Course of “Industrial Automation”

Discrete-Time LTI systems analysis – part 1

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- Let's assume the LTI time discrete system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(k_0) = x_0 \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

- The solution of the linear difference equation (1) defines the ***time evolution of the state variables*** and it is given

$$x(k) = A^{k-k_0}x_0 + \sum_{h=k_0}^{k-k_0-1} A^{k-k_0-h-1}Bu(h), \quad k \geq k_0 \tag{2}$$

- The time evolution of the outputs is

$$y(k) = CA^{k-k_0}x_0 + C\sum_{h=k_0}^{k-k_0-1} A^{k-k_0-h-1}Bu(h) + D\,u(k), \quad k \geq k_0 \tag{3}$$

Free and forced responses of LTI system

- The ***time evolution of the state and output variables*** can be conceptually divided in two parts:

$$x(k) = \underbrace{A^{k-k_0}x_0}_{\textit{Free evolution}} + \underbrace{\sum_{h=k_0}^{k-k_0-1} A^{k-k_0-h-1}Bu(h)}_{\textit{Forced evolution}}, \quad k \geq k_0$$

$$y(k) = \underbrace{CA^{k-k_0}x_0}_{\textit{Free evolution}} + \underbrace{C\sum_{h=k_0}^{k-k_0-1} A^{k-k_0-h-1}Bu(h) + D\,u(k)}_{\textit{Forced evolution}}, \quad k \geq k_0$$

- The ***free evolution*** indicate the evolution of state and output vectors that would be obtained in the absence of input ($u(t) = 0$).

- The ***forced evolution*** indicate the evolution of state and output vectors that would be obtained in the presence of input and null initial conditions ($x_0 = 0$)

Free evolution

- The free evolution can be achieved by setting in (1) $u(k) = 0$

$$x(k+1) = Ax(k), \quad x(k_0) = x_0$$
$$y(k) = Cx(k)$$

- By iterations, $k = k_0, k = k_0 + 1, \ldots k = k_0 + n,$ it follows

$$x(k_0+1) = Ax(k_0) = Ax_0$$
$$x(k_0+2) = Ax(k_0+1) = A^2x_0$$
$$\ldots$$
$$x(k_0+n) = A^n x_0$$

- The solution for a general k is given by

$$x(k) = A^{k-k_0}x_0$$

Free evolution: matrix ‘A’ diagonalizable

- The free evolution of a discrete-time LTI system is determined by A^k
- In case the matrix A has real and distinct eigenvalues, it is diagonalizable and A^k

$$A^k = U \Lambda^k U^{-1} = U\, \mathrm{diag}\{\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\} V$$

where $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigenvalues of the A matrix, U is eigenvector matrix and $V = U^{-1}$ is the left eigenvector matrix.

Free evolution: real egeinvalues

- The free evolution of a discrete LTI system when the matrix A is diagonalizable turns out to be

$$
\begin{aligned}
A^k x_0 &= U\,\mathrm{diag}\{\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k\} V x_0 \\
&= \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{pmatrix} \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} x_0 \\
&= \sum_{i=1}^{n} \lambda_i^k u_i v_i^T x_0 \\
&= \sum_{i=1}^{n} \lambda_i^k u_i c_i
\end{aligned}
$$

An aperiodic modes

where the coefficient $c_i \in R^n$ are the projection of the initial state x_0 on the eigenvector u_i.

Aperiodic evolution modes (1/4)

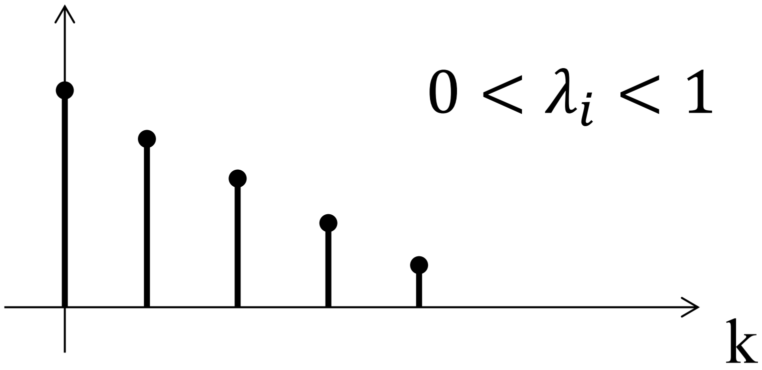
- ***An aperiodic mode*** is an evolution mode of a linear system related to a real eigenvalue of the matrix A of multiplicity 1. It can be written in the form

$$c_i \lambda_i^k u_i$$

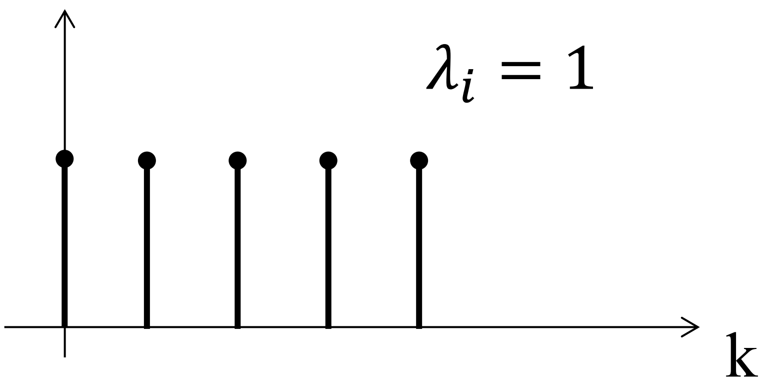
- It gives us the evolution of the state along the direction defined by the eigenvector u_i starting from an initial value c_i (projection of the initial state x_0 on the eigenvalue u_i).

- Depending on the value of the eigenvalue λ_i, an aperiod evolution modes can be
  - convergent for $|\lambda_i| < 1$;
  - constant for $|\lambda_i| = 1$;
  - divergent for $|\lambda_i| > 1$.

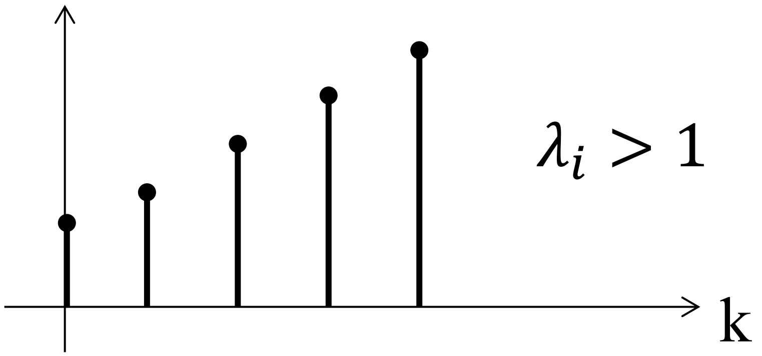
- ***Convergent aperiodic mode***



- ***Constant aperiodic mode***

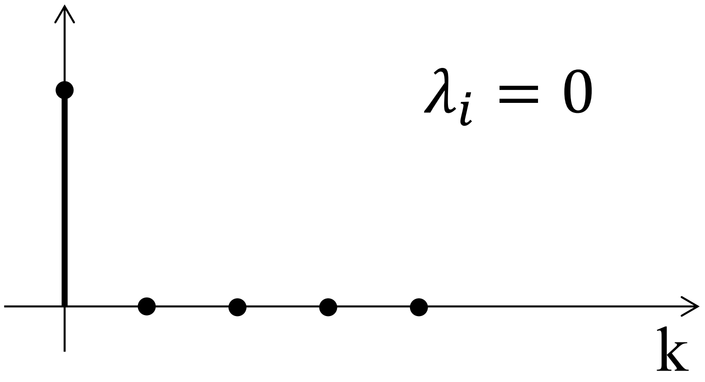


- ***Divergent aperiodic mode***



Aperiodic evolution modes (3/4)

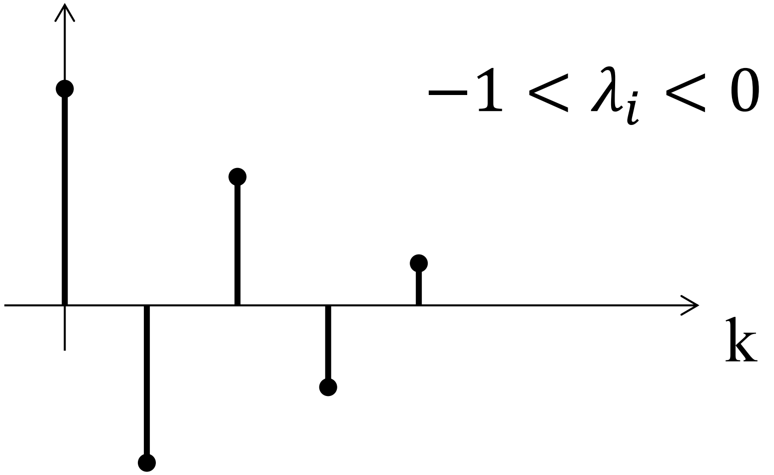
- *Convergent mode in one step*



- In the case of negative eigenvalues, the mode exhibits an alternating behavior

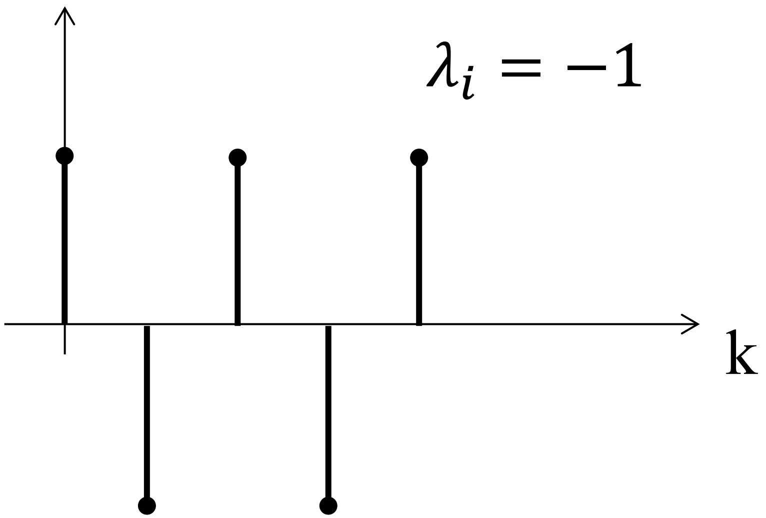
$$\lambda^k = (-|\lambda|)^k = (-1)^k |\lambda|^k$$

- *Aperiodic convergent mode with oscillation behavior*

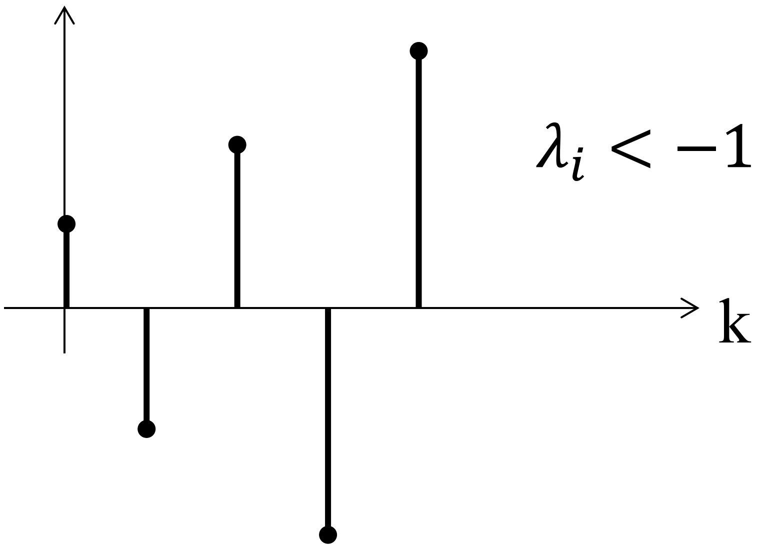


Aperiodic evolution modes (4/4)

- *Constant mode with oscillation behavior*



- *Divergent aperiodic mode with oscillation behavior*

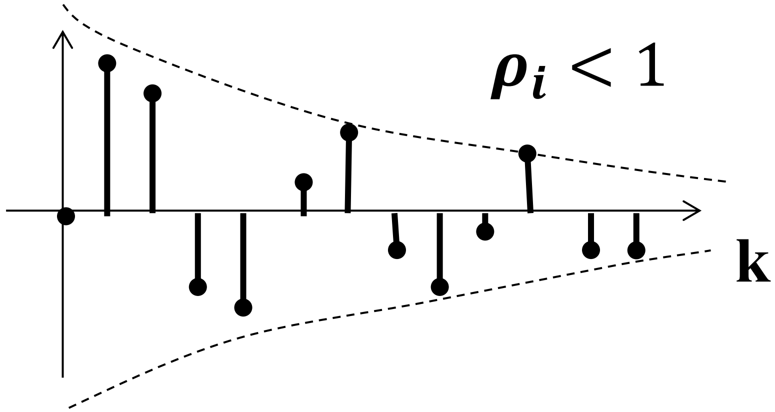


Free evolution: complex egeinvaues

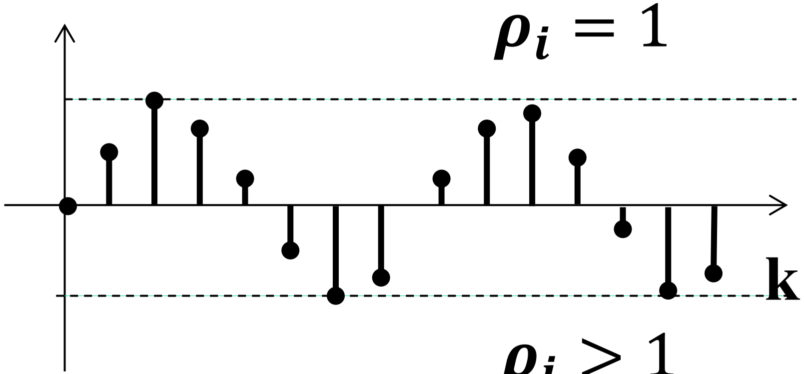
- In the case of complex conjugate eigenvalues $\rho_1 e^{\pm j\theta_1}$, $\rho_2 e^{\pm j\theta_2}$, … $\rho_\nu e^{\pm j\theta_\nu}$, complex conjugate eigenvalues, the free evolution exhibits a pseudo-periodic mode

- Convergent iff $\rho_i < 1$.

- *Convergent pseudo-periodic mode*



- *Constant pseudo-periodic mode*



- *Divergent pseudo-periodic mode*