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Frequency response

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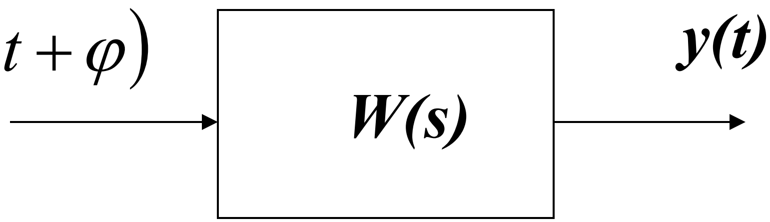
Steady state response of continuous LTI systems to sinusoidal inputs - 1

- Let us consider an asymptotically stable LTI system with a transfer function $W(s)$ subject to a sinusoidal input signal

$$u(t) = U_0 \sin(\omega_0 t + \varphi)$$



- The evaluation of the steady state response of LTI system to sinusoidal inputs is very important:

any signal can be decomposed in the sum of a finite (periodic signal) and infinite number (aperiodic signal) of sinusoids by means of the Fourier series.

Steady state response of continuous LTI systems to sinusoidal inputs - 2

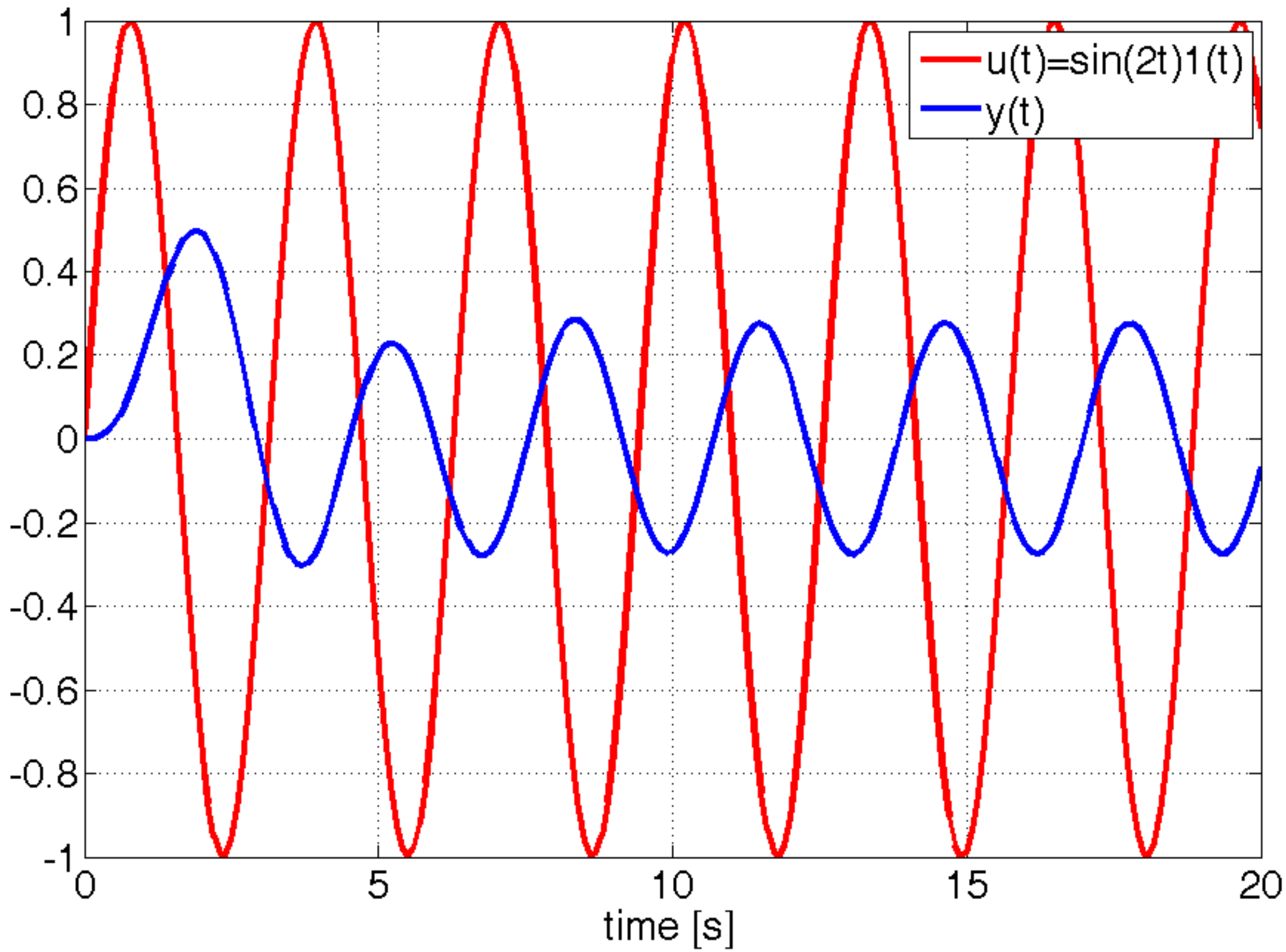
- It is possible to prove that the steady state response of an LTI system with transfer function $W(s)$ to a sinusoidal inputs $u(t) = \mathrm{U}_0 \sin(\omega_0 t + \phi)$ can be written in the time domain as

$$y_{ss}(t) = U_0 \left| W(s) \right|_{s=j\omega_0} \sin\left(\omega_0 t + \varphi + \angle W(s)_{s=j\omega_0}\right)$$

where

- $|W(s)|_{s=j\omega_0}$ is the magnitude of the Laplace transform of $W(s)$ evaluated in $s = j\omega_0$.
- $\angle W(s)|_{s=j\omega_0}$ is the phase of the Laplace transform of $W(s)$ evaluated in $s = j\omega_0$.

Total response of system $W(s) = 1/(s^2+s+1)$ to the input $u(t) = \sin(2t) \cdot 1(t)$.



- The proposed result can be summarized as follows:
  - The magnitude of a sinusoidal input signal $u(t) = \sin(\omega_0 t + \phi)$ is ***amplified or reduced*** by a linear system depending on the value of $|W(s)|_{s=j\omega_0}$.
  - An input signal $u(t) = \sin(\omega_0 t + \phi)$ is ***phase shifted*** by a linear system depending on the value of $\angle W(s)|_{s=j\omega_0}$.
- In other terms, ***a linear system can be designed as a filter*** able to amplify without distortion a certain set of input signals Ω_1 and reduce or eliminate the other signals.

- This result underlines the importance of the function $W(j\omega)$ for the analysis of the forced response of LTI systems.

- The function $W(j\omega)$ is called ***harmonic response function*** of the system.

- In the following we present a method able to rapidly evaluate the magnitude and the phase $W(j\omega)$ as a function of ω.

$W(j\omega)$ general form - 1

- Given an asymptotically stable LTI system, the ***harmonic response function*** $W(j\omega)$ is given by the ratio of polynomial with real and complex conjugate roots

$$W(j\omega) = W(s)\big|_{s=j\omega} = K \left. \frac{s^{\nu} \prod_i (1+\sigma_i s)^{m_i} \prod_q \left(1 + \frac{2\xi_q}{\omega_{nq}} s + \frac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_j (1+\tau_j s)^{n_j} \prod_p \left(1 + \frac{2\zeta_p}{\omega_{np}} s + \frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}} \right|_{s=j\omega}$$

$W(j\omega)$ general form - 2

- ***Bode diagrams*** allows to extract the magnitude and the phase of $W(j\omega)$ as a function of ω

- Bode diagrams are a main tool for the closed loop control design

- For the closed loop control problems, we are interested to analyze magnitude and phase of transfer functions $W(s)$ also in case of stable and unstable systems

- In that cases, $W(s)|_{s=j\omega}$ is not the harmonic function.

Bode diagrams definition

- The y-axis of the magnitude and phase Bode diagrams indicate respectively
  - ***the magnitude of the transfer function in dB (decibel)***

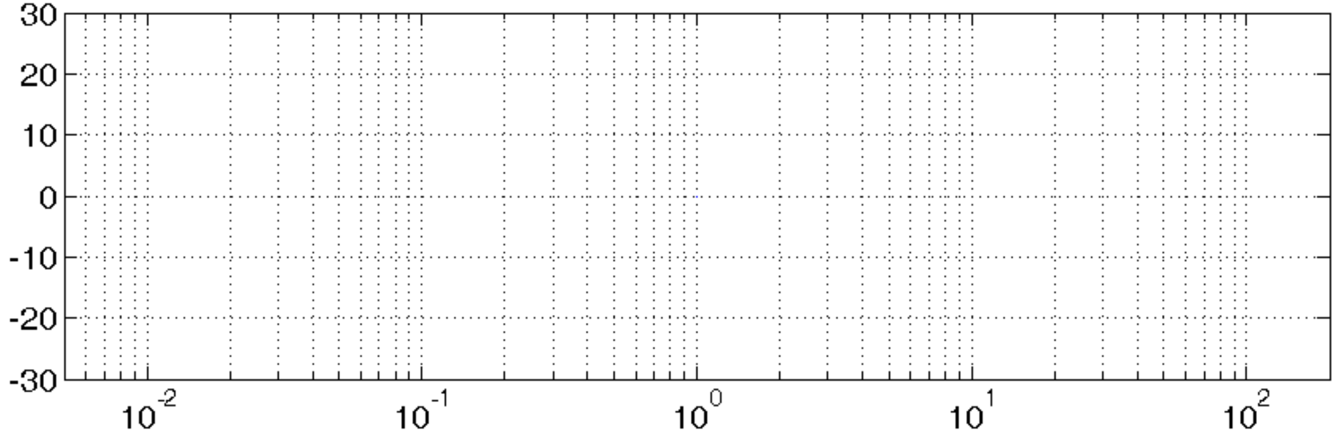
$$|\mathrm{W}(\mathrm{j}\omega)|_{\mathrm{db}} = 20\log_{10}|\mathrm{W}(\mathrm{j}\omega)|$$

  - ***the phase of the transfer function in degrees or radians***

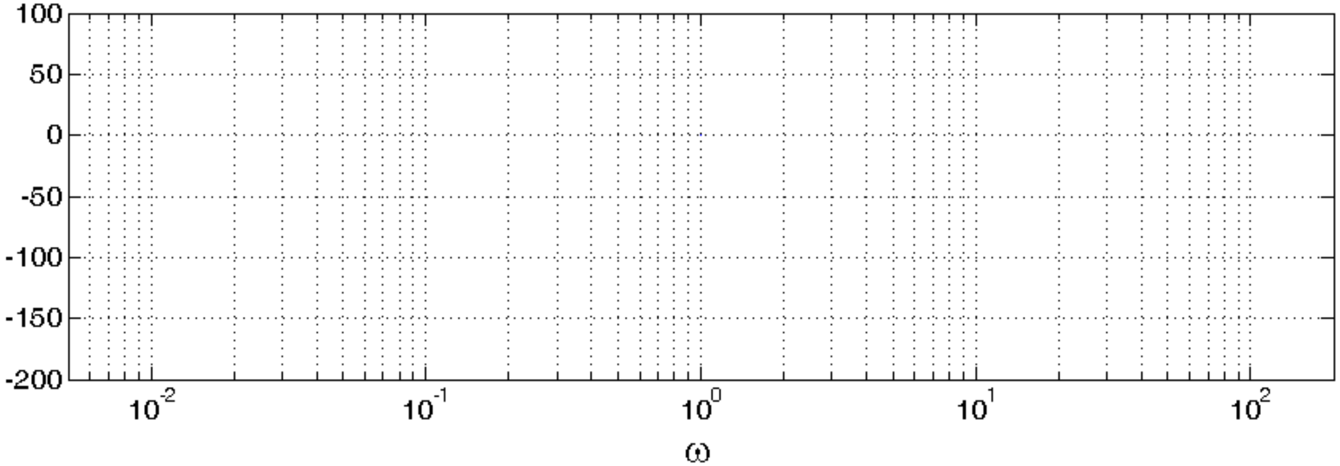
$$\angle W(j\omega)$$

Magnitude and phase diagrams

magnitude [dB]



Phase [deg]



Bode diagrams of the magnitude

- The magnitude of the Bode diagrams is expressed in decibel firstly because the logarithmic scale allows to consider ***large magnitude intervals with limited space*** (ex: $|10|_{db} = 20$, $|100|_{db} = 40$, $|1000|_{db} = 60$)

- Moreover, the magnitude of $W(s)|_{s=j\omega}$ in decibel can be written has

$$\left|W(j\omega)\right|_{db} = 20\log_{10}\left(K \frac{s^{\nu}\prod_i (1+\sigma_i s)^{m_i} \prod_q \left(1+\frac{2\xi_q}{\omega_{nq}}s+\frac{s^2}{\omega_{nq}^2}\right)^{\eta_q}}{\prod_j (1+\tau_j s)^{n_j} \prod_p \left(1+\frac{2\zeta_p}{\omega_{np}}s+\frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}} \Bigg|_{s=j\omega} \right) =$$

and using the main properties of the logarithm….

Bode diagrams of the magnitude

$$\left|W(j\omega)\right|_{db} = 20\log_{10} K +$$ ***Constant term***

$$+20\log_{10} s^{\nu} +$$ ***Monomial term***

$$+\sum_i 20\log_{10}\left(1+\sigma_i s\right)^{m_i} - \sum_j 20\log_{10}\left(1+\tau_j s\right)^{n_j}$$ ***Binomial terms***

Trinomial terms $$+\sum_q 20\log_{10}\left(1+\frac{2\xi_q}{\omega_{nq}}s+\frac{s^2}{\omega_{nq}^2}\right)^{\eta_q} - \sum_p 20\log_{10}\left(1+\frac{2\zeta_p}{\omega_{np}}s+\frac{s^2}{\omega_{np}^2}\right)^{\kappa_p}$$

- The magnitude of $W(s)|_{s=j\omega}$ in decibel is given by the sum of four terms: ***constant, monomial, binomial and trinomial terms***
- The phase function has the some product property of the logarithm. Hence in the following we will construct the magnitude and phase Bode diagrams considering these four terms separately.

Basic terms for the Bode diagrams

- ***Constant term:*** K

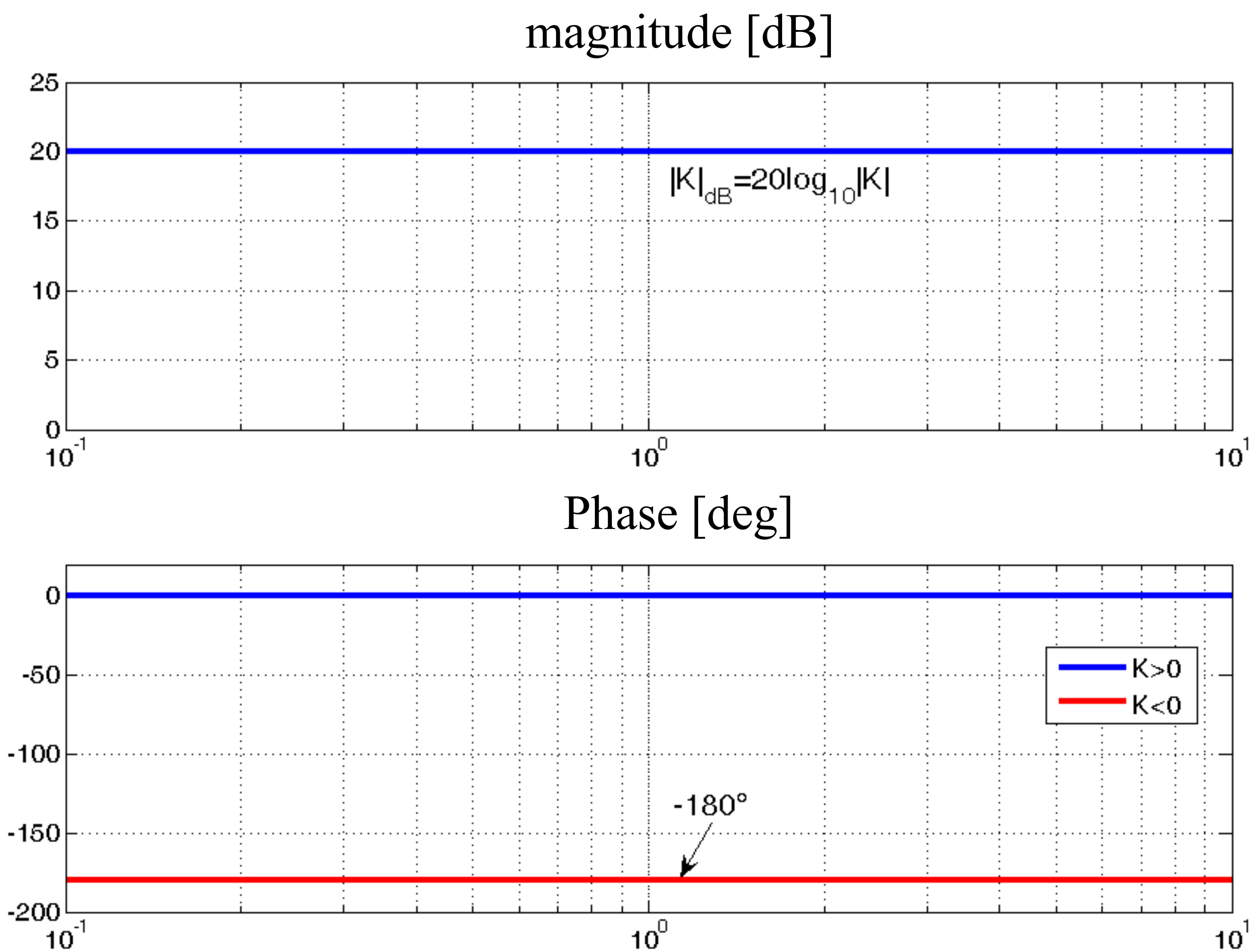
- ***Monomial term:*** Zero/Pole in the origin of multiplicity ν: $\mathbf{20\log_{10} s^{\nu}}$

- ***Binomial term*** **:** Real zero/pole of multiplicity ν: $\mathbf{20\log_{10}(1+\tau s)^{\pm\nu}}$

- ***Trinomial term*** **:** Complex conjugate zero/pole of multiplicity ν:

$$20\log_{10}\left(1+\frac{2\zeta s}{\omega_n}+\frac{s^2}{\omega_n^2}\right)^{\pm\nu}$$

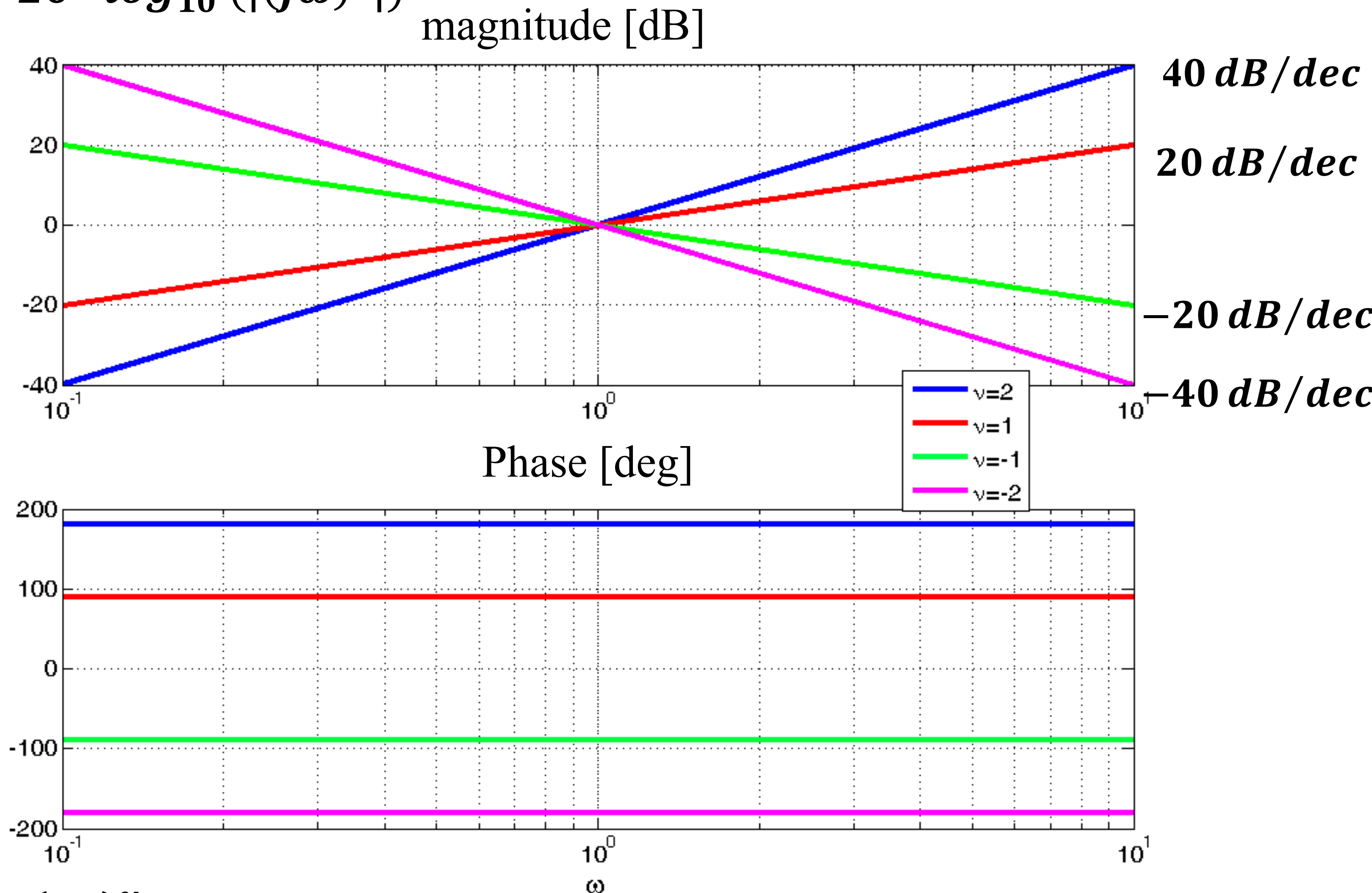
Constant term: K



$\lvert K \rvert$	$\lvert K \rvert_{dB}$
0.01	-40
0.1	-20
1	0
2	6
3	10
5	14
10	20
100	40
1000	60

Monomial terms: $(j\omega)^\nu$

$$|\mathbf{W}(\mathbf{j}\boldsymbol{\omega})|_{\mathbf{dB}} = \mathbf{20} \cdot \boldsymbol{log_{10}}\,(|(\boldsymbol{j\omega})^{\boldsymbol{\nu}}|)$$



$$\angle\mathbf{W}(\mathbf{j}\boldsymbol{\omega}) = \angle\,(\boldsymbol{j\omega})^{\boldsymbol{\nu}}$$

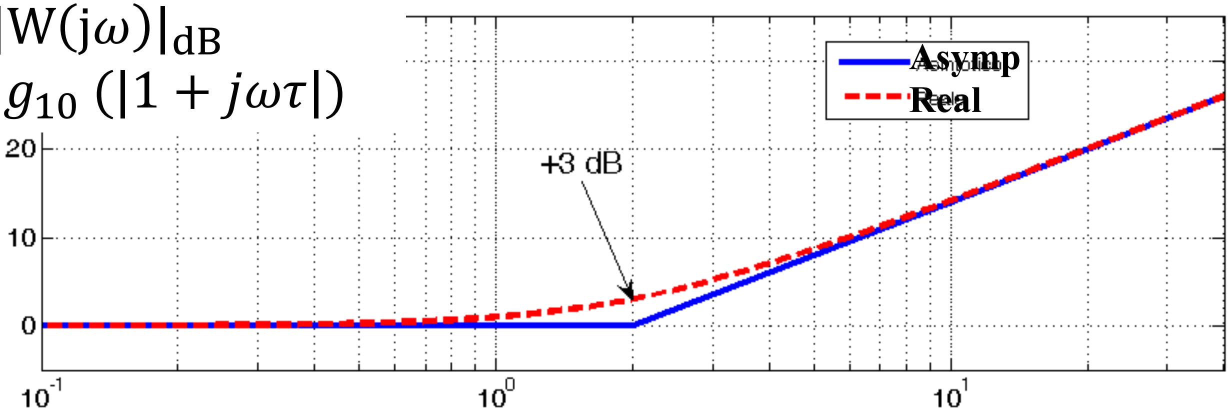
Real Bode diagrams: binomial term

Zero of multiplicity one $\boldsymbol{W(s) = (1 + s\tau)}$

magnitude in decibel

$$|W(j\omega)|_{dB} = 20 \cdot log_{10}\,(|1 + j\omega\tau|)$$

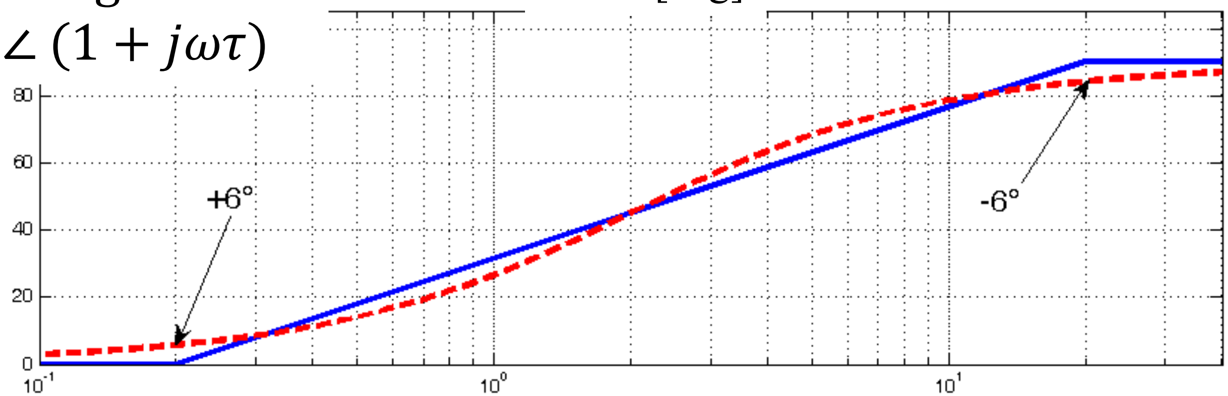
Magnitude [dB]



Phase in degree

$$\angle W(j\omega) = \angle\,(1 + j\omega\tau)$$

Phase [deg]



This result can be easily generalized to a generic binomial term

Complex conjugate poles of multiplicity one $\boldsymbol{W(s) = \left(1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}\right)^{-1}}$

magnitude in decibel $|W(j\omega)|_{dB}$

$$= 20 \cdot log_{10}\left(1 + j\frac{2\zeta\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right)$$

Peak module

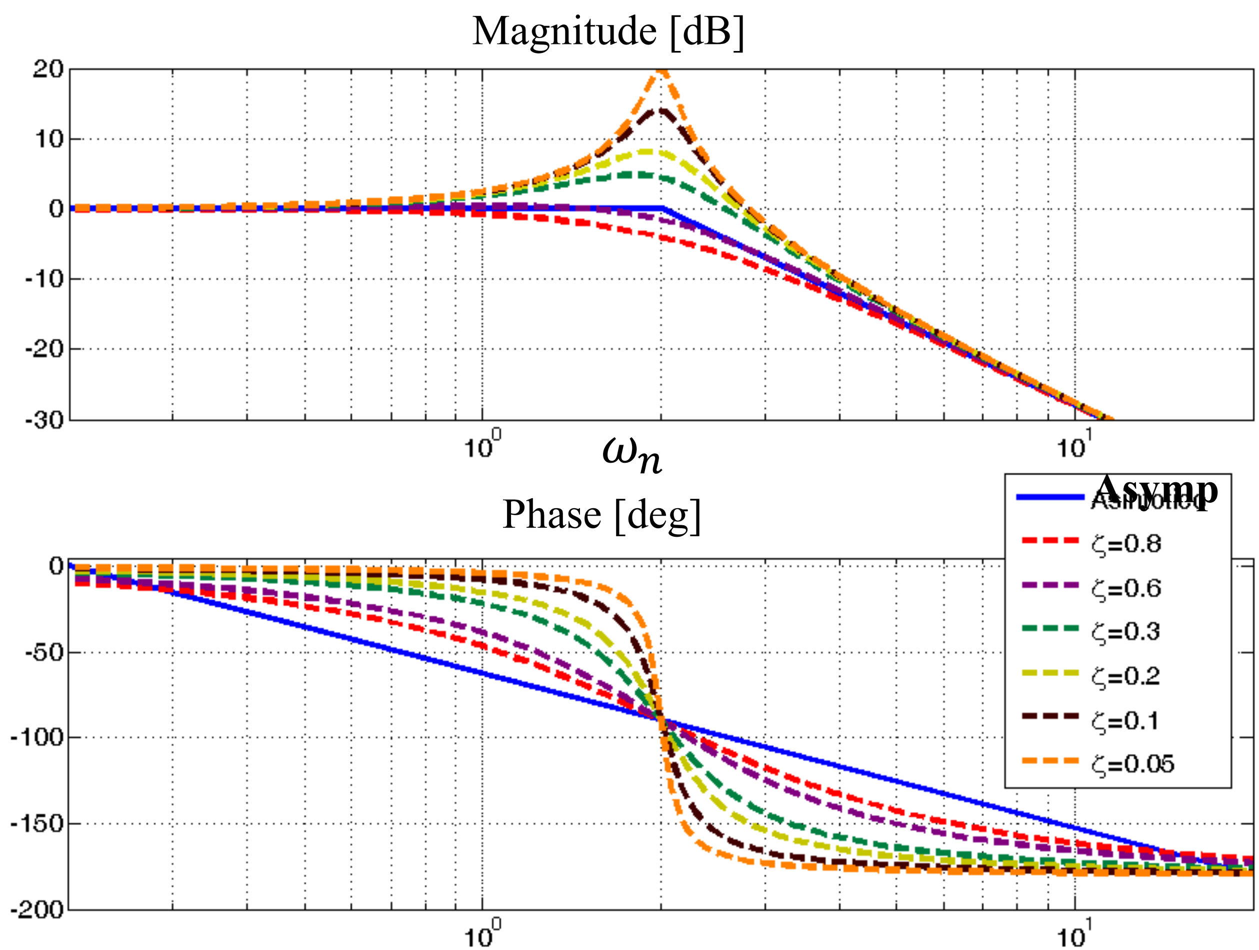
$$\boldsymbol{M_p = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}}$$

Peak frequency

$$\boldsymbol{\omega_p = \omega_n\sqrt{1-2\zeta^2}}$$

Phase in degree, $\angle W(j\omega)$

$$= \angle\left(1 + j\frac{2\zeta\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right)$$



This result can be easily generalized to a generic trinomial term

Bode magnitude table

- Monomial terms of multiplicity 1. The slope is constant in $\omega \in [0 \;\; \infty[$

Zero in the origin	**+20 dB/decade**
Pole in the origin	**-20 dB/decade**

- Binomial and trinomial terms of multiplicity 1. The slope changes on the break point

	<u>Indipendent from the sign of the real part</u>
Real Zero	**+20 dB/decade**
Real Pole	**-20 dB/decade**
Comp. Conjug. zeros	**+40 dB/decade**
Comp. Conjug. poles	**-40 dB/decade**

- When the term has a multiplicity greater than one, the slopes should be multiplied by the multiplicity.

Bode phase table

- Constant and monomial terms of multiplicity 1. The slope is constant in $\omega \in [0\ \ \infty[$

K < 0	**-180° per $\omega \in [0,\infty)$**
Zero in the origin	**+90° per $\omega \in [0,\infty)$**
Pole in the origin	**-90° per $\omega \in [0,\infty)$**

- Binomial and trinomial terms of multiplicity 1. The slope changes one decade before and after the breaking point.

	Negative real part	**Positive real part**
Real Zero	**+90° +45→ -45 °/decade**	**-90° -45→ +45 °/decade**
Real Pole	**-90° -45→ +45 °/decade**	**+90° +45→ -45 °/decade**
Comp. Conjug. zeros	**+180° +90→ -90 °/decade**	**-180° -90→ +90 °/decade**
Comp. Conjug. poles	**-180° -90→ +90 °/decade**	**+180° +90→ -90 °/decade**

- When the term has a multiplicity greater than one, the phase variation should be multiplied by the multiplicity.

- **Any periodic function *f*(*t*) with period *T*,**

$$f(t) = f(t+T),$$

can be written as

$$f(t) = F_0 + \sum_{n=1}^{\infty}\left[F_{cn}\cos(n\omega_0 t) + F_{sn}\sin(n\omega_0 t)\right]$$

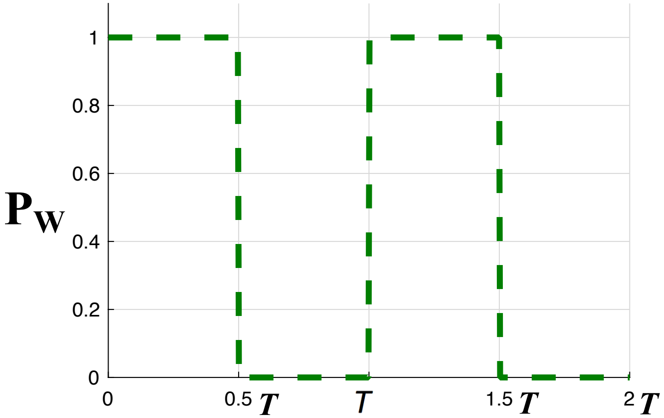
where $\omega_0 = \frac{2\pi}{\mathrm{T}}$,

$$F_0 = \frac{1}{T}\int_T f(t)dt \qquad F_{cn} = \frac{2}{T}\int_T f(t)\cos(n\omega_0 t)dt \qquad F_{sn} = \frac{2}{T}\int_T f(t)\sin(n\omega_0 t)dt \cdot$$

F_0 **is the average value of** f **over a single period.**

The component with ω_0 **is the fundemental armonic or 1**st **harmonic, that with** $n\omega_0$ **is n-th harmonic.**

Example: square wave



$$P_{\mathrm{w}}(t) = \begin{cases} 1 & \text{if} \quad 0 < t \leq T/2 \\ 0 & \text{if} \quad T/2 < t \leq T \end{cases}$$

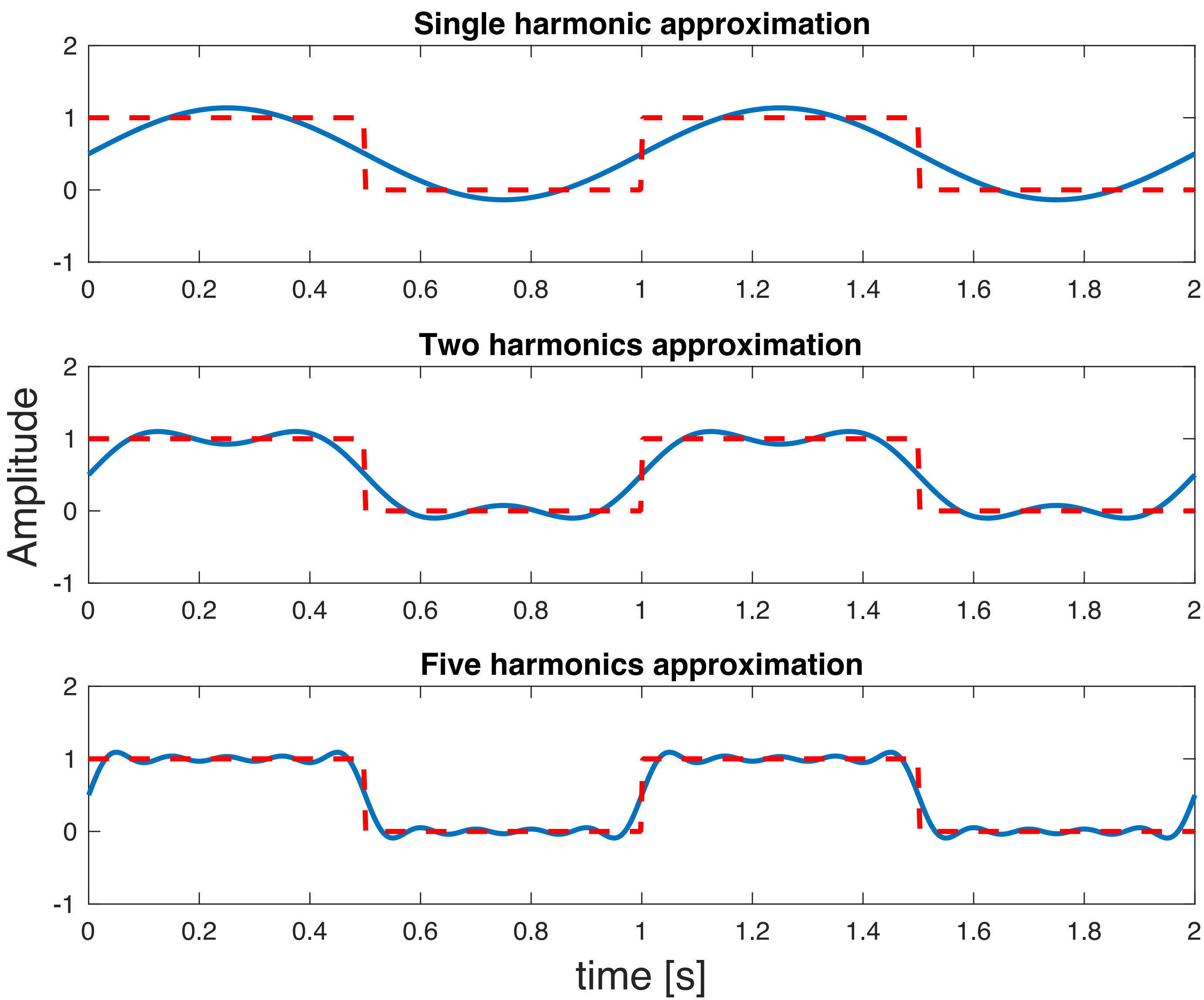
Using Fourier analysis:

$$F_0 = \frac{1}{2}\,, \quad F_{\mathrm{cn}} = 0 \quad \forall n \in \mathbb{N}\,, \quad F_{\mathrm{sn}} = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

Therefore, the square wave can be written

$$P_{\mathrm{w}}(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \frac{2}{5\pi}\sin(5\omega_0 t) + \cdots$$

Example: approximation of a square wave



$\omega_0 = 2\pi$ rad/s

$\omega_0, 3\omega_0$

$\omega_0, 3\omega_0,$
$5\omega_0, 7\omega_0,$
$9\omega_0,$

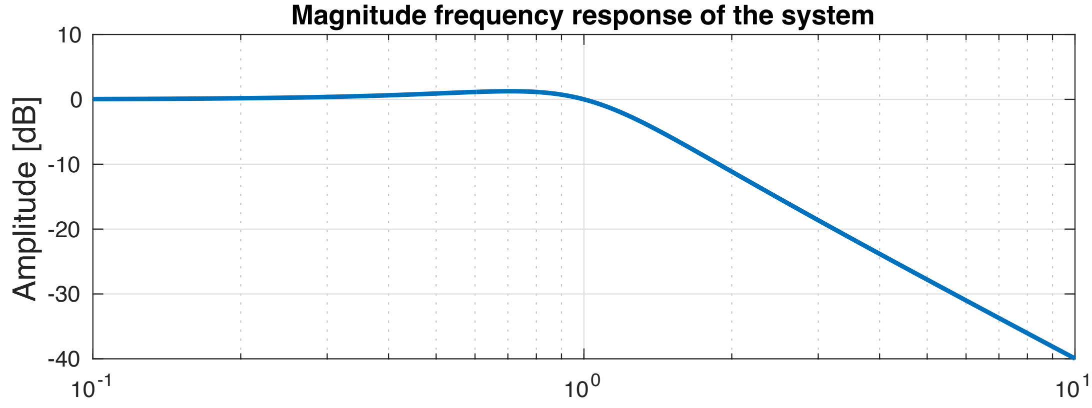
Example: steady state response to a square wave - 1

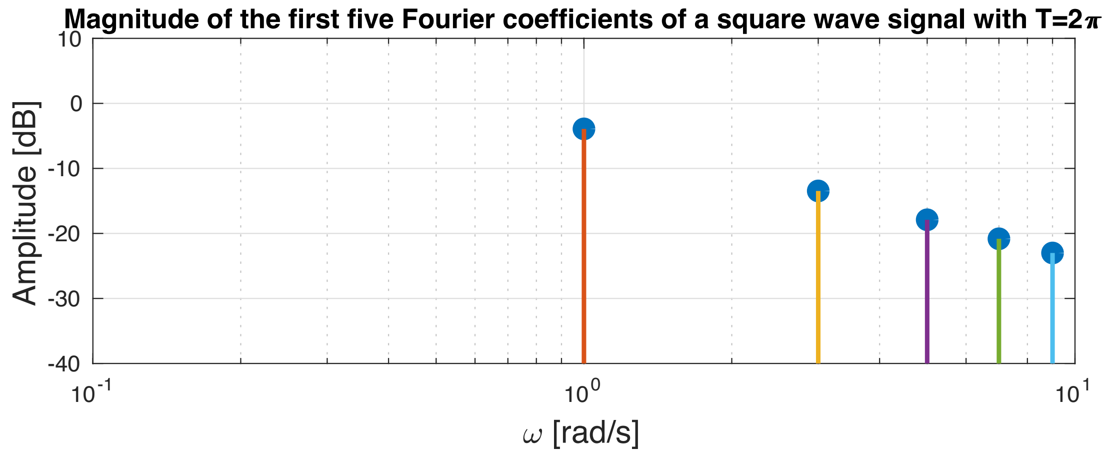
Let us consider the system with transfer function:

$$G(s) = \frac{1}{s^2 + s + 1}$$

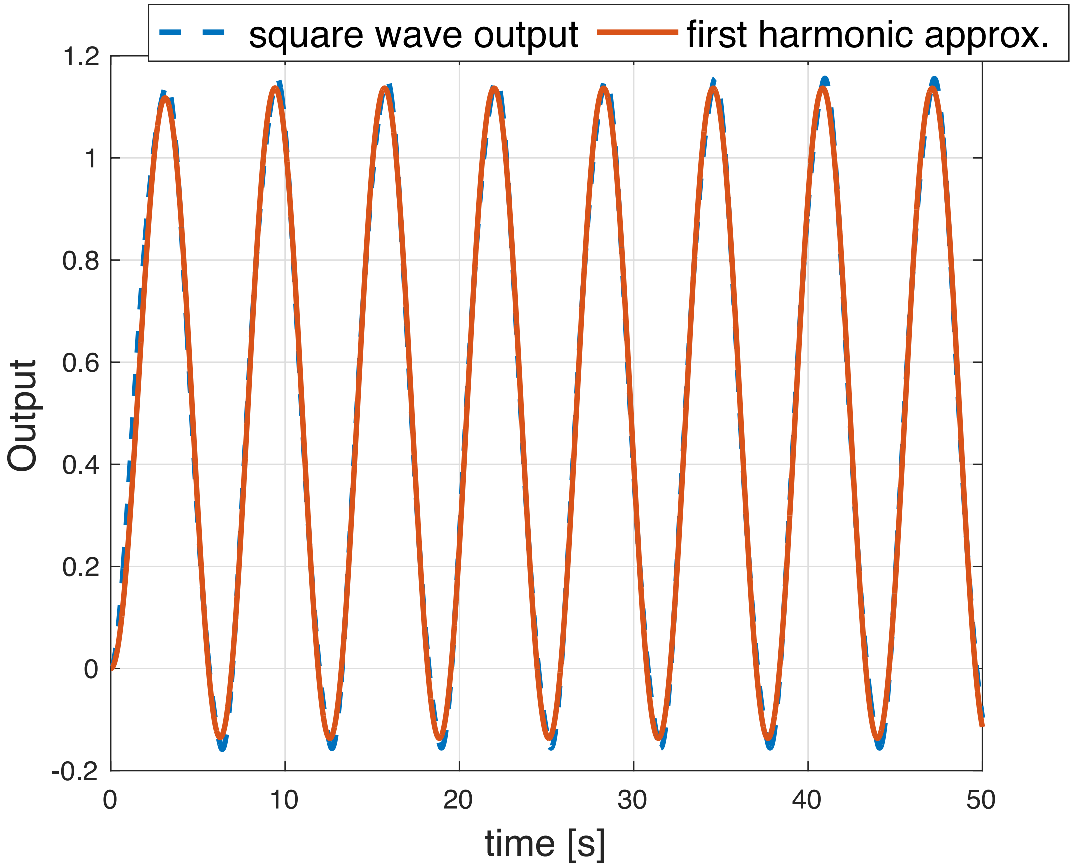
and assume we want to compute the steady-state response to the square wave with period T=2π.

- $u(t) = \frac{1}{2} + \frac{2}{\pi}\sin t + \frac{2}{3\pi}\sin(3t) + \frac{2}{5\pi}\sin(5t) + \cdots$





Example: steady state response to a square wave - 2



The stead state response of the system with transfer function

$$G(s) = \frac{1}{s^2 + s + 1}$$

is practically identical to the response assuming just the first two terms of the Fourier expansion (the average value plus the first harmonic)

Fourier transform

- So far we have assumed that the signal are periodic. In this case, the frequency spectrum (i.e., the coefficients of the Fourier series) of the signal is discrete (i.e., it is defined only a certain frequencies)

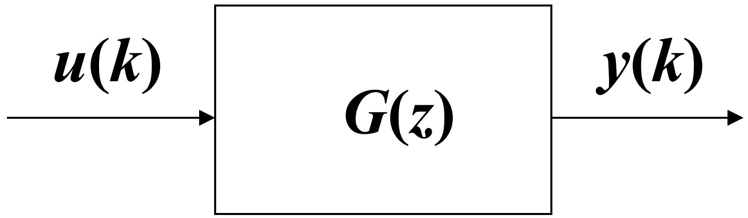
- When the signal is aperiodic, we can assume it as a signal with period $T = \infty$. Thus, the interval between two consecutive harmonics $n\omega_0 = n\frac{2\pi}{T}$ tends to zero and the frequency spectrum becomes a continuous function of w (i.e. defined for all the frequency values)

- Formally, given a aperiodic signal $f(t)$, it can be analysed in the frequency domain by applying the Fourier transform, defined as

$$\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt$$

Steady state response of discrete LTI systems to sinusoidal inputs - 1

- Let us consider an asymptotically stable discrete LTI system with a transfer function $G(z)$



- The steady state response to a sinusoidal input $u(\boldsymbol{k})$,

$$u(\boldsymbol{k}) = \boldsymbol{U}\sin(\boldsymbol{\theta_0 k})\mathbf{1}(\boldsymbol{k})$$

is given by

$$\boldsymbol{y}(\boldsymbol{k}) = \boldsymbol{U}\left|\boldsymbol{G}\left(\boldsymbol{e}^{j\boldsymbol{\theta_0}}\right)\right|\sin\left(\boldsymbol{\theta_0 k} + \arg\left(\boldsymbol{G}\left(\boldsymbol{e}^{j\boldsymbol{\theta_0}}\right)\right)\right)$$

where $\boldsymbol{G}(\boldsymbol{z})\big|_{\boldsymbol{z}=\boldsymbol{e}^{j\boldsymbol{\theta_0}}}$ is the frequency response of the discrete-time system

Steady state response of discrete LTI systems to sinusoidal inputs - 2

Indeed, by applying z-Transform

$$Y(z) = G(z)\frac{z\sin(\omega T)U}{(z-e^{j\omega T})(z-e^{-j\omega T})}$$

and by decomposition

$$y_k = \mathcal{Z}^{-1}\left[z\left(\underbrace{\sum_{i=1}^{n}\frac{R_i}{z+p_i}}_{Y^t(z)} + \underbrace{\frac{Q}{z-e^{j\omega T}} + \frac{\bar{Q}}{z-e^{-j\omega T}}}_{Y^r(z)}\right)\right] = y_k^l + y_k^f$$

where

$$Q = G(e^{j\omega T})\frac{U}{2j}\,, \quad \bar{Q} = -G(e^{-j\omega T})\frac{U}{2j} = -\bar{G}(e^{j\omega T})\frac{U}{2j}$$

In the case of asymptotically stable system, $\lim_{k\to\infty} y_k^l = 0$

$$y_k^r = \mathcal{Z}^{-1}\left[G(e^{j\omega T})\frac{U}{2j}\frac{z}{z-e^{j\omega T}} - \bar{G}(e^{j\omega T})\frac{U}{2j}\frac{z}{z-e^{-j\omega T}}\right]$$

Steady state response of discrete LTI systems to sinusoidal inputs - 3

$$y_k^r = \frac{U}{2j}\left(G(e^{j\omega T})e^{j\omega Tk} - \bar{G}(e^{j\omega T})e^{-j\omega Tk}\right)$$

$$= \frac{U}{2j}\left(\left[Re\big(G(e^{j\omega T})\big) + jIm\big(G(e^{j\omega T})\big)\right]\left[\cos(\omega Tk) + j\sin(\omega Tk)\right] + \right.$$

$$\left.\left[jIm\big(G(e^{j\omega T})\big) - Re\big(G(e^{j\omega T})\big)\right]\left[\cos(\omega Tk) - j\sin(\omega Tk)\right]\right) =$$

$$U|G(e^{j\omega T})|\Big(\cos\angle G(e^{j\omega T})\sin(\omega Tk) + \sin\angle G(e^{j\omega T})\cos(\omega Tk)\Big) =$$

$$|G(e^{j\omega T})|U\sin(\omega Tk + \angle G(e^{j\omega T}))\,.$$

From this results, given a sample time $\boldsymbol{T}$, the response of a discrete time LTI to a sinusoidal input is the same at each frequency $\omega + m\frac{2\pi}{T}$, with $m \in \mathbb{N}$

Frequency response

- This result, as for the continuous time LTI systems, underlines the importance of the function $G(e^{j\theta})$ for the analysis of the forced response of discrete time LTI systems.

$$G(e^{j\theta}) = C\left(e^{j\theta}I - A\right)^{-1}B + D$$

- The function $G(e^{j\theta})$ is the ***frequency response*** of the discrete time system, with $\theta \in [0\ 2\pi[$, and $e^{j\theta}$ is not a pole of $G(z)$.

Fourier analysis – discrete time signal

- **Any periodic function $u(k)$, denoted here with u_k with period N, (positive integer)**

$$u_{k+N} = u_k \,, \quad \forall\, k$$

can be written as

.

$$u_k = \sum_{N} U_n e^{jn\theta_0 k} \,, \quad \theta_0 = \frac{2\pi}{N}$$

The response y_k of a discrete time LTI to u_k .

$$\tilde{y}_k = \sum_{N} Y_n e^{jn\theta_0 k} \,, \quad Y_n = G(e^{jn\theta_0}) U_n$$