Course of “Industrial Automation”
2023/24

Analysis of the Sample and Hold - Sampled-Data Systems

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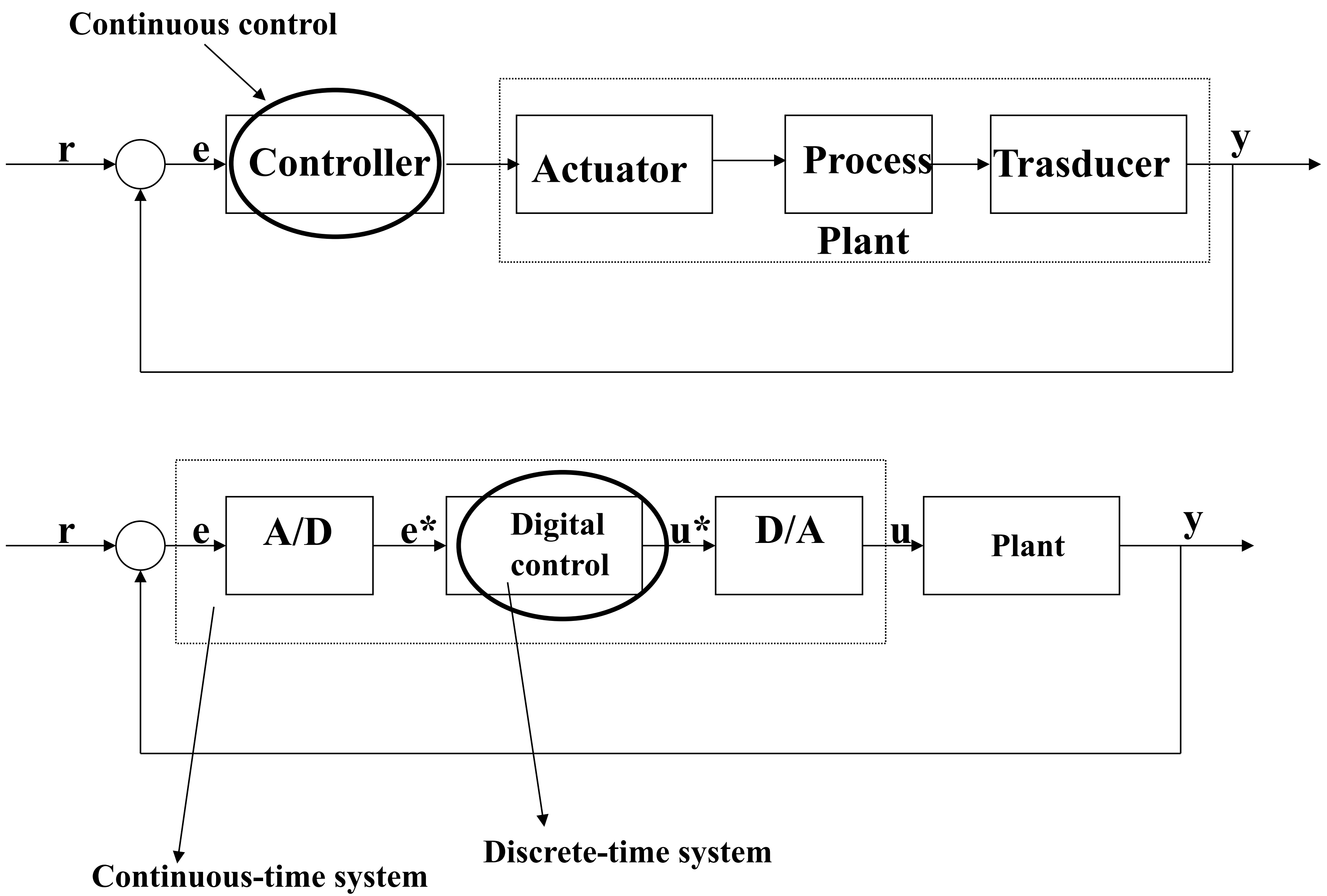
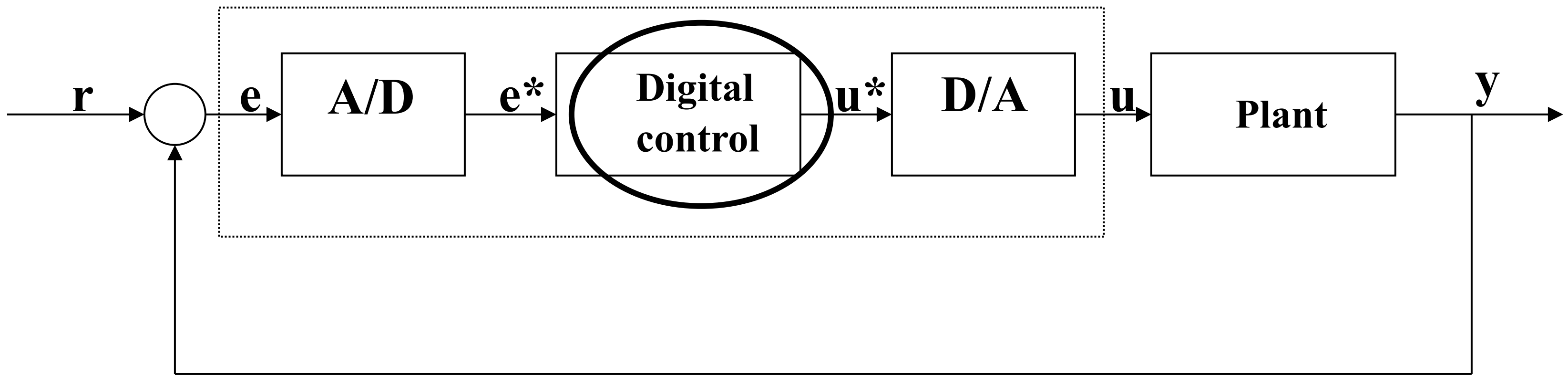
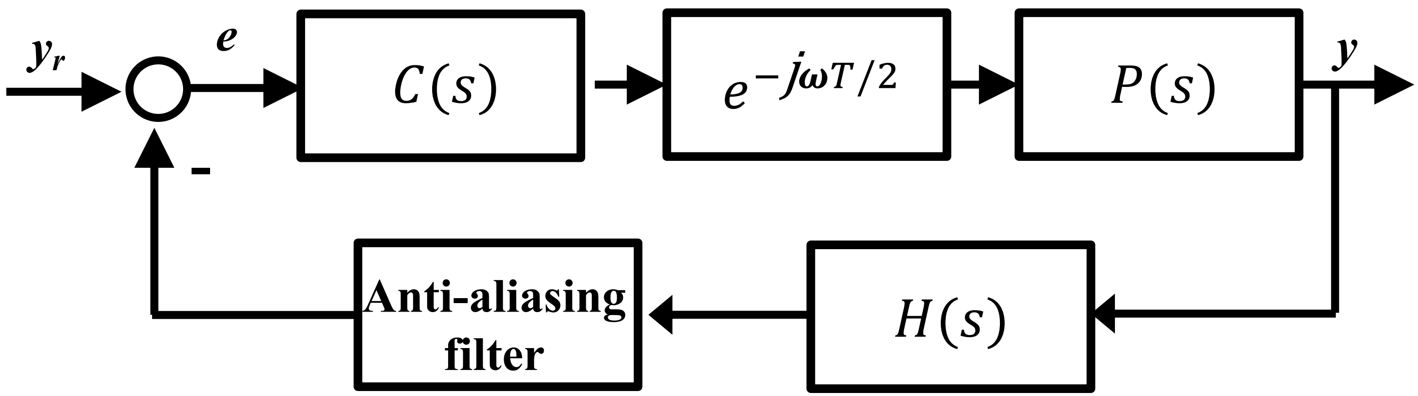
Team code: **vgxlryz**

Continuous vs. digital



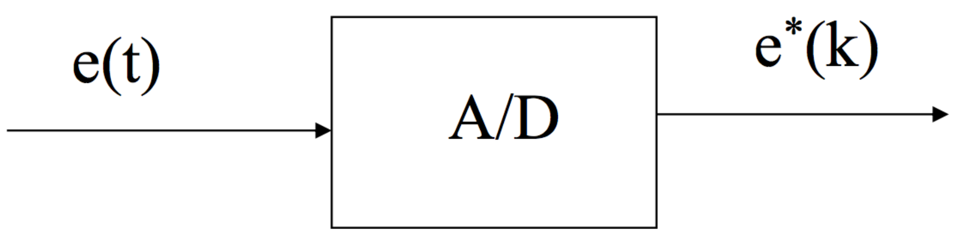
Scheme of the digital control system in continuous-time





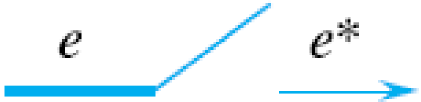
Analog-to-digital converter (ADC, A/D, or A-to-D)

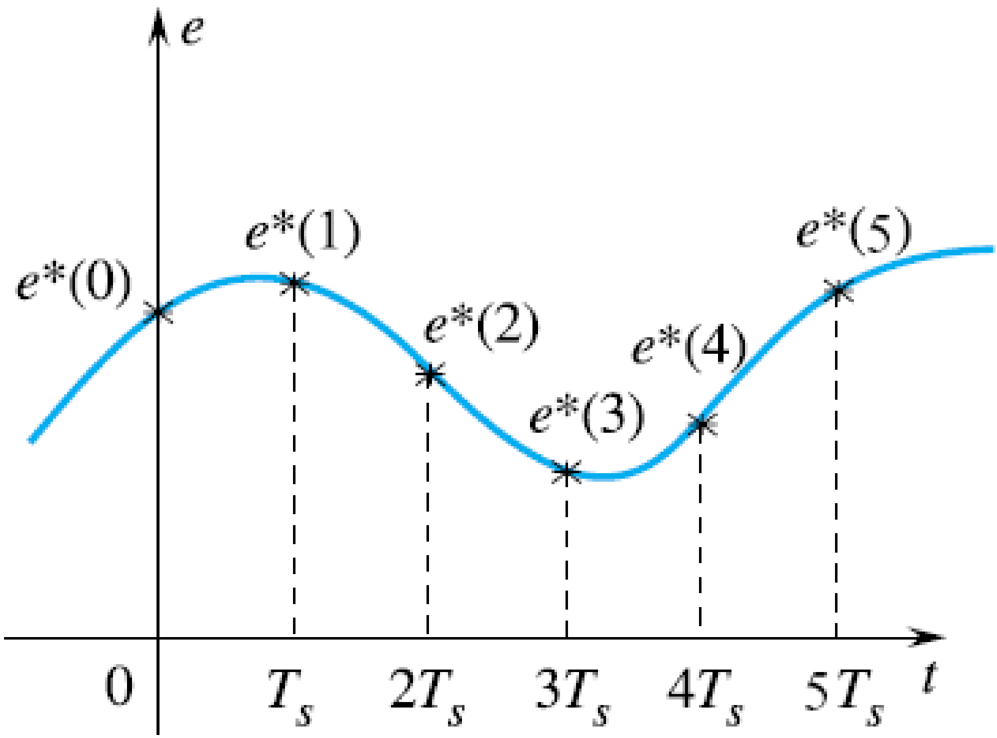
- The digital controller is a discrete-time system and the plant to be controlled is a continuous-time system.
- It is needed a device that transforms a continuous signal into a discrete one.



- Such device is the analog-to-digital converter (A/D).

Ideal sampler





- Periodic sampling: the sampling instants are equally spaced, or k, i.e. $t_k = kT_s$ (k=0,1,2,..), with T_s representing the sampling time.
- The hold circuit holds the value of the sampled signal over a specified period of time.

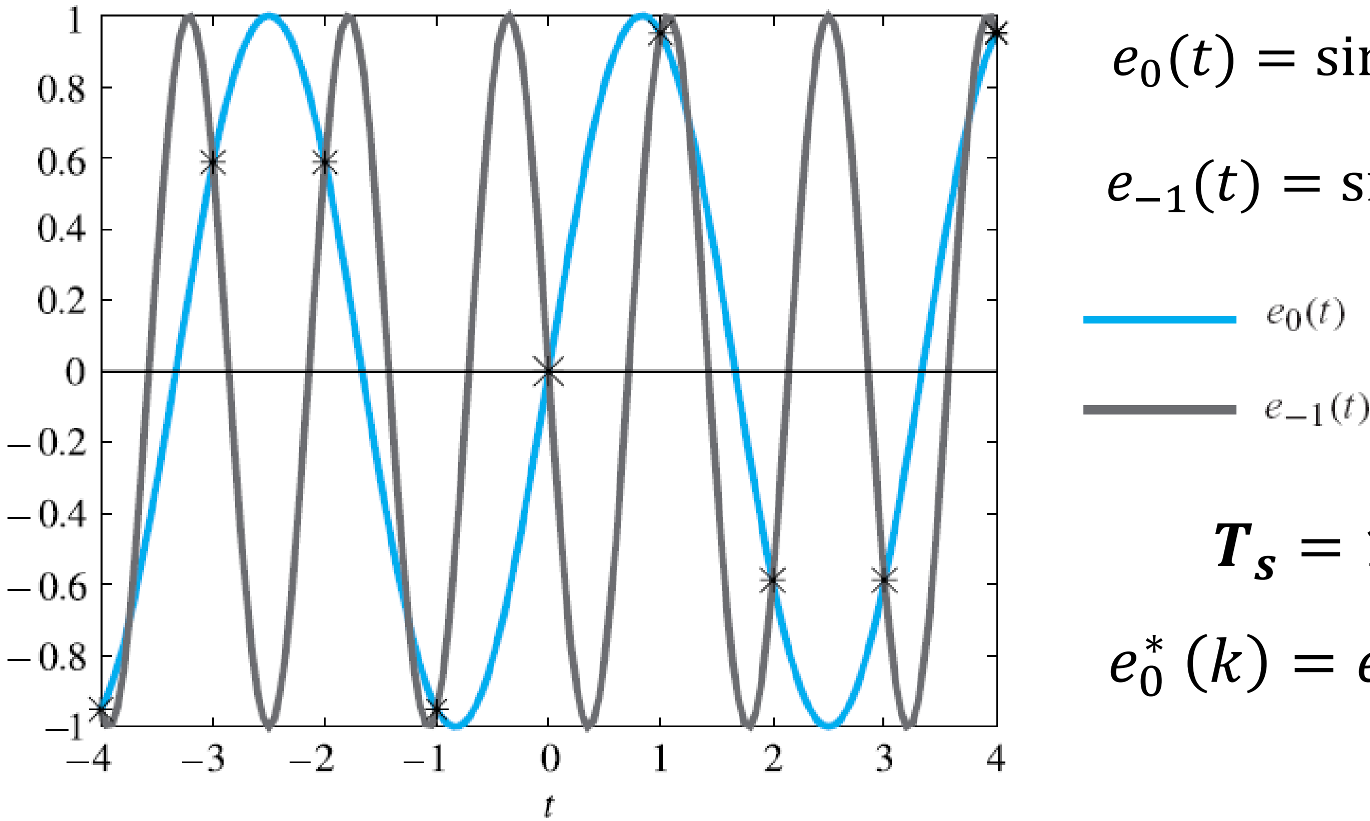
Sampling operation

- The common problem when sampling a signal is the loss of information.

- Indeed, it is obvious that the same signal $e^*(k)$ can be generated by infinite continuous-time functions $e(t)$.

- Hence, given a signal $e^*(k)$ it is impossible to go back to the original signal $e(t)$.

Given a set of signals, $e_h(t) = \sin\big((\alpha + 2h)\pi t\big)$



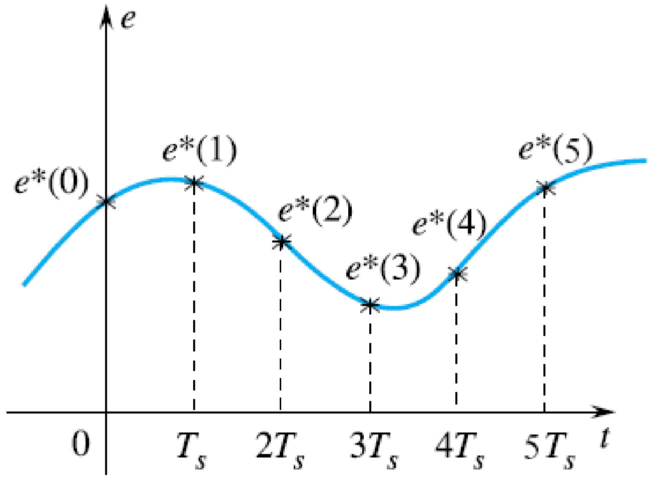
$e_0(t) = \sin(0.6\pi t)$

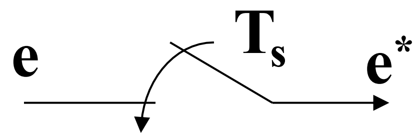
$e_{-1}(t) = \sin(-1.4\pi t)$

$\boldsymbol{T_s = 1}$

$e_0^*(k) = e_1^*(k)$

Ideal sampler –Spectrum of a sampled signal





$$\omega_S = 2\pi f_S = \frac{2\pi}{T_S}$$

$$e^*(k) = e(kT_s)$$

Assume $e_s(t)$ as the mathematical representation of the sampling operation:

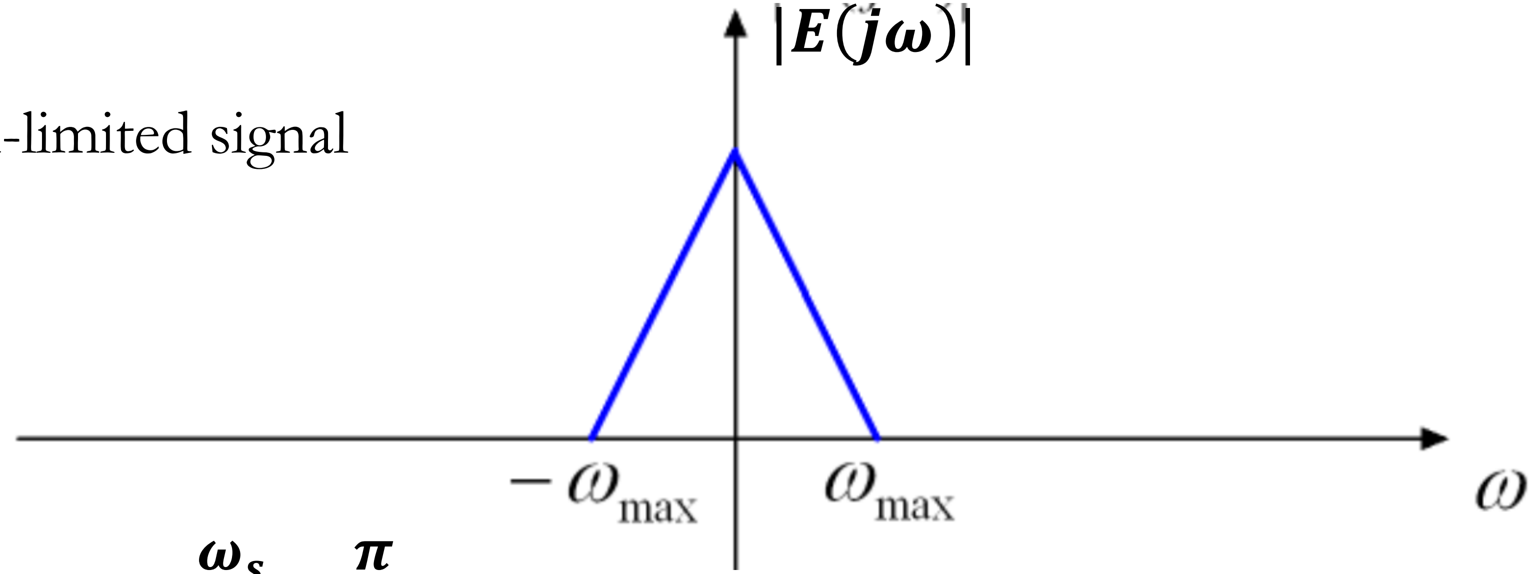
$$e_s(t) = e(t)\sum_{k=0}^{\infty}\delta(t - kT_s) = \sum_{k=0}^{\infty} e(kT_s)\delta(t - kT_s)$$

$\mathcal{F}$ (Fourier transform)

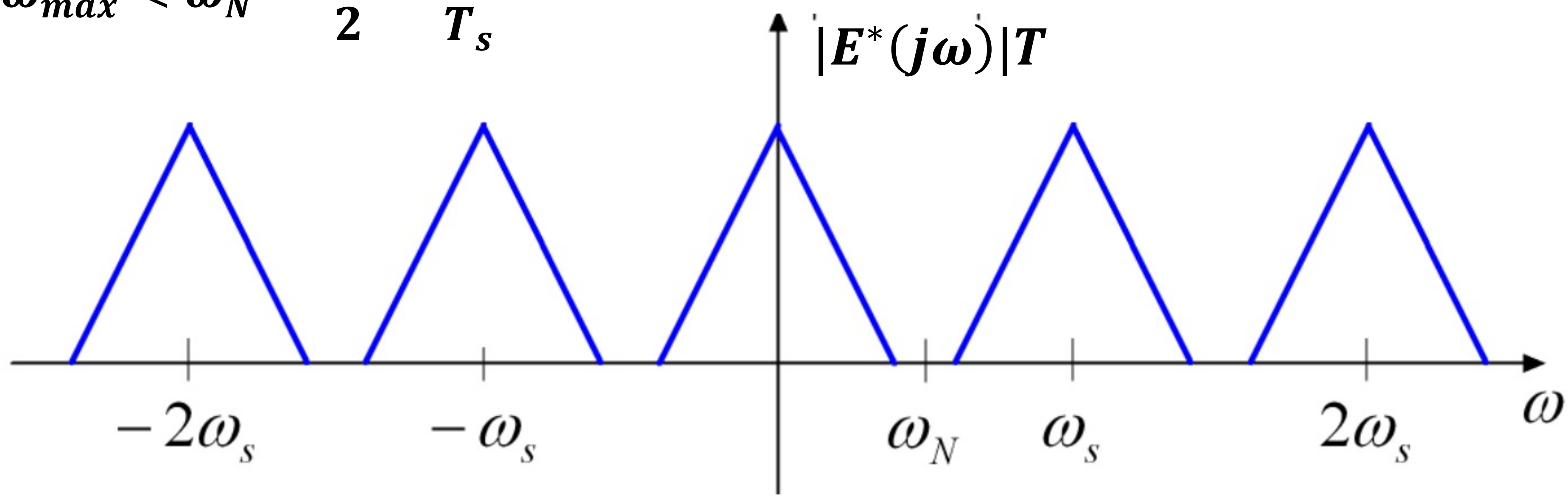
$$E^*(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} E(j\omega + jn\omega_s)$$

Ideal sampler –Spectrum of a sampled signal

Band-limited signal

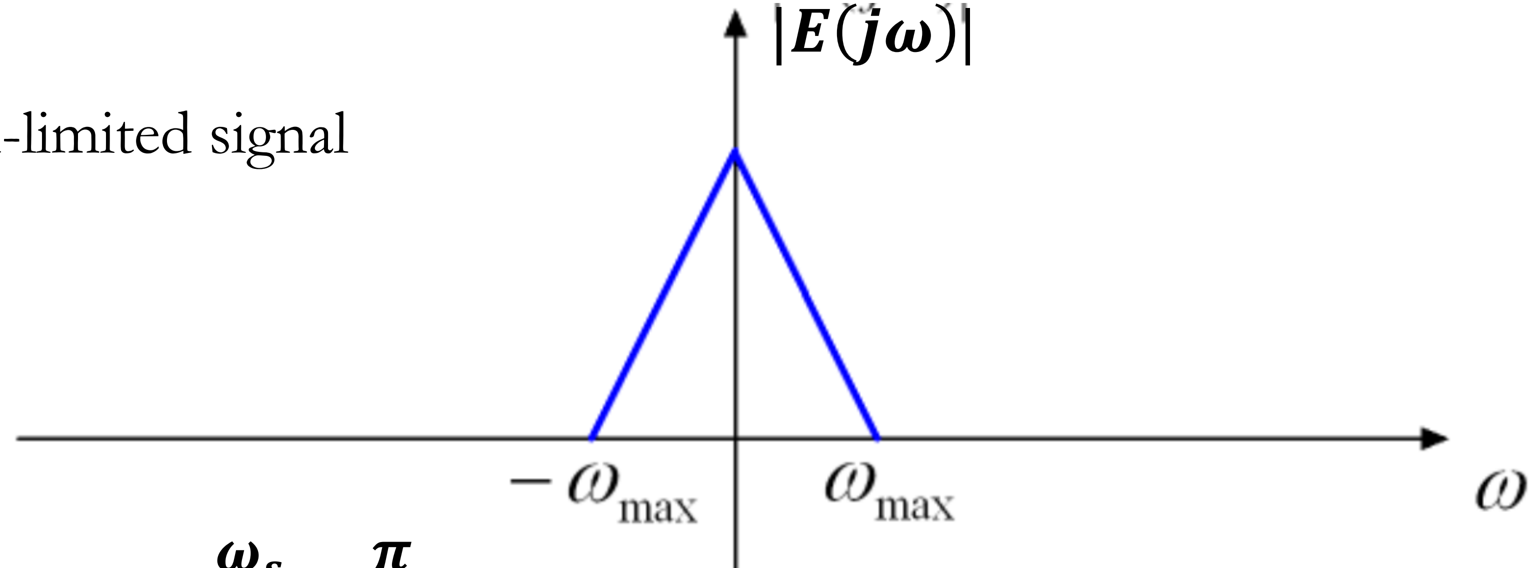


$$\omega_{max} < \omega_N = \frac{\omega_s}{2} = \frac{\pi}{T_s}$$



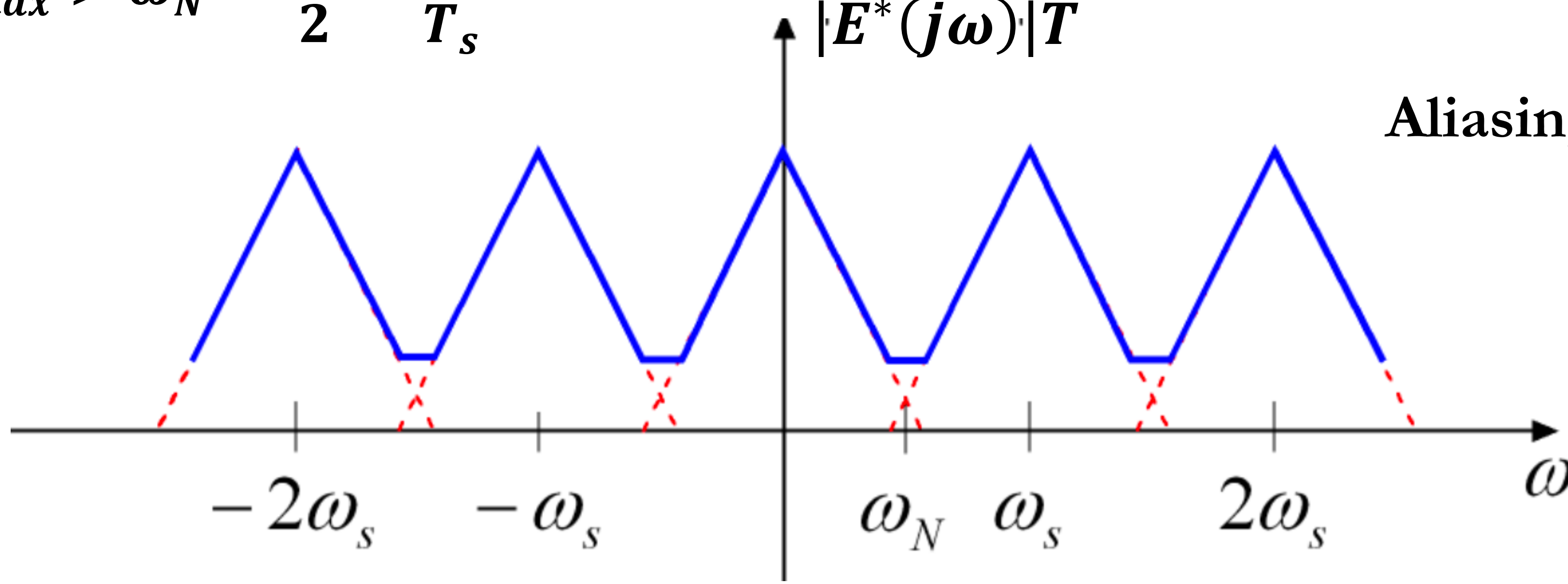
Ideal sampler –Spectrum of a sampled signal

Band-limited signal



$$\omega_{max} > \omega_N = \frac{\omega_s}{2} = \frac{\pi}{T_s}$$

Aliasing



Shannon's Theorem

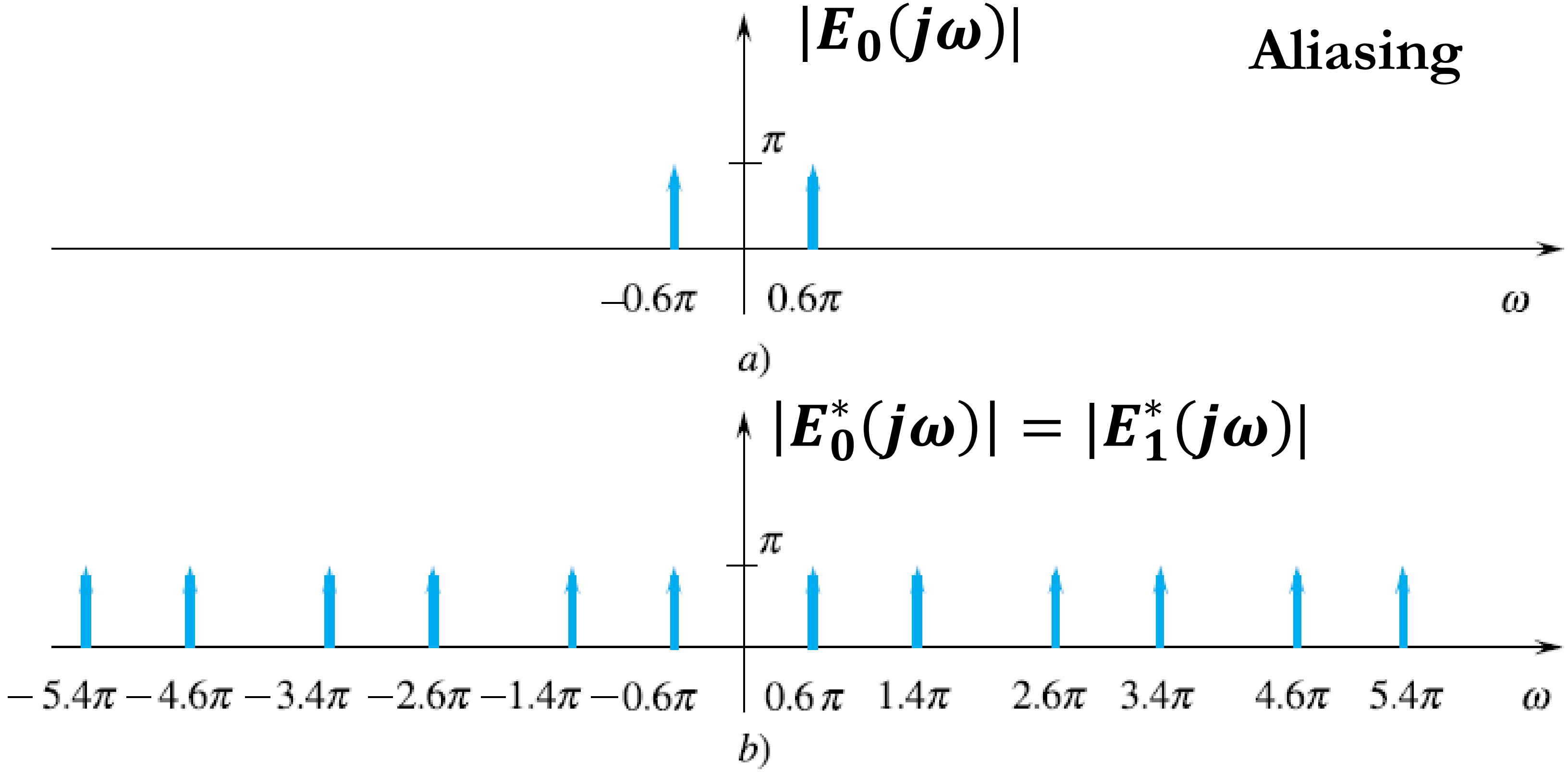
In order for an analog signal (*e*(*t*)) to be reconstructed from its sampled version (*e(*k*)), by Shannon's theorem, it must have a strictly limited bandwidth and $\boldsymbol{\omega_s > 2\omega_{max}}$**

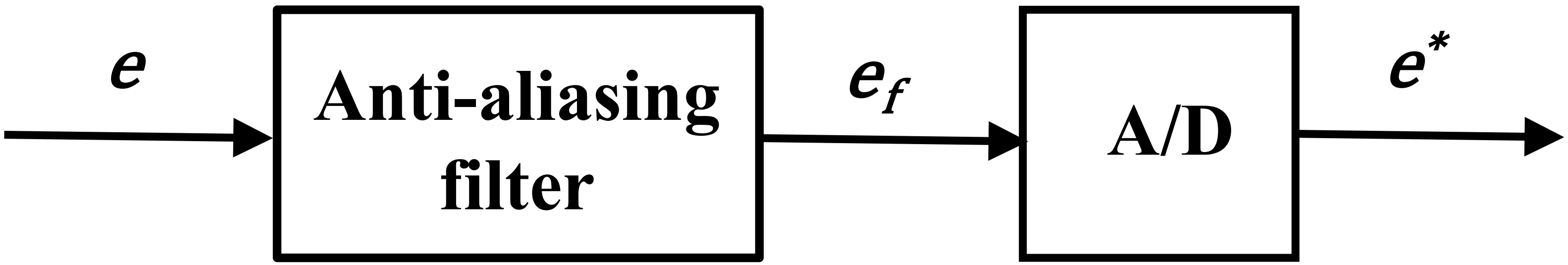
$$\boldsymbol{\omega_s > 2\omega_{max}}$$

$$e_0(t) = \sin(0.6\pi t)$$

$$e_{-1}(t) = \sin(-1.4\pi t)$$

Aliasing





In this way the spectrum of the signal to be sampled doesn't have significant components for $\omega > \omega_f$

$$\omega_f < \omega_N = \frac{\omega_s}{2}$$

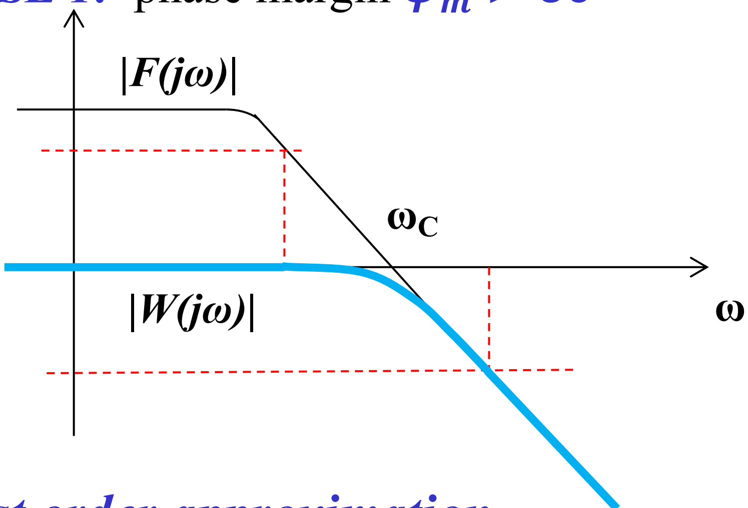
$$\omega_s > 2\omega_f$$

Bandwidth of the control system

Taking into accout the bandwidth of the control system, approximated by ω_C, then for Shannon's Theorem

$$\omega_s > 2\omega_c$$

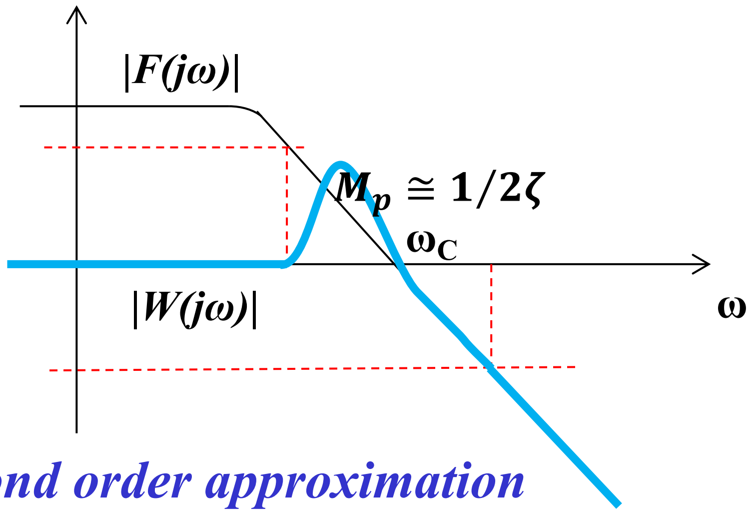
CASE 1: phase margin $\varphi_m > 60°$



First order approximation

$$W_a(s) = \frac{1}{1 + s/\omega_c}$$

CASE 2: phase margin $\varphi_m < 60°$



Second order approximation

$$W_a(s) = \frac{1}{1 + 2\zeta s/\omega_c + s^2/\omega_c^2} \quad \text{and } \zeta \cong \frac{\varphi_m}{100}$$

Antialiasing filter – Bandwidth of the control system

The antialiasing filter can reduce the phase margin by destabilizing the entire control system, then

$$\boldsymbol{\omega_f \gg \omega_C}$$

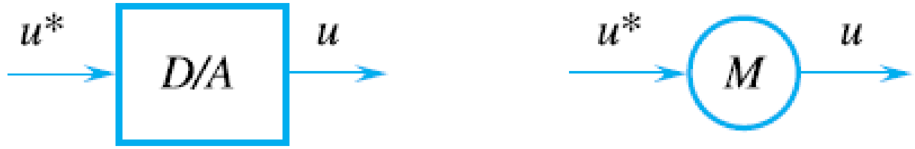
By taking into account all the constraints

$$\boldsymbol{\omega_s > 2\omega_f \gg \omega_C}$$

In general $\boldsymbol{\omega_s}$ is given by

$$\boldsymbol{\alpha\omega_C < \omega_s < 10\alpha\omega_C}$$

with $5 \leq \boldsymbol{\alpha} \leq \mathbf{10}$



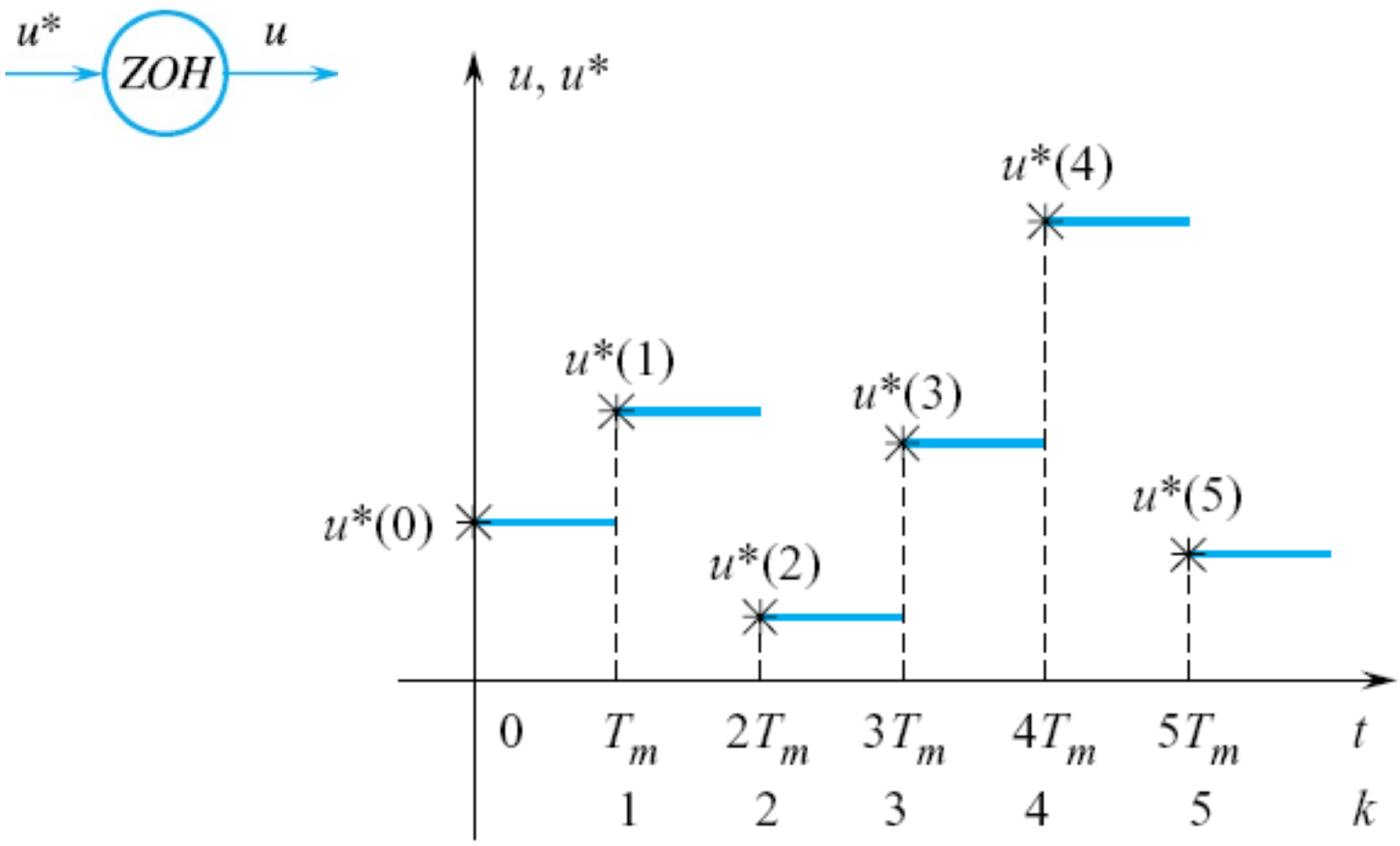
- It is a device that transforms a digital input (binary numbers) to an analog output.

- The most commonly used D/A converter is the zero order hold (ZOH), which operates as follows:

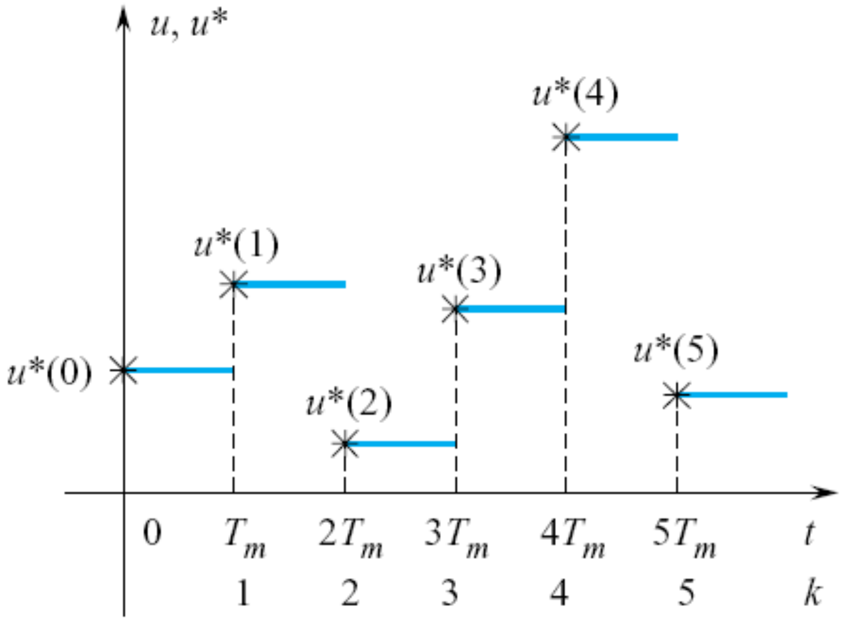
$$u(t) = u^{*}(k) \quad t \in \left[kT_m, (k+1)T_m\right]$$

- T_m is the sample time

ZOH circuit



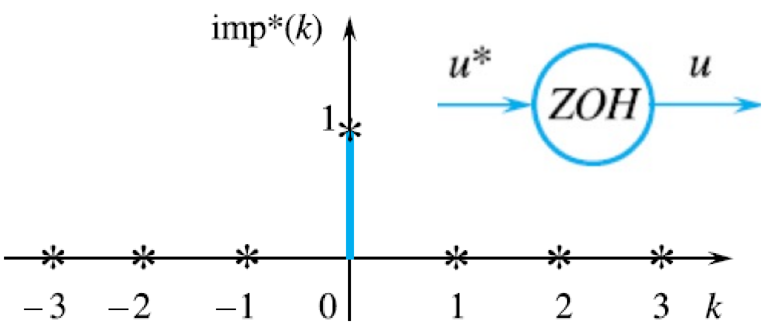
ZOH - 1

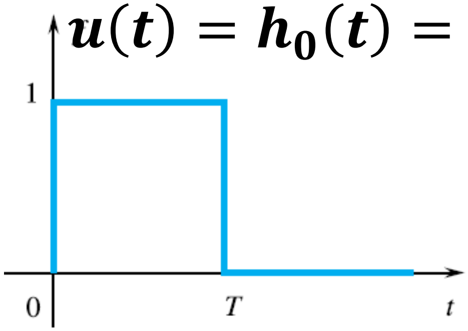


$$\omega_m = 2\pi f_m = \frac{2\pi}{T_m}$$

$$u(t) = u^*(k) \quad t \in [kT_m, (k+1)T_m]$$

Relationship in terms of transform between the input samples $u^*(k)$ and the output signal $u(t)$ $(T = T_m)$





$$u(t) = h_0(t) = 1(t) - 1(t - T)$$

$\Big\downarrow \mathcal{L}$

$$H_0(s) = \frac{1 - e^{-sT}}{s}$$

ZOH - 2

In general, for an input sequence $u^*(k)$ with zeta-Transform $U^*(z)$, the corresponding output signal

$$u(t) = \sum_{k=0}^{\infty} u^*(k)\, h_0(t-kT) \xrightarrow{\mathcal{L}} U(s) = \sum_{k=0}^{\infty} u^*(k) e^{-skT}\, H_0(s) = H_0(s) U^*\left(e^{sT}\right).$$

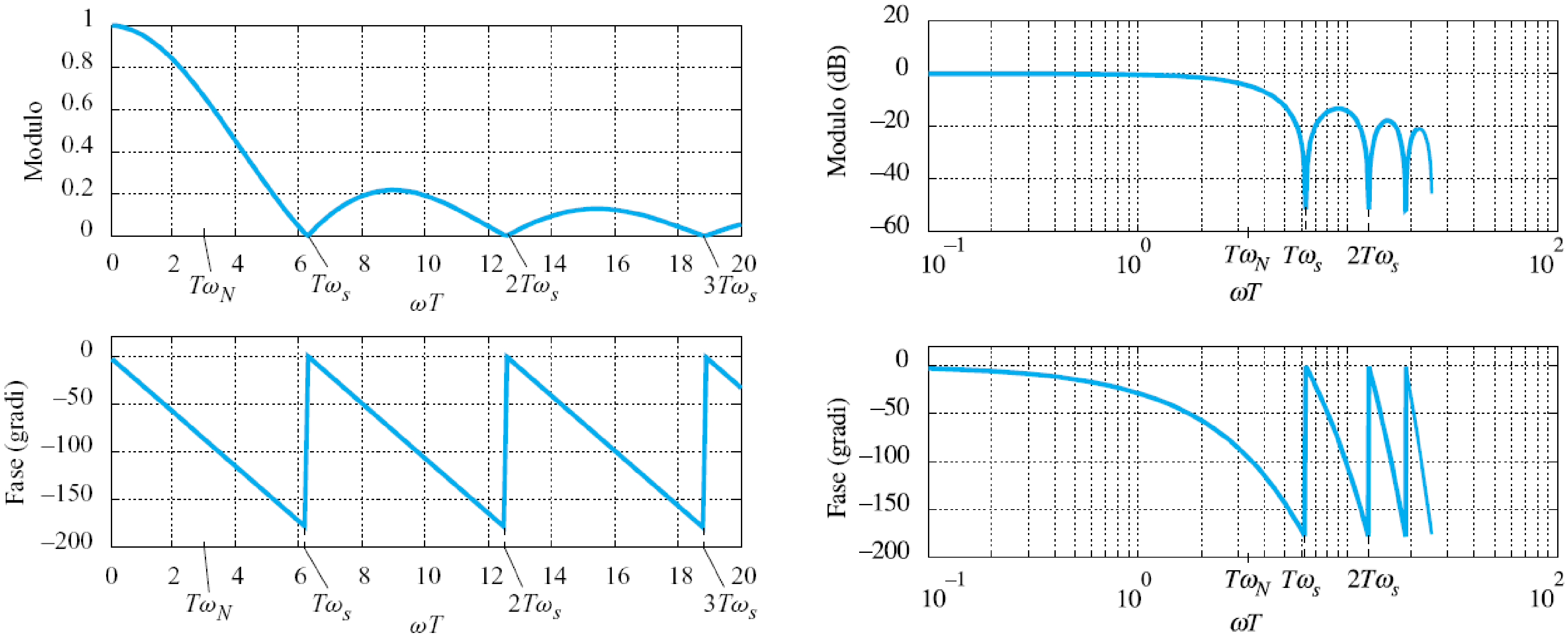
$$\Rightarrow \quad H_0(s) = \frac{U(s)}{U^*(e^{sT})},$$ **it can be assumed as the tf of ZOH**

Moreover, for $s = j\omega$, the ZOH frequency response can be evalueted

$$H_0(j\omega) = \frac{1-e^{-j\omega T}}{j\omega} = e^{-j\omega T/2} \frac{e^{-j\omega T/2} - e^{-j\omega T/2}}{2j} 2j = T e^{-j\omega T/2} \frac{\sin \omega T/2}{\omega T/2}$$

Frequency response of **ZOH:**

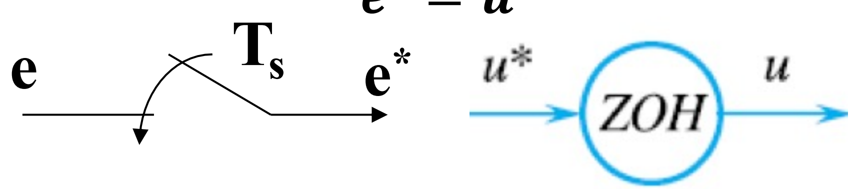
Bode Diagrams of $\boldsymbol{H_0(j\omega)}/\mathrm{T}$ in linear and log. scale ($\boldsymbol{\omega_N = \pi/T}$)

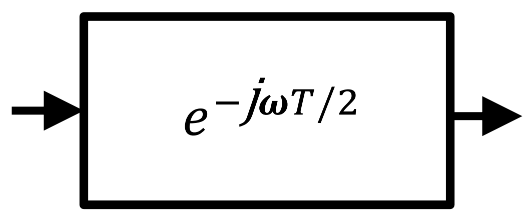


Sampler – ZOH series

- By working in the frequency range $\omega < \frac{\omega_S}{8}$, it is possible to approximate the sampler-zoh series (hp $T_s = T_m = T$) with a delay element

$$e^* = u^*$$



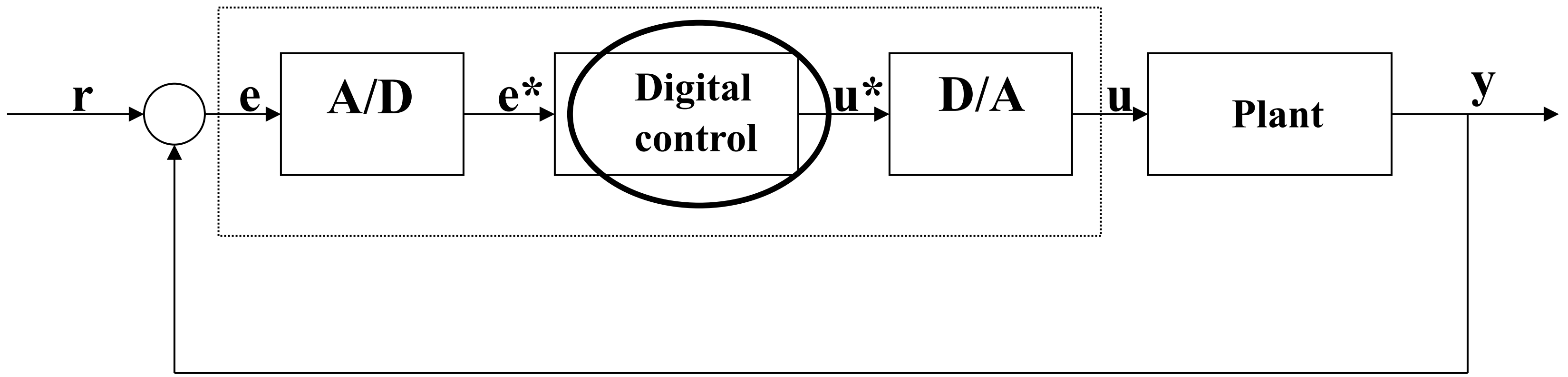


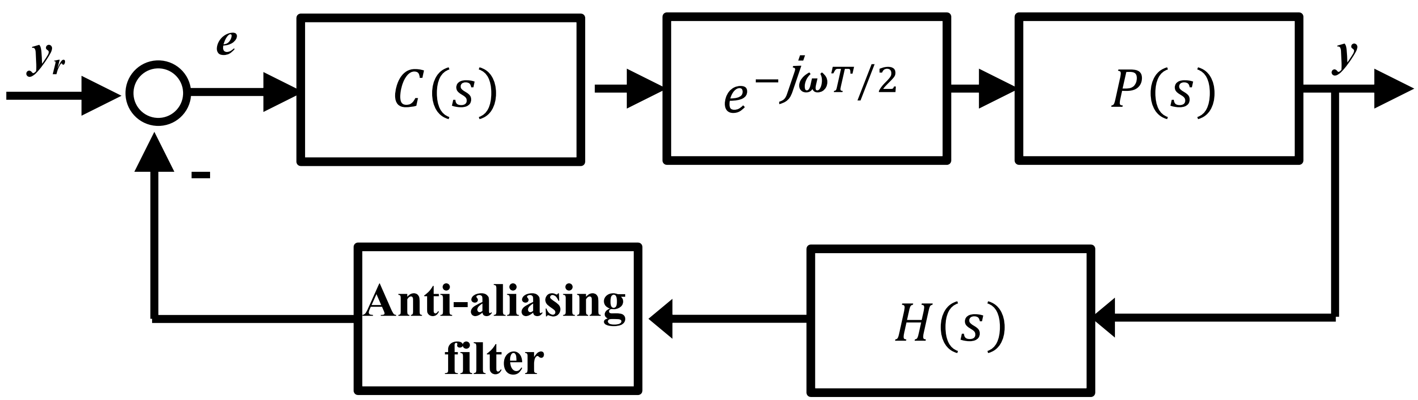
$$\frac{\omega T}{2}\Big|_{\omega=\frac{\omega_S}{8}} \approx 22°$$

- where this term introduces a maximum delay equal to

- The presence of a numerical control tends to destabilize the entire system.

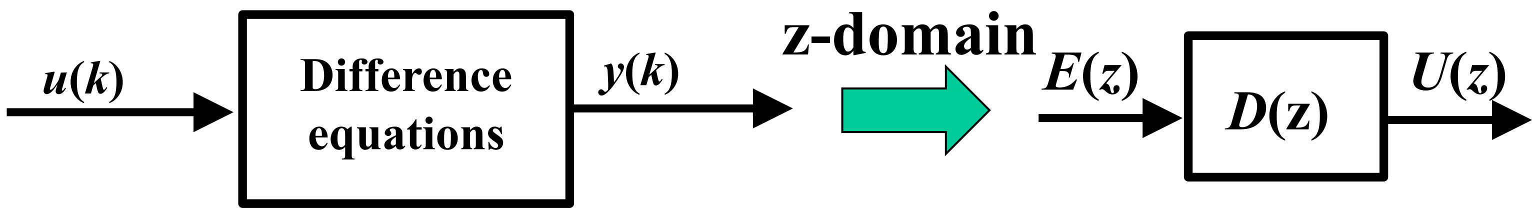
Scheme of the digital control system in continuous-time





From analog to digital

- Implementation of the digital control:



- From $C(s)$ we want to find an equivalent $D(z)$:
- A transformation allows the transition from continuous time to discrete time such that $$C(s) \cong D(z)$$

same static and dynamic perfomance

$$\boldsymbol{s \longrightarrow z}$$

- Several transformations:

The Laplace trasform of an ideally sampled signal corresponds to the Zeta of the sampled sequence with the sostitution

$$z = e^{sT}$$

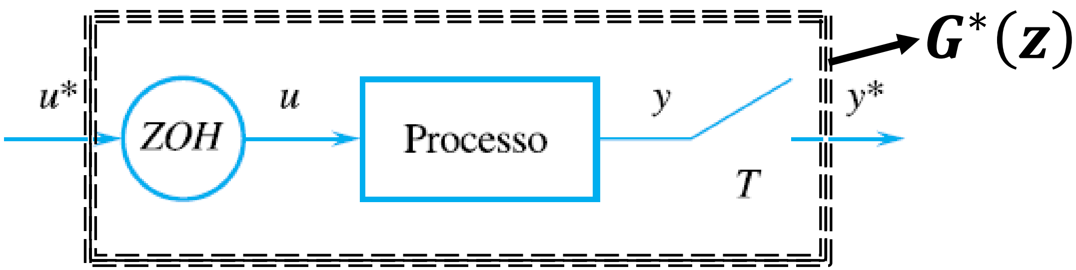
By a first-order Padé approximant of the natural logarithm function,

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

Others: By Euler's method,

$s = \frac{z-1}{T}$ (forward rectangular rule) and $s = \frac{z-1}{zT}$ (backward)

Here, we develop the analysis needed to compute the discrete transfer function between the samples that come from the digital computer to the D/A converter and the samples picked up by A/D converter.



This scheme represents a discrete time system describing the continuous time systems at the sampling times (it describes the e relationship between $\boldsymbol{u}^*$ and $\boldsymbol{y}^*$)

The discrete tf, $\boldsymbol{G}^*(\boldsymbol{z})$, corresponds to the z-Transform of the output samples y_δ^* ($Y_\delta^*(z)$) when the input samples are the unit pulse $\delta(k)$ at $k = 0$:

$$u_\delta^*(k) = \delta(k) \rightarrow U_\delta^*(z) = Z\big(\delta(k)\big) = 1 \rightarrow \boldsymbol{G}^*(\boldsymbol{z}) = \frac{Y_\delta^*(z)}{U_\delta^*(z)} = Y_\delta^*(z)$$

Sampled -data system - 2

For $u^*(k) = \delta(k) \rightarrow u(t) = h_0(t) = 1(t) - 1(t-T)$

$$\xrightarrow{\mathcal{L}} \quad H_0(s) = \frac{\mathbf{1 - e^{-sT}}}{\boldsymbol{s}}$$

Then $y_\delta(t)$ is the difference between the step response of $G(s)$ (i.e., response to 1(t), denoted with $y_s(t)$) and the delayed (T) step response, denoted with $y_s(t-T)$. Thus, by applying Laplace,

$$Y_\delta(s) = H_0(s)G(s) = \frac{\mathbf{1-e^{-sT}}}{\boldsymbol{s}}G(s) = \left(\mathbf{1 - e^{-sT}}\right)\frac{G(s)}{s} = \left(\mathbf{1 - e^{-sT}}\right)Y_s(s)$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y_\delta(t) = y_s(t) - y_s(t-T)$$

By sampling: $y_\delta^*(kT) = y_s(KT) - y_s((k-1)T)$

And by applying z-transform ($Y(z) = Z\left(y_s(KT)\right)$):

$$Y_\delta^*(z) = Y(z) - \frac{Y(z)}{z} = \frac{z-1}{z}Y(z)$$

Sampled -data system - 3

$$\Rightarrow \quad G^*(z) = Y_\delta^*(z) = \frac{z-1}{z} Y(z) = \frac{z-1}{z} Z\left(\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\Bigg|_{t=KT}\right)$$

Summarizing, by this procedure it is possible to obtain the discrete tf of the sampled-data system, $G^*(z)$:

1. **Determine the step response of the continuous LTI system in the Laplace domain, $Y_s(s)= \frac{G(s)}{s}$.**
2. **Antitransform $Y_s(s)$ thereby determining the samples of the output $y_s(kT)$.**
3. **Compute the z-transform of the output samples $y_s(kT)$: $\boldsymbol{Z}\left(y_s(kT)\right)$**
4. **Determine the tf of $G^*(z)= \frac{z-1}{z}\boldsymbol{Z}\left(y_s(kT)\right)$**

Sampled -data system - Example

What is the discrete transfer function of

$$\boldsymbol{G(s) = \frac{1}{s(s+1)}},$$

preceded by a ZOH?

Solution:

1. $Y(s) = \frac{G(s)}{s} = \frac{\mathbf{1}}{\boldsymbol{s^2(s+1)}} = \frac{\mathbf{1}}{\boldsymbol{s^2}} - \frac{\mathbf{1}}{\boldsymbol{s}} + \frac{\mathbf{1}}{\boldsymbol{s+1}}$
2. $y_s(kT) = \mathcal{L}^{-1}(Y(s))|_{t=KT} = (t - 1 + e^{-t})1(t)|_{t=KT} = \left(kT - 1 + e^{-kT}\right)1(kT)$
3. $Z\left(y_s(kT)\right) = \frac{\boldsymbol{Tz}}{\boldsymbol{(z-1)^2}} - \frac{\boldsymbol{z}}{\boldsymbol{z-1}} + \frac{\boldsymbol{z}}{\boldsymbol{z-e^{-T}}}$
4. $G^*(z) = \frac{z-1}{z}\left(\frac{\boldsymbol{Tz}}{\boldsymbol{(z-1)^2}} - \frac{\boldsymbol{z}}{\boldsymbol{z-1}} + \frac{\boldsymbol{z}}{\boldsymbol{z-e^{-T}}}\right) = \frac{\boldsymbol{T}}{\boldsymbol{z-1}} - 1 + \frac{\boldsymbol{z-1}}{\boldsymbol{z-e^{-T}}}$