Course of "Automatic Control Systems"
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Stability of LTI systems

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Stability

- A linear system is said ***stable*** if no evolution mode is divergent (only convergent and constant evolution modes).

- It happens if all the eigenvalues of the matrix A (pole of the $W(s)$) have a negative or null real part and the eigenvalues with null real part have multiplicity 1.

- In a stable system
  - ***the free evolution doesn't tend to infinity***
  - ***the free evolution doesn't converge to zero*** if the constant evolution mode is excited

Example 1

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In order to evaluate the eigenvalues of the matrix A, we can calculate the roots of the characteristic polynomial
- In Matlab, it is possible to use the command ***eig(A)***
- In this example we have $p_1 = p_2 = -1$.
- This system is ***asymptotically stable*** because it has all eigenvalues with negative real part

Example 2

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In this example we have $p_1 = p_2 = 1$.

- The system is ***unstable*** because it has two eigenvalues with positive real part

Example 3

- Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- In this example we have $p_1 = 0$, $p_2 = -1$.

- The system is ***stable*** because it has
  - a null eigenvalue
  - an eigenvalue with negative real part

Example 4

- Let us consider the transfer function of an LTI system

$$W(s) = \frac{s+1}{s^2(s+5)}$$

- This system is ***unstable*** because it has two null poles.

Routh-Hurwitz criterion

- Routh-Hurwitz criterion is used to study the sign of the real part of a polynomial roots.

- It is particularly useful in case of high order polynomials or polynomials with uncertain parameters where the evaluation of the roots can be difficult.

- Routh-Hurwitz criterion is of interest to study the stability of LTI systems both in the state-space form and in the Laplace domain

$$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

where the poles of $W(s)$ coincide with the eigenvalues of the matrix A

Necessary condition

- Let us consider a polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$

and without loss of generality let us assume that

  - $a_n > 0$
  - $a_0 \neq 0$

- **Stodola criterion (Necessary condition):**

A necessary condition for the roots of the polynomial D(s) to have negative real parts is that

$$\boldsymbol{sign(a_0) = sign(a_1) = \cdots = sign(a_n).}$$

This condition is also sufficient for polynomials of degree $\mathbf{n = 1}$ ***,*** $\mathbf{n = 2}$.

Routh table

- Let us consider the polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$

- ***The Routh table*** is defined as follows

n	a_n	a_{n-2}	a_{n-4}	…
$n-1$	a_{n-1}	a_{n-3}	a_{n-5}	…
$n-2$	b_{n-2}	b_{n-4}	b_{n-6}	…
$n-3$	c_{n-3}	c_{n-5}	…	…
…	…	…	…	

$$b_{n-2} = -\frac{1}{a_{n-1}} \det\begin{pmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{pmatrix}$$

$$b_{n-4} = -\frac{1}{a_{n-1}} \det\begin{pmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{pmatrix}$$

$$c_{n-3} = -\frac{1}{b_{n-2}} \det\begin{pmatrix} a_{n-1} & a_{n-3} \\ b_{n-2} & b_{n-4} \end{pmatrix}$$

Routh table: Example

- Routh table, $n + 1$ rows and the last row has 1 element different from zero.

- Let us define the Routh table of the function

$$f(s) = s^4 + 2s^3 + 3s^2 + 5s + 10$$

$$\Downarrow$$

4	*1*	*3*	*10*
3	*2*	*5*	*0*
2	*0.5*	*10*	*0*
1	*-35*	*0*	*0*
0	*10*	*0*	*0*

Routh criterion

- Let us consider the Routh table of the polynomial $D(s)$

n	a_n	a_{n-2}	a_{n-4}	$\dots$
$n-1$	a_{n-1}	a_{n-3}	a_{n-5}	$\dots$
$n-2$	b_{n-2}	b_{n-4}	b_{n-6}	$\dots$
$n-3$	c_{n-3}	c_{n-5}	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	

- ***The roots of the polynomial $D(s)$ have all negative real parts iff the elements of the first column of the Routh table are all positive.***

- **Each sign variation of the element of the first column of the Routh table correspond to a root of $D(s)$ *with a positive real part.***

Routh criterion: example

- Let us consider the polynomial

$$f(s) = s^4 + 2s^3 + 3s^2 + 5s + 10$$

- The Routh table of $f(s)$ is

4	*1*	*3*	*10*	*0*
3	*2*	*5*	*0*	*0*
2	*0.5*	*10*	*0*	*0*
1	*-35*	*0*	*0*	*0*
0	*10*	*0*	*0*	*0*

Roots of *f(s)*
0.544 + j1.60
0.544 - j1.60
-1.54 + j1.06
-1.54 + j1.06

- Let us consider a transfer function $W(s)$ of an LTI system where the poles of $W(s)$ depends on un uncertain parameter p,

$$W(s) = \frac{s+1}{2s^3 + 5ps^2 + (3+p)s + 1}$$

- From the Routh table we have that

3	**2**	**3+p**
2	**5p**	**1**
1	$\frac{5p^2+15p-2}{5p}$	**0**
0	**1**	**0**

$$\begin{cases} 5p > 0 \\ 5p^2 + 15p - 2 > 0 \end{cases}$$

$$\begin{cases} p > 0 \\ p < -3.13 \wedge p > 0.128 \end{cases} \Rightarrow p > 0.128$$

Routh criterion: Singular Cases

- In the design of the Routh table two singular cases can be found
  - *a) The first term of a row is null*
  - *b) All the terms of a row are null*

- In these cases, some mathematical manipulations of the Routh table can be adopted. However, it is not of interest for this course.