



Course of "Automatic Control Systems"
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Stability of LTI systems

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Stability

- ✧ A linear system is said *stable* if no evolution mode is divergent (only convergent and constant evolution modes).
- ✧ It happens if all the eigenvalues of the matrix A (pole of the $W(s)$) have a negative or null real part and the eigenvalues with null real part have multiplicity 1.
- ✧ In a stable system
 - ✧ *the free evolution doesn't tend to infinity*
 - ✧ *the free evolution doesn't converge to zero* if the constant evolution mode is excited



Example 1

- ✧ Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- ✧ In order to evaluate the eigenvalues of the matrix A , we can calculate the roots of the characteristic polynomial
- ✧ In Matlab, it is possible to use the command *eig(A)*
- ✧ In this example we have $p_1 = p_2 = -1$.
- ✧ This system is *asymptotically stable* because it has all eigenvalues with negative real part



Example 2

✧ Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

✧ In this example we have $p_1 = p_2 = 1$.

✧ The system is *unstable* because it has two eigenvalues with positive real part



Example 3

✧ Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

✧ In this example we have $p_1 = 0$, $p_2 = -1$.

✧ The system is *stable* because it has

- ✧ a null eigenvalue
- ✧ an eigenvalue with negative real part



Example 4

- ✧ Let us consider the transfer function of an LTI system

$$W(s) = \frac{s + 1}{s^2(s + 5)}$$

- ✧ This system is *unstable* because it has two null poles.



Routh-Hurwitz criterion

- ✧ Routh-Hurwitz criterion is used to study the sign of the real part of a polynomial roots.
- ✧ It is particularly useful in case of high order polynomials or polynomials with uncertain parameters where the evaluation of the roots can be difficult.
- ✧ Routh-Hurwitz criterion is of interest to study the stability of LTI systems both in the state-space form and in the Laplace domain

$$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

where the poles of $W(s)$ coincide with the eigenvalues of the matrix A

✧ Let us consider a polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$$

and without loss of generality let us assume that

$$\star a_n > 0$$

$$\star a_0 \neq 0$$

✧ Stodola criterion (Necessary condition):

A necessary condition for the roots of the polynomial $D(s)$ to have negative real parts is that

$$\text{sign}(a_0) = \text{sign}(a_1) = \dots = \text{sign}(a_n).$$

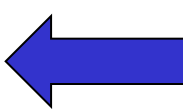
This condition is also sufficient for polynomials of degree $n = 1$, $n = 2$.

Routh table

✧ Let us consider the polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$$

✧ *The Routh table* is defined as follows

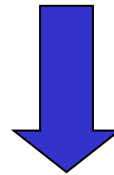
<i>n</i>	<i>a_n</i>	<i>a_{n-2}</i>	<i>a_{n-4}</i>	...		$b_{n-2} = -\frac{1}{a_{n-1}} \det \begin{pmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{pmatrix}$
<i>n-1</i>	<i>a_{n-1}</i>	<i>a_{n-3}</i>	<i>a_{n-5}</i>	...		
<i>n-2</i>	<i>b_{n-2}</i>	<i>b_{n-4}</i>	<i>b_{n-6}</i>	...		$b_{n-4} = -\frac{1}{a_{n-1}} \det \begin{pmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{pmatrix}$
<i>n-3</i>	<i>c_{n-3}</i>	<i>c_{n-5}</i>		
...		$c_{n-3} = -\frac{1}{b_{n-2}} \det \begin{pmatrix} a_{n-1} & a_{n-3} \\ b_{n-2} & b_{n-4} \end{pmatrix}$



Routh table: Example

- ✧ Routh table, $n + 1$ rows and the last row has 1 element different from zero.
- ✧ Let us define the Routh table of the function

$$f(s) = s^4 + 2s^3 + 3s^2 + 5s + 10$$



<i>4</i>	<i>1</i>	<i>3</i>	<i>10</i>
<i>3</i>	<i>2</i>	<i>5</i>	<i>0</i>
<i>2</i>	<i>0.5</i>	<i>10</i>	<i>0</i>
<i>1</i>	<i>-35</i>	<i>0</i>	<i>0</i>
<i>0</i>	<i>10</i>	<i>0</i>	<i>0</i>

✧ Let us consider the Routh table of the polynomial $D(s)$

n	a_n	a_{n-2}	a_{n-4}	\dots
$n-1$	a_{n-1}	a_{n-3}	a_{n-5}	\dots
$n-2$	b_{n-2}	b_{n-4}	b_{n-6}	\dots
$n-3$	c_{n-3}	c_{n-5}	\dots	\dots
\dots	\dots	\dots	\dots	

- ✧ *The roots of the polynomial $D(s)$ have all negative real parts iff the elements of the first column of the Routh table are all positive.*
- ✧ *Each sign variation of the element of the first column of the Routh table correspond to a root of $D(s)$ with a positive real part.*

Routh criterion: example

✧ Let us consider the polynomial

$$f(s) = s^4 + 2s^3 + 3s^2 + 5s + 10$$

✧ The Routh table of $f(s)$ is



	4	3	2	1	0	
	1	2	3	5	10	Roots of $f(s)$
	2	5	10	0	0	$0.544 + j1.60$
	0.5	10	0	0	0	$0.544 - j1.60$
	-35	0	0	0	0	$-1.54 + j1.06$
	10	0	0	0	0	$-1.54 - j1.06$

Routh Criterion: uncertain parameters

- ✧ Let us consider a transfer function $W(s)$ of an LTI system where the poles of $W(s)$ depends on an uncertain parameter p ,

$$W(s) = \frac{s+1}{2s^3 + 5ps^2 + (3+p)s + 1}$$

- ✧ From the Routh table we have that

3	2	$3+p$		$\left\{ \begin{array}{l} 5p > 0 \\ 5p^2 + 15p - 2 > 0 \end{array} \right.$
2	$5p$	1		
1	$\frac{5p^2 + 15p - 2}{5p}$	0		
0	1	0	<div style="text-align: center;">  </div> $\left\{ \begin{array}{l} p > 0 \\ p < -3.13 \wedge p > 0.128 \end{array} \right. \xrightarrow{\text{blue arrow}} p > 0.128$	



Routh criterion: Singular Cases

- ✧ In the design of the Routh table two singular cases can be found
 - a) The first term of a row is null*
 - b) All the terms of a row are null*
- ✧ In these cases, some mathematical manipulations of the Routh table can be adopted. However, it is not of interest for this course.