Course of “Automatic Control System”
2023/24

Laplace transform

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Laplace transform definition

- ***The Laplace transform*** of a function ***f(t)*** is defined as

$$f(t) \to F(s) = L(f(t)) = \int_0^{+\infty} f(t)e^{-st}dt$$

where $t \in R$ is a real variable, while $s = \alpha + j\omega \in C$ is a complex variable.

- Vice versa, given a function $F(s)$ in the Laplace domain, the original function in the time domain can be obtained using the ***Laplace anti-transformation***

$$F(s) \to f(t) = \lim_{\omega \to \infty} \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds$$

- ***The Laplace transform is a bilateral only if the function f(t) is null for*** $t < 0$

Laplace transform main properties (1/2)

- ***Linearity***

$$L\left(af(t)+bg(t)\right)=aF(s)+bG(s)$$

- ***Translation in the Laplace domain***

$$L\left(e^{\alpha t}f(t)\right)=F(s-\alpha)$$

- ***Translation in the time domain***

$$L\left(f(t-T)\right)=F(s)e^{-sT}$$

Laplace transform main properties (2/2)

- ***Time domain derivation***

$$L\left(\frac{df(t)}{dt}\right) = sF(s) - f(0)$$

- ***Time domain integration***

$$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}F(s)$$

- ***Time domain convolution***

$$L\left(f(t) * g(t)\right) = F(s)G(s)$$

Additional properties useful in control theory

- ***Initial value theorem***

$$f(0) = \lim_{s\to\infty} sF(s)$$

- ***Final value theorem***

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- ***Initial value theorem of the derivate of the function***

$$\left.\frac{df(t)}{dt}\right|_{t=0} = \lim_{s\to\infty} s^2F(s) - sf(0)$$

Selected Laplace transforms

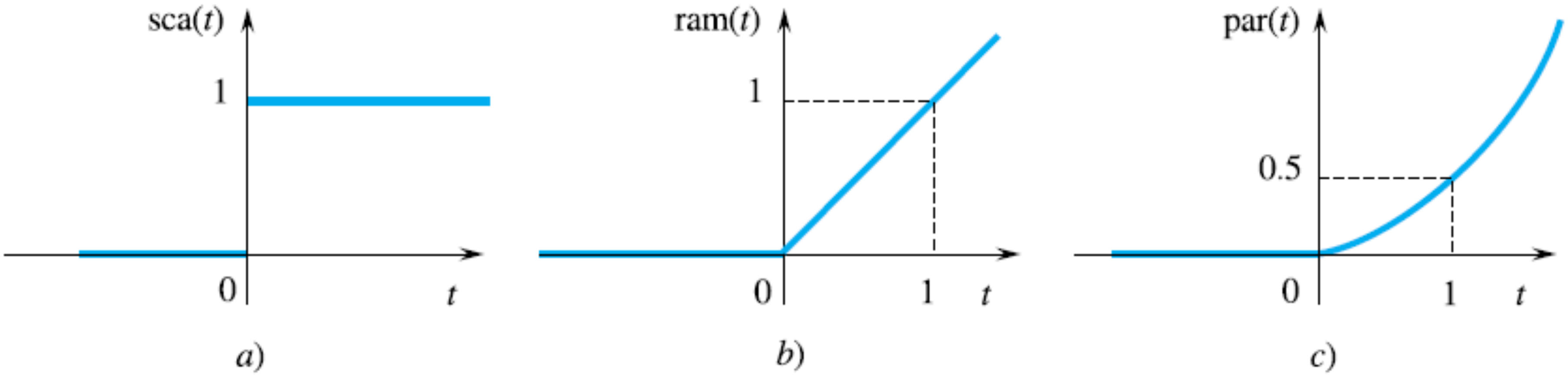
- In the system theory, we will mainly use the Laplace transform for the evaluation of the forced response of LTI systems to selected sets of input :

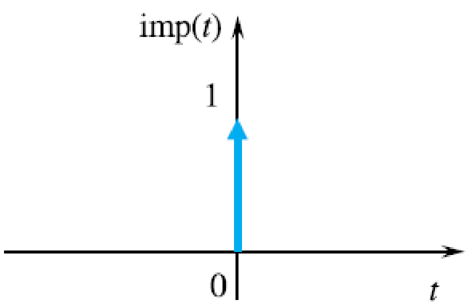
  - ***Polynomial inputs*** $\boldsymbol{u(t) = t^n 1(t)}$

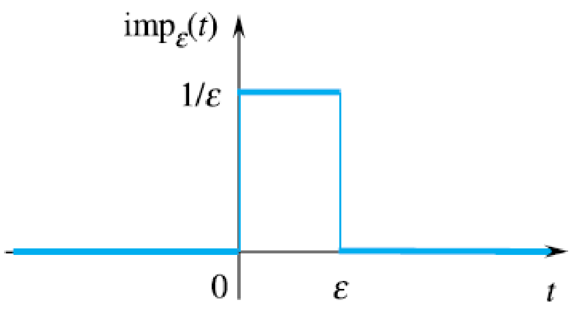
  - ***Sinusoidal inputs*** $\boldsymbol{u(t) = sin(\omega t)\, 1(t)}$

    $\boldsymbol{u(t) = cos(\omega t) 1(t)}$

Selected Laplace transforms







Selected Laplace transforms: polynomial signals

- In order to evaluate the Laplace transform of polynomial signals, let us firstly consider the Laplace transform of the impulse

  - ***Impulse $\delta(t)$*** $\longrightarrow$ $L(\delta(t)) = 1$ *(from the Laplace transform definition)*

- Then, using the ***time domain integration property***, we have

  - ***Step $1(t)$*** $\longrightarrow$ $L(1(t)) = \dfrac{1}{s}$

  - ***Ramp $t \cdot 1(t)$*** $\longrightarrow$ $L(t \cdot 1(t)) = \dfrac{1}{s^2}$

  - ***Polinomial function $t^n \cdot 1(t)$*** $\longrightarrow$ $L(t^n \cdot 1(t)) = \dfrac{n!}{s^{n+1}}$

Selected Laplace transforms: sinusoidal signals

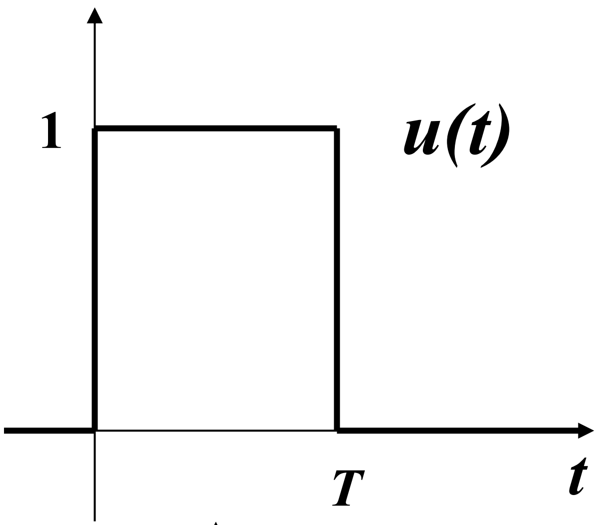
- The Laplace transform of sinusoidal functions
  - *Sine* $\sin(\omega t)1(t)$ $\longrightarrow$ $$L(\sin(\omega t)\cdot 1(t)) = \frac{\omega}{s^2+\omega^2}$$
  - *Cosine* $\cos(\omega t)1(t)$ $\longrightarrow$ $$L(\cos(\omega t)\cdot 1(t)) = \frac{s}{s^2+\omega^2}$$
- Finally, in the control theory the following transformations are of interest for the definition of the Laplace domain of the evolution modes of LTI systems

$$L(e^{\alpha t}1(t)) = \frac{1}{s-\alpha}$$

$$L\left(e^{\alpha t}\cos(\omega t)\cdot 1(t)\right) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2}$$

$$L\left(e^{\alpha t} sen(\omega t)\cdot 1(t)\right) = \frac{\omega}{(s-\alpha)^2+\omega^2}$$

Example: Laplace transform of a window signal



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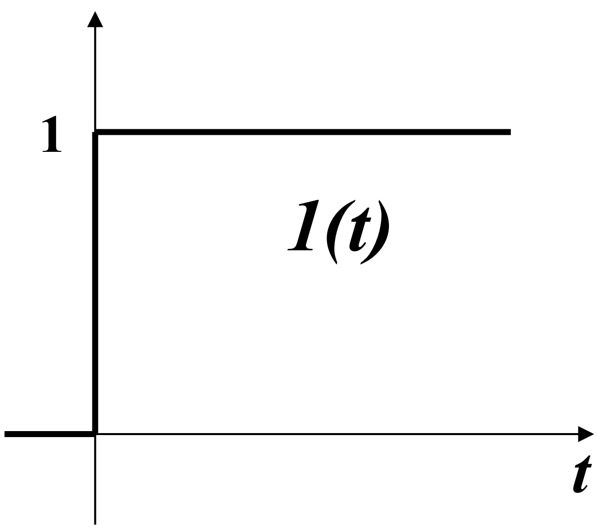
$$\boxed{u(t)=1(t)-1(t-T)}$$



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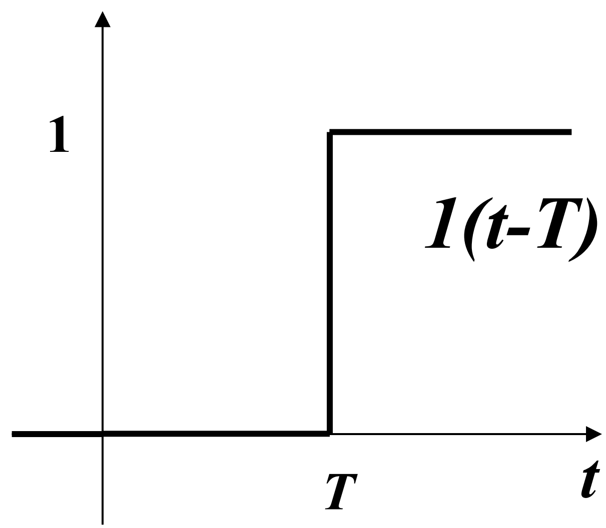

Example: Laplace transform of a window signal

- The Laplace transform of a window signal can be evaluated from the Laplace transforms of two steps.

$$
\begin{aligned}
L(u(t)) &= L(1(t) - 1(t-T)) \\
&= L(1(t)) - L(1(t-T)) \\
&= \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1-e^{-sT}}{s}
\end{aligned}
$$

Solution of first order linear differential equation

Let us consider a first order differential equation, linear with constant coefficients,

$$\dot{y}(t) + a_0 y(t) = bu(t), \quad y(t_0) = y_0$$

By applying Laplace trasform, assuming a step input signal, ***u*(*t*)=*U*$_0$1(*t*),** with amplitude U_0

$$L\big(\dot{y}(t) + a_0 y(t)\big) = L(bU_0 1(t))$$

$$Y(s) = L(y(t))$$

$$L(bU_0 1(t)) = \frac{bU_0}{s}$$

$$\Rightarrow \quad sY(s) - y_0 + a_0 Y(s) = \frac{U_0}{s} \quad \Rightarrow \quad Y(s) = \underbrace{\frac{y_0}{s + a_0}}_{Y_{free}} + \underbrace{\frac{bU_0}{s(s + a_0)}}_{Y_{forced}}$$

Solution of first order linear differential equation

$$Y_{free}(s) = \frac{y_0}{s+a_0} \xrightarrow{\mathcal{L}^{-1}} y_{free}(t) = e^{-\boldsymbol{a_0 t}} y_0 1(t)$$

$$Y_{forced}(s) = \frac{bU_0}{s(s+a_0)} = \frac{A}{s} + \frac{B}{s+a_0}$$

Compute ***A*** and ***B*** by substitution:

$$Y_{\boldsymbol{forced}}(s) = \frac{A(s+a_0)+Bs}{s(s+a_0)}$$
$$= \frac{(A+B)s + Aa_0}{s(s+a_0)}$$

$$\begin{cases} A+B=0 \\ Aa_0 = bU_0 \end{cases} \Rightarrow \quad \boldsymbol{A} = \frac{bU_0}{a_0} \qquad \boldsymbol{B} = -\frac{bU_0}{a_0}$$

Or by residual method:

$$A = (s-0)Y_{forced}(s)|_{s=0}$$
$$= \frac{bU_0}{s+a_0}|_{s=0} = \frac{bU_0}{a_0}.$$

$$B = (s-(-a_0))Y_f(s)|_{s=-a_0}$$
$$= \frac{bU_0}{s}|_{s=-a_0} = -\frac{bU_0}{a_0}.$$

Solution of first order linear differential equation

$$Y_{forced}(s) = \frac{bU_0}{s(s+a_0)} = \frac{A}{s} + \frac{B}{s+a_0} = \frac{\frac{bU_0}{a_0}}{s} + \frac{-\frac{bU_0}{a_0}}{s+a_0}$$

$$\xrightarrow{\mathcal{L}^{-1}} y_{forced}(t) = \frac{bU_0}{a_0}1(t) - \frac{bU_0}{a_0}e^{-\boldsymbol{a_0 t}}1(t) = \frac{bU_0}{a_0}(1 - e^{-\boldsymbol{a_0 t}})1(t)$$

Then,

$$y(t) = y_{free}(t) + y_{forced}(t) = \left(e^{-\boldsymbol{a_0 t}}y_0 1(t) + \frac{bU_0}{a_0}(1 - e^{-\boldsymbol{a_0 t}})\right)1(t)$$