Propagation in anisotropic media

Electromagnetics
and
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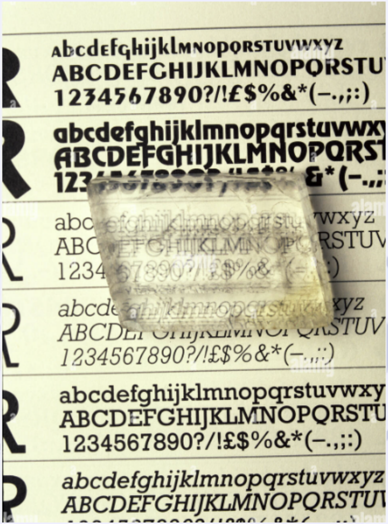
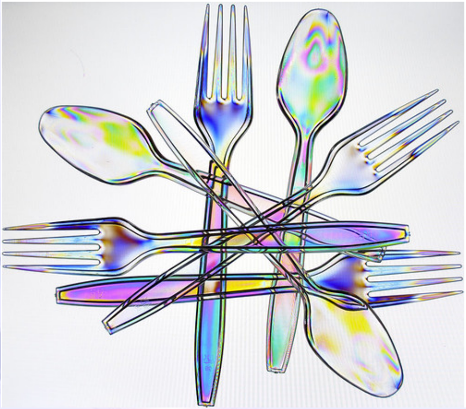
Outline

Birefringence

Birefringence

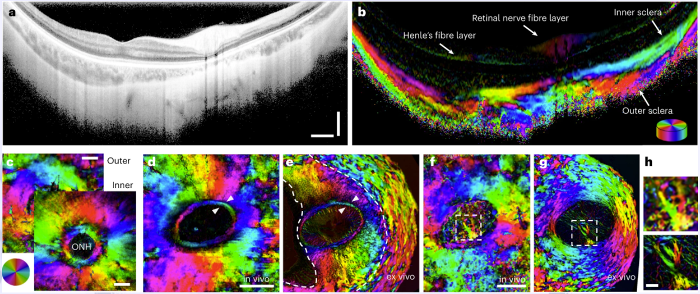


Birefringence

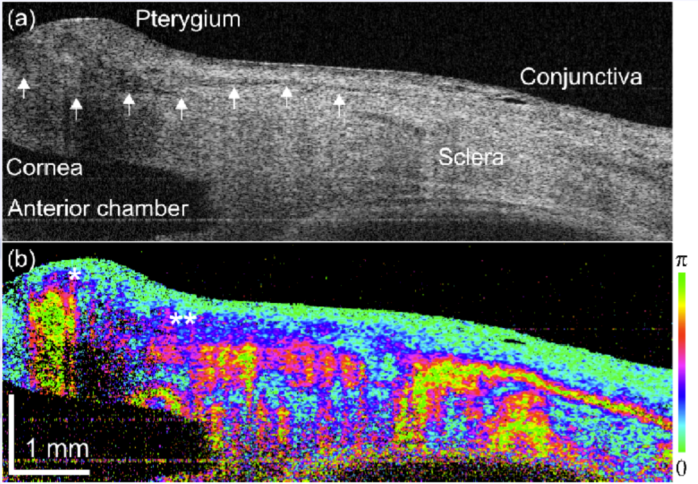


The property exhibited by some transparent solids, whereby they become doubly refractive, or "birefringent", when subjected to stress

Birefringence measurement of cornea and anterior segment



Double refraction



Definition

All this phenomena share a common physical rationale: the double refraction of light in a molecularly ordered material which is manifested by the existence of orientation-dependent differences in refractive index.

Anisotropic material

The non-uniform spatial distribution of material properties results - very often - in different values being obtained when specimens are probed from several directions.

Outline

Linear anisotropic media

A linear anisotropic medium calls for the following constitutive relations:

$$\begin{aligned} \mathbf{D} &= \bar{\epsilon} \cdot \mathbf{E} \\ \mathbf{B} &= \bar{\mu} \cdot \mathbf{H} \\ \mathbf{J} &= \bar{\sigma} \cdot \mathbf{E} \end{aligned}$$

with the constitutive parameters being now dyadic operators.

- A linear isotropic medium is such that $\bar{\epsilon} = \epsilon \mathbf{i}$, $\bar{\mu} = \mu \mathbf{i}$, $\bar{\sigma} = \sigma \mathbf{i}$, with **i** being the identity dyadic.
- A linear anisotropic dielectric medium is such that $\bar{\epsilon} \neq \epsilon \mathbf{i}$, $\bar{\mu} = \mu \mathbf{i}$, $\bar{\sigma} = \sigma \mathbf{i}$.
- Similarly for a linear anisotropic conductor and magnetic medium.

Maxwell's equations

Maxwell's equations can be written straightforwardly

$$
\begin{aligned}
\nabla \times \mathbf{E} &= -j\omega\bar{\mu}\cdot\mathbf{H} \\
\nabla \times \mathbf{H} &= j\omega\bar{\epsilon}\cdot\mathbf{E} + \bar{\sigma}\cdot\mathbf{E} + \mathbf{J}_o = j\omega\bar{\epsilon}_c\cdot\mathbf{E} + \mathbf{J}_o
\end{aligned}
$$

The lossless case is achieved when:

- $\bar{\sigma} = 0$ this corresponds to have a real dielectric constant $\bar{\epsilon}$;
- $\bar{\epsilon}_c = \bar{\epsilon}_c^{\dagger}$ this correspond to still have a complex dielectric constant $\bar{\epsilon}_c$

In the lossless case, there is always a basis for which the dielectric constant dyadic can be diagonalized.

Reciprocity

Starting from the reciprocity theorem, it can be shown that

- Reciprocity holds IF and only IF:

$$\bar{\epsilon}_c = \bar{\epsilon}_c^\top \tag{1}$$

  with $\top$ being the transpose operator.

- This equality implies that the necessary and sufficient condition for a linear anisotropic medium to be reciprocal is that $\bar{\epsilon}_c$ is symmetric.
- Symmetry is a property that is invariant under orthogonal transformations, i.e., a change of basis in the real 3D space, NOT invariant under the most general unitary,i.e., complex transformation.

Lossless vs Reciprocity

Lossless

- The lossless case is described by Hermitian dyadic operator.
- Hermitian is a “special” symmetry property that is invariant under the most general unitary transformation.

We can speak about a lossless medium without specifying the reference system we are using.

Lossless vs Reciprocity

Reciprocity

- The reciprocal medium is described by a symmetric dyadic operator.
- Symmetry is invariant under the orthogonal transformation.
- Symmetry is not invariant under the general unitary transformation.

We can speak about a reciprocal medium without any ambiguity as long as we refer to linearly polarized fields.

Outline

Wave equations

Dielectric anisotropic medium

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} &= -j\omega\mu(j\omega\bar{\epsilon}_c \cdot \mathbf{E} + \mathbf{J}_o) \quad (2) \\ \nabla^2\mathbf{E} - \nabla\nabla \cdot \mathbf{E} + \omega^2\mu\bar{\epsilon}_c \cdot \mathbf{E} &= j\omega\mu\mathbf{J}_o \end{aligned}$$

- Eq.3 is NOT the same as the Helmholtz equation for isotropic media because $\nabla \cdot (\bar{\epsilon}_c \cdot \mathbf{E}) = 0$ does not imply $\nabla \cdot \mathbf{E} = 0$.
- In this medium, at least without making assumptions on $\bar{\epsilon}_c$, it is not possible to derive an equation where only **H** appears.
- Following a similar guideline it is possible to deal with magnetic anisotropic media.

Outline

Generalities

Let's assume:

- A linear lossless anisotropic dielectric medium. Note that the lossy case can be dealt with heavier maths.
- Under this hypothesis, $\bar{\epsilon}_c \leftarrow \bar{\epsilon}$.

- Since $\bar{\epsilon}$ is a real 2D tensor, it is also symmetric, i.e., $\bar{\epsilon} = \bar{\epsilon}^\top$
- This implies that there always exists an orthogonal transformation that makes $\bar{\epsilon}$ a diagonal matrix.

E and **D** are no longer parallel

- In this reference frame, which is termed as the system of principal axes, the electric induction is linked to the electric field by the following relation

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \epsilon_o \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad (3)$$

- If the medium is also homogeneous, the system of principal axes does not change in space and the ϵ_i are termed as principal relative permittivity.
- Eq.(3) clearly shows that **D** and **E** are not parallel in this medium.

E and **D** are no longer parallel

- According to eq.(3), the principal directions are the only ones that may results in parallel **D** and **E** vectors.
- This happens, for instance when **D** is linearly polarized along either $\hat{x}$, $\hat{y}$ or $\hat{z}$.
- If, for instance **E** points along the $\hat{x}$ direction, **D** must also point along with that direction.

Outline

Linking **D** and **E**

Assumptions

- To grasp detailed info about the relationship between **D** and **E**, assumptions on the propagating wave must be done.
- Let's assume (assumption to be confirmed later one) that a linearly polarized plane wave (which is also uniform since the medium is lossless) is allowed to propagate in this medium.
- This implies that $\mathbf{E} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$; $\mathbf{H} = \mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$

Maxwell's equations

Maxwell's equations can be rewritten as follows:

$$\begin{cases} -j\mathbf{k} \times \mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}} = j\omega\bar{\epsilon} \cdot \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \\ -j\mathbf{k} \times \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} = -j\omega\mu\mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \end{cases} \tag{4}$$

$$\begin{cases} -j\mathbf{k} \times \mathbf{H} = j\omega\mathbf{D} \\ -j\mathbf{k} \times \mathbf{E} = -j\omega\mu\mathbf{H} \end{cases} \tag{5}$$

- According to (5):
- **H** is orthogonal to **k** and **E**.
- **D** is orthogonal to **k** and **H**.
- This implies that **H** is orthogonal to **k**, **E** and **D**.
- Therefore, **k**, **E** and **D** lie in the same plane.

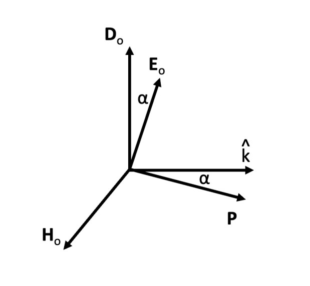
The **k**, **E**, **H** and **D** vectors

To visualize those vectors:

Let's assume - consistently with the previous assumption of linear polarization - $\mathbf{H} = \mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$ with a real $\mathbf{H}_o$. Hence:

$$\mathbf{E}_o = \frac{\mathbf{E}}{e^{-j\mathbf{k}\cdot\mathbf{r}}} \qquad \mathbf{D}_o = \frac{\mathbf{D}}{e^{-j\mathbf{k}\cdot\mathbf{r}}} \tag{6}$$

are also real and can be depicted as follows:



The Poynting vector

The Poynting vector

- defined as $\mathbf{P} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ is also real and - since by definition it is orthogonal to $\mathbf{H}_o$, it lies in the same plane of $\mathbf{k}$, $\mathbf{E}_o$, $\mathbf{D}_o$.
- $\mathbf{P}$ is not directed along $\hat{k}$ and it makes an angle with $\hat{k}$ that is equal to the angle between $\mathbf{E}_o$ and $\mathbf{D}_o$.

Power flow

In a birefringent medium the power flows, in general, in a direction (termed as ray direction) that does not coincide with the direction normal to the constant-phase planes (the so-called normal direction).

Linking **E** and **D** vectors

Establishing a link between **E** and **D** for any direction $\hat{k}$

- Let's define **k** using the refractive index n (assuming $\mu = \mu_o$):

$$\mathbf{k} = \beta\hat{k} = \omega\sqrt{\epsilon\mu}\hat{k} = \omega\sqrt{\epsilon_r}\sqrt{\epsilon_o\mu_o}\hat{k} = n\beta_o\hat{k} \tag{7}$$

- Starting from the second equation in (5) and using (7) one obtains:

$$\mathbf{H} = \frac{\mathbf{k}\times\mathbf{E}}{\omega} = \frac{n\beta_o\hat{k}\times\mathbf{E}}{\omega} \tag{8}$$

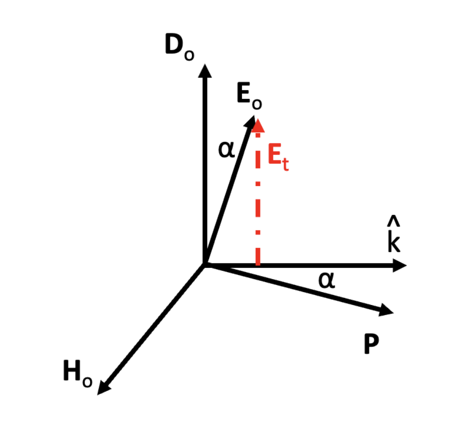
- which, replaced into the first equation in (5), gives:

$$\mathbf{D} = \frac{\mathbf{H}\times\mathbf{k}}{\omega} = \frac{n\beta_o\hat{k}\times\mathbf{E}\times n\beta_o\hat{k}}{\omega^2} = n^2\epsilon_o\mathbf{E}_t \tag{9}$$

Linking **E** and **D** vectors

Establishing a link between **E** and **D** form any direction $\hat{k}$

- $\mathbf{E}_t$ is the component of **E** transverse to the direction $\hat{k}$



Outline

Refraction index

To evaluate n as a function of the parameters of the medium

- It is useful to introduce the inverse permittivity:

$$\mathbf{D} = \bar{\epsilon}\mathbf{E} \rightarrow \epsilon_o \mathbf{E} = \mathbf{bD} \tag{10}$$

- with the inverse permittivity tensor defined as:

$$\mathbf{b} = \epsilon_o \bar{\epsilon}^{-1} \tag{11}$$

- By projecting eq.(10) onto a constant-phase plane (by definition orthogonal to $\hat{k}$) one obtains:

$$\epsilon_o \mathbf{E}_t = \mathbf{b}_t \mathbf{D} \tag{12}$$

- Note that **D** is by construction orthogonal to $\hat{k}$. This is the reason why in birefringent media the **D** vector is preferred to the **E** one.

Birefringent media

To evaluate n as a function of the parameters of the medium

- By using eq.(9) into eq.(12) one obtains:

$$\frac{1}{n^2}\mathbf{D} = \mathbf{b}^{-1}\mathbf{D} \tag{13}$$

Eigenvalue problem

This is an eigenvalue problem which implies that, since $\mathbf{b}_t$ is also symmetric, there are only two admissible n values for any given $\hat{k}$ direction.

This is the reason why those media are termed as birefringent

Retarder

- The two eigenvalues n_1 and n_2 call for two orthogonal eigenvectors.
- This implies that there are only two mutually orthogonal linear polarization states for the **D** vector of the uniform plane wave traveling in this medium.
- Any other polarization state cannot be written in the form $e^{-j\mathbf{k}\cdot\mathbf{r}}$.

Birefringent media

Two polarizations

- To see how a wave (known at a given point), in an unspecified polarization state, propagates in the medium:
  - First, the wave is decomposed according to the two linear polarization provided by the eigenvectors of .(13).
  - The two linearly-polarized waves will propagate in the medium independently of each other and with two different propagation constants (and, therefore, two different velocities).
  - The total wave, at a given distance, is the composition of the two waves and is characterized by a polarization state - in general - different from the one at the origin.
- This channel is called retarder.

The goal of the Fresnel equation of wavenormal

- The goal is to evaluate $n1$ and $n2$ in a simpler way.
- The approach is structured according to two steps:
  - Evaluation of the refraction indices as a function of $\hat{k}$ and $\bar{\epsilon}$.
  - Evaluation of the linear polarization allowed for the uniform plane wave to propagate along with the $\hat{k}$ direction.

Outline

Evaluating $n1$ and $n2$

To evaluate $n1$ and $n2$

- The straightforward approach is solving eq.(13).
- Unfortunately, this is not at all simple since it requires: a) inverting $\bar{\epsilon}$; b) changing the coordinate system.
- A faster way is possible that consists of starting from:

$$\mathbf{D} = \bar{\epsilon} \cdot \mathbf{E} \tag{14}$$

- which, according to (9), implies that:

$$\bar{\epsilon} \cdot \mathbf{E} = \epsilon_o n^2 \mathbf{E}_t \tag{15}$$

- since $\mathbf{E}_t$ is the transverse component, i.e. $\mathbf{E}_t = (\mathbf{E} \cdot \hat{k})\hat{k}$:

$$\bar{\epsilon} \cdot \mathbf{E} = \epsilon_o n^2 \mathbf{E} - \epsilon_o n^2 (\mathbf{E} \cdot \hat{k})\hat{k} \tag{16}$$

Evaluating $n1$ and $n2$

To evaluate $n1$ and $n2$

- Eq.(16) can be arranged as follows:

$$(\bar{\epsilon} - \epsilon_o n^2 \mathbf{i}) \cdot \mathbf{E} = -\epsilon_o n^2 (\mathbf{E} \cdot \hat{k})\hat{k} \tag{17}$$

- This equation can be easily written by components in the principal axes reference frame where $\bar{\epsilon}$ is diagonal:

$$(n^2 - \epsilon_x)E_x = n^2 \nu_x (\hat{k} \cdot \mathbf{E}) \tag{18}$$
$$(n^2 - \epsilon_y)E_y = n^2 \nu_y (\hat{k} \cdot \mathbf{E}) \tag{19}$$
$$(n^2 - \epsilon_z)E_z = n^2 \nu_z (\hat{k} \cdot \mathbf{E}) \tag{20}$$

- with $\hat{k} = (\nu_x, \nu_y, \nu_z)$.

Fresnel equation of wavenormal

The Fresnel equation of wavenormal is to evaluate $n1$ and $n2$

- Multiplying the first, the second and the third for ν_x, ν_y and ν_z, respectively.
- Then, dividing both sides for $\hat{k} \cdot \mathbf{E} \neq 0$ and summing up the three equations one gets:

$$\frac{\nu_x^2}{n^2 - \epsilon_x} + \frac{\nu_y^2}{n^2 - \epsilon_y} + \frac{\nu_z^2}{n^2 - \epsilon_z} = \frac{1}{n^2} \tag{21}$$

This equation is termed as Fresnel's equation of the wave normal.

The eq.(21):

- allows an easy computation of n^2, once the principal permittivity and the propagation direction (ν_x, ν_y, ν_z) are given;
- is biquadratic in n, because of the identity $\nu_x^2 + \nu_y^2 + \nu_z^2 = 1$. Hence, it always gives two solutions for n^2, in agreement with eq.(13);
- it holds also in isotropic media ($\epsilon_x = \epsilon_y = \epsilon_z = \epsilon_r$), and, in this case, it gives (as it should be) solutions $n = \pm\sqrt{\epsilon_r}$ that are independent of the direction of propagation.

Outline

Optical indicatrix

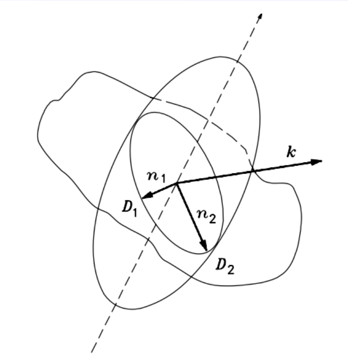
The directions of the two linear polarizations allowed for **D**

- They can be easily evaluated by simple geometry
- In the principal-axes reference frame (x, y, z), the ellipsoid (centered in the origin) is defined by the following equation:

$$\frac{x^2}{\epsilon_x} + \frac{y^2}{\epsilon_y} + \frac{z^2}{\epsilon_z} \tag{22}$$

This surface is called index ellipsoid, or the optical indicatrix.

Fresnel equation of wavenormal



For a wave traveling parallel to a unit vector $\hat{k} = (\nu_x, \nu_y, \nu_z)$, the two directions that are allowed for linearly polarized **D** vectors are the major and minor axes of the ellipse which is the line where the ellipsoid Eq.(22) intersects the plane orthogonal to $\hat{k}$ passing through the origin

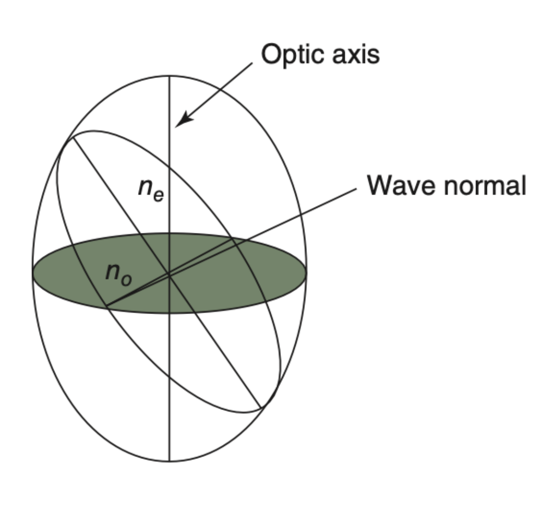
Outline

Fresnel equation of wavenormal

When two out of the three principal permittivities are identical:

- i.e., $\epsilon_x = \epsilon_y \neq \epsilon_z$.
- The index ellipsoid (22) is rotationally symmetric along the z axis.
- For any direction $\hat{k}$, one of the axes of the ellipse where the ellipsoid intersects the plane normal to $\hat{k}$ is, at the same time, a diameter of the circle where the ellipsoid intersects the plane $z = 0$ (which, in turn, is either a minimum or a maximum in length among all the intersection ellipses, depending on whether ϵ_x is smaller or larger than ϵ_z).
- The cases $\epsilon_z > \epsilon_x$ and $\epsilon_z < \epsilon_x$ lead to optically positive and negative birefringence, respectively.

Uniaxial media



Uniaxial media

In case of uniaxial media:

- The two refractive indices $n1$ and $n2$ are termed as "ordinary" index (n_o) and extraordinary index (n_e).
- From the eigenvectors of eq.(13), it can be shown that:
  - The ordinary index is associated to waves polarized in the plane transverse to the oprical axis of the medium (z).
  - The extraordinary wave is associated to waves polarized along the optical axis.

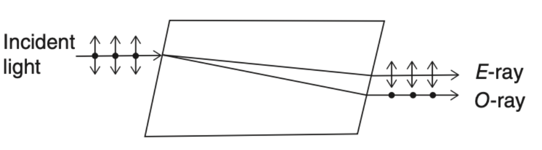
Uniaxial media

In case of uniaxial media:

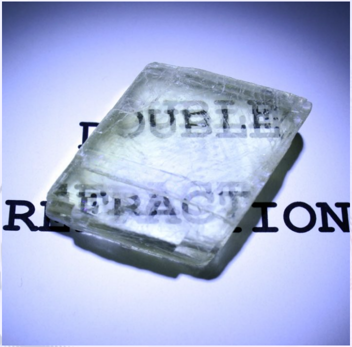
- A wave propagating along the optical axis of the medium, i.e. $\hat{k} = \hat{z}$, propagates in the medium without any distortion to either (any) component of its electric vector.
- A wave traveling in any other direction through the medium experiences two different refractive indices, i.e., n_o and n_e, and is split into components that travel at different speeds and have perpendicular polarizations.
- n_o is independent of the direction $\hat{k}$.
- n_e depends on the direction $\hat{k}$.

Double refraction

- The ordinary wave experinces that conventional Fresnel's refraction law and travels with the same velocity in every direction through the crystal.
- The extraordinary wave travels with a velocity that is dependent upon the propagation direction within the medium.

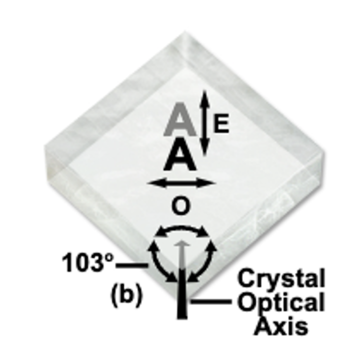


Double refraction



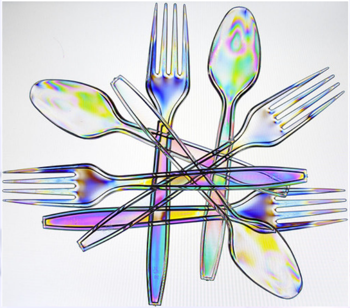
One of the images appears as would normally be expected when observing an object through clear glass or an isotropic crystal (ordinary ray), while the other image appears slightly displaced, due to the nature of doubly-refracted light (extraordinary ray).

Birefringent Calcite Crystal



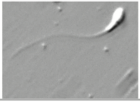
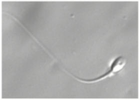
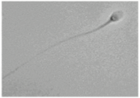
The crystal in panel (b) is positioned over the capital letter A on a white sheet of paper demonstrating double refraction. If you rotate slowly the crustal around the letter, one of the images of the letter will remain stationary (ordinary), while the other precesses in a 360-degree circular orbit around the first.

Photoelasticity



The intensity of the light is modulated by the retardation exhibited by the sample. Strain can be estimated through birefringence.

Observation of morphologically normal sperm cells by polarized microscopy

Total birefringence		The whole sperm head shows a uniform birefringence.
Partial birefringence		The birefringence is localized in the post-acrosomal region.
No birefringence		The sperm head does not show any birefringence.

Total birefringence: the birefringent aspect of the whole head is related to a cell that still has an intact acrosome. Partial birefringence: the localization of the birefringence in the post-acrosomal region is indicative of a cell that has already undergone the acrosome reaction. No birefringence: the absence of birefringence indicates a cell with a compromised inner structure.

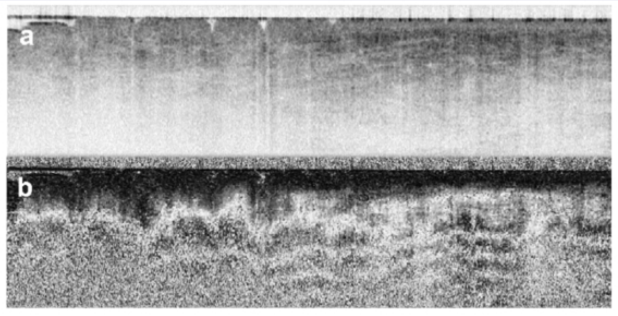
Observing tissues

Birefringence in tissues

Tissues often exhibit birefringence, especially those with a significant collagen content, including tendon, ligament, and skin. The collagen content of the skin is particularly important. Loss of collagen structure and integrity is frequently associated with abnormalities of the skin, including skin cancer and thermal injury, suggesting that birefringence measurements may prove valuable as a diagnostic indicator of certain dermatologic pathologies.

Arm tissue of a human volunteer

Scar vs normal arm tissue



Conventional (a) and polarization-sensitive (b) images of skin at the site of scar tissue on the arm of a human volunteer. The images are 5 $\times$ 1.2 mm. To the left side of the image is normal skin, to the right is mature scar tissue.

The effect of wound healing on skin birefringence

The image traverses the boundary between normal skin (left) and scar tissue (right), with the scar region providing a slightly stronger (darker) backscattering signal compared to the normal region in panel (a).
Examination of the corresponding polarization-sensitive image (panel (b)) reveals multiple blackâwhite bands within the scar region that are absent in the normal (left) side of the image. The appearance of strong multiple banding within both young and old scar tissue has been observed consistently during imaging of scar sites on the skin and they are related to the presence of newly formed collagen at these locations, which exhibit increased birefringence.

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For further reading

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- G. Gerosa and P. Lampariello, Lezioni di Campi Elettromagnetici, Edizioni Ingegneria 2000, Roma, 2006
- J.G. Van Bladel, Electromagnetic fields, IEEE Press, Piscataway, NJ, 2007
- Pierce, et al. "Birefringence measurements in human skin using polarization-sensitive optical coherence tomography," J. Biomed. Opt. 9(2),2004.

For further reading

- https://www.strainoptics.com/training/
- Magli et al., "Birefringence properties of human immotile spermatozoa and ICSI outcome", Reproductive BioMedicine Online, Volume 46, Issue 3,2023