Course of "Industrial Automation"

z-Transform domain analysis of LTI systems

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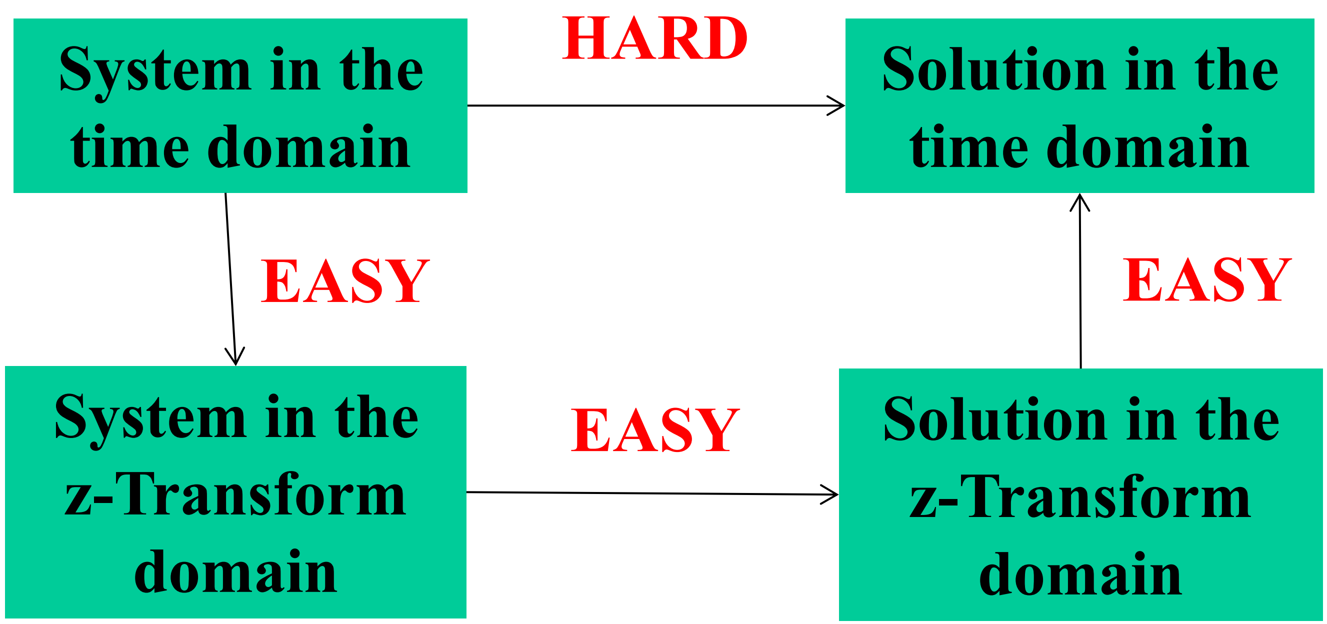
Evaluation of an LTI system response

- Let us consider a Discrete Linear Time Invariant (LTI) system in the state space form

$$x(k+1) = Ax(k) + Bu(k), \quad x(k_0) = x_0 \qquad (1.a)$$

$$y(k) = Cx(k) + Du(k) \qquad (1.b)$$

- The Evaluation of an LTI system response in a transformed domain is convenient only if



LTI systems in the z-domain - 1

- Let us indicate with $\boldsymbol{X(z), U(z)}$ *e* $\boldsymbol{Y(z)}$ the z-Transforms of $\boldsymbol{x(k), u(k)}$ and $\boldsymbol{y(k)}$.

- Transforming both the sides of the equation (1), we have

$$\mathcal{Z}(x(k+1) = \mathcal{Z}\big(Ax(k) + Bu(k)\big)$$

$$\mathcal{Z}(y(k)) = \mathcal{Z}\big(Cx(k) + Du(k)\big)$$

- By using the properties of z-Transform (e.g., linearity, time shift)

$$X(z) = z(zI - A)^{-1}x_0 + (zI - A)^{-1}BU(z) \qquad (2.1)$$

$$Y(z) = Cz(zI - A)^{-1}x_0 + C(zI - A)^{-1}BU(z) + DU(z) \qquad (2.2)$$

- ***Note that in the z-domain the dependency of the state variables $\boldsymbol{X(s)}$ from the input U(s) is expressed by a matrix product instead of a convolution***

LTI systems in the z-domain - 2

- The matrix function $\Phi(z) = z(zI - A)^{-1}$ is called ***Transition matrix*** whose dimension is given by the dimension of the A matrix.

- The matrix function $W(z) = C(zI - A)^{-1}B + D$ is called ***Transfer function***

$$X(z) = \Phi(z)x_0 + \Phi(z)BU(z) \tag{3.1}$$

$$Y(z) = C\Phi(z)x_0 + W(z)U(z) \tag{3.2}$$

- For ***Single Input Single Output (SISO)*** systems the transfer function $\mathrm{W}(z)$ is a scalar function;

- For ***Multiple Input Multiple Output (MIMO)*** systems the transfer function $\mathrm{W}(z)$ is a matrix whose element $\mathrm{W}(z)_{ij}$ will connect the output i with the input j.

Transfer function

- For SISO systems the scalar ***transfer function*** is given by the ratio of two polynomial functions

$$W(z) = \frac{num(z)}{den(z)} = \frac{a_m\, z^m + a_{m-1}\, z^{m-1} + \cdots + a_1\, z + a_0}{z^n + b_{n-1}\, z^{n-1} + \cdots + b_1\, z + b_0}$$

where $m \leq n$.

- ***If $\boldsymbol{m < n}$ the system is said strictly proper***. It happens when the D matrix of the LTI system in the state space is zero.

- ***If $\boldsymbol{m = n}$ the system is said proper***. It happens when the D matrix of the LTI system in the state space is different from zero.

Poles and zeros

- Given a ***transfer function***

$$W(z) = \frac{N(z)}{D(z)} = \frac{a_m\, z^m + a_{m-1}\, z^{m-1} + \cdots + a_1\, z + a_0}{z^n + b_{n-1}\, z^{n-1} + \cdots + b_1\, z + b_0}$$

- The roots of the $\mathrm{N(z)}$ are said ***zeros.***

- The roots of the $\mathrm{D(z)}$ are said ***poles***

- The polynomial $\mathrm{D(z)}$ is defined as $D(z) = \det(zI - A)$, hence
  - ***D(z) coincides with the characteristic polynomial of the system***
  - ***the poles coincide with the eigenvalues of the system***

- Then in the z-domain, ***the free evolution in the Laplace domain is given by the ratio of polynomial functions***

$$Y_{free}(z) = C\Phi(z)x_0$$

- ***This is also true for the forced evolution in case we restrict our attention to the case of polynomial and sinusoidal inputs***

$$Y_{forced}(z) = W(z)U(z)$$

- It is convenient to antitransform $Y(z)$ by reducing the ratio of high degree polynomial functions to the sum of selected signals transform