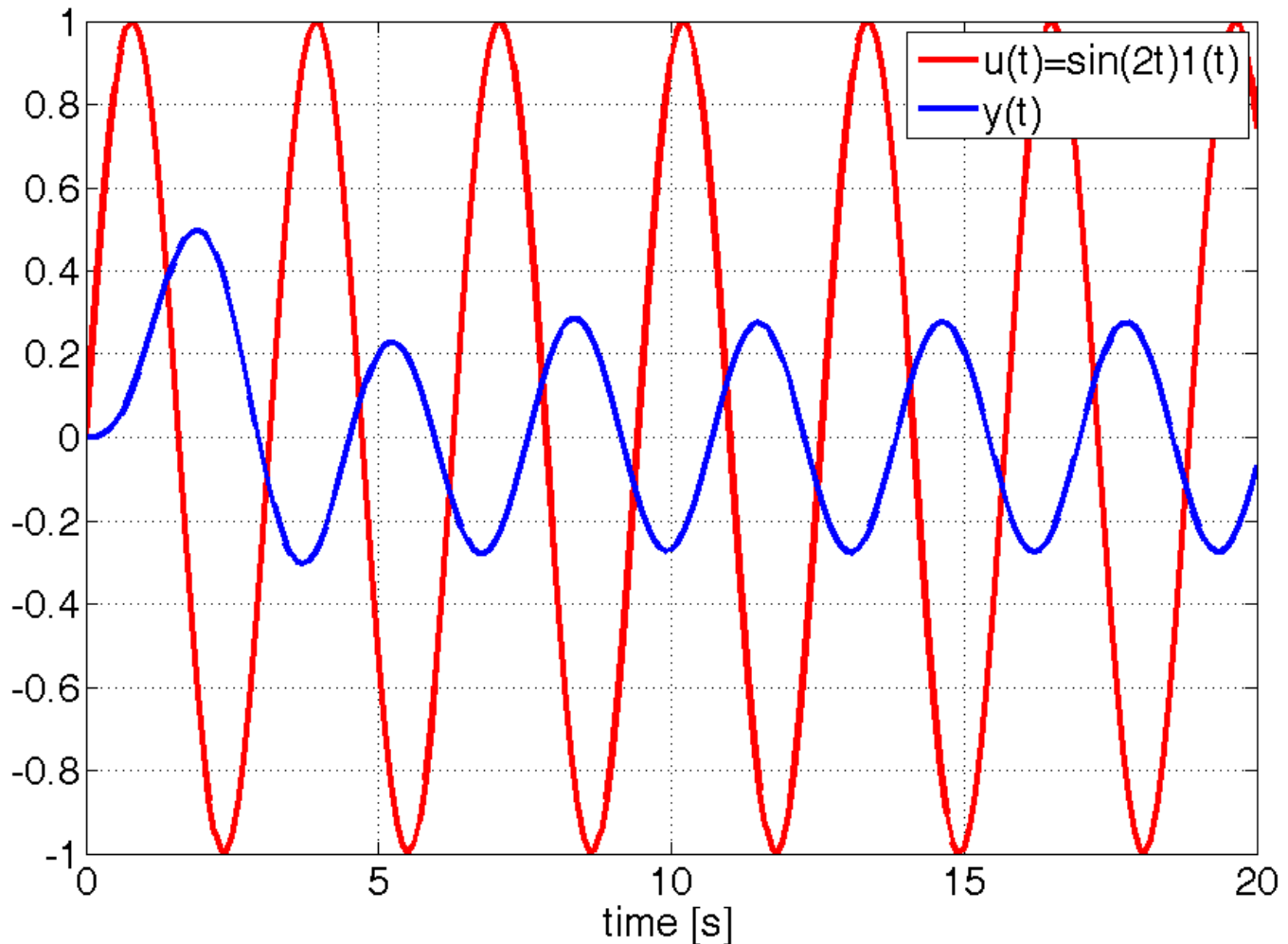
$$y(t) = Y\sin(\omega_0 t + \psi), t > 0$$

with $\quad Y = |W(j\omega_0)|U,$

where $W(j\omega_0) = C(j\omega_0 I - A)^{-1}B + D,$ and $\quad \psi = \varphi + \arg(W(j\omega_0)).$

If the system is a.s. $y(t)$ (and $x(t)$) represents the asymptotic movement of the output (state).

Total response of system $W(s) = 1/(s^2 + s + 1)$ to the input $u(t) = \sin(2t) \cdot 1(t)$.

⚑ The proposed result can be summarized as follows:

    ✦ The magnitude of a sinusoidal input signal $u(t) = \sin(\omega_0 t + \phi)$ is ***amplified or reduced*** by a linear system depending on the value of $|W(s)|_{s=j\omega_0}$.

    ✦ An input signal $u(t) = \sin(\omega_0 t + \phi)$ is ***phase shifted*** by a linear system depending on the value of $\angle W(s)|_{s=j\omega_0}$.

⚑ In other terms, ***a linear system can be designed as a filter*** able to amplify without distortion a certain set of input signals $\Omega_1$ and reduce or eliminate another signals.

⚑ Possible structures of filters will be discussed in the following lessons.

&#8593; This result underlines the importance of the function $W(j\omega)$ for the analysis of the forced response of LTI systems.

&#8593; The function $W(j\omega)$ is called ***harmonic response function*** of the system.

&#8593; In the following we present a method able to rapidly evaluate the magnitude and the phase $W(j\omega)$ as a function of $\omega$.

✣ Given an asymptotically stable LTI system, the **_harmonic response function_** $W(j\omega)$ is given by the ratio of polynomial with real and complex conjugate roots

$$W(j\omega) = W(s)\big|_{s=j\omega} = K \frac{s^{\nu} \prod_i (1 + \sigma_i s)^{m_i} \prod_q \left(1 + \dfrac{2\xi_q}{\omega_{nq}} s + \dfrac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_j (1 + \tau_j s)^{n_j} \prod_p \left(1 + \dfrac{2\zeta_p}{\omega_{np}} s + \dfrac{s^2}{\omega_{np}^2}\right)^{\kappa_p}}\Bigg|_{s=j\omega}$$

&#x2721; ***Bode diagrams*** allows to extract the magnitude and the phase of $W(j\omega)$ as a function of $\omega$

&#x2721; Bode diagrams are a main tool for the closed loop control design

&#x2721; For the closed loop control problems we will be interested to analyze magnitude and the phase of transfer functions $W(s)$ also in case of stable and unstable systems

&#x2721; In that cases, $W(s)|_{s=j\omega}$ will be not the harmonic function.