Course of “Automatic Control Systems”
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Analysis of LTI systems in the time domain

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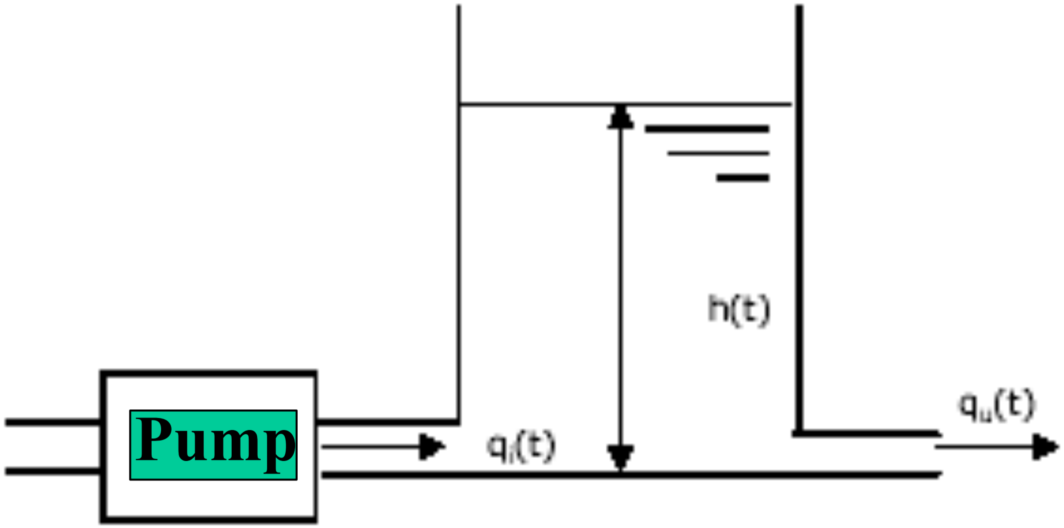
- ***Linear time invariant (LTI) systems*** in the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}, \quad x(t_0) = x_0$$

with $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$, where $x(t)$ is the state vector, $u(t)$ is the input vector and $y(t)$ is the output vector of the system.

Example of first-order LTI system: hydraulic system



input: $u(t) = q_i(t)$

output: $y(t) = h(t)$

$$\frac{dV}{dt} = q_i(t) - q_u(t)$$

hp. laminar flow

Input-output representation:

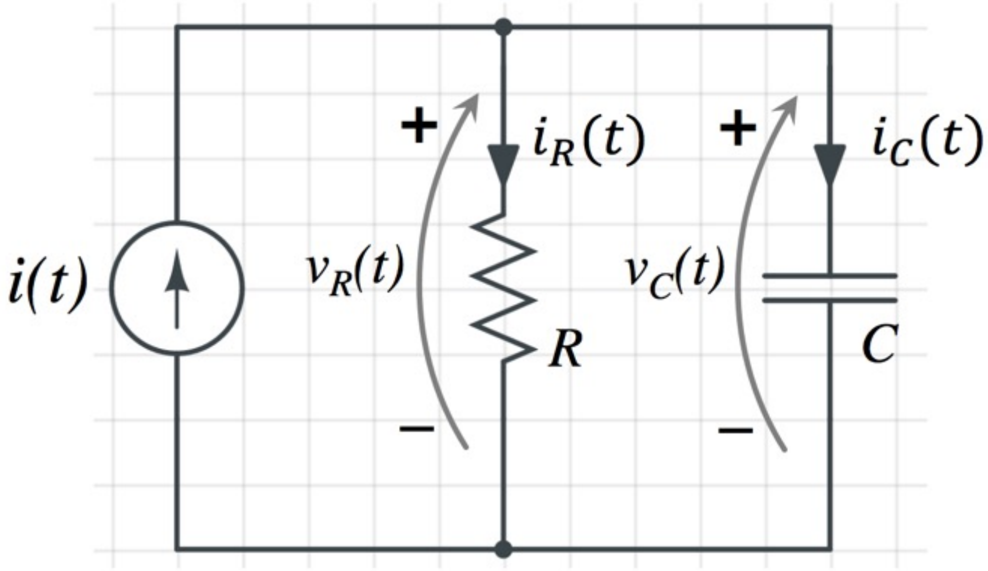
$$S\dot{y}(t) = u(t) - ky(t) \Longrightarrow S\dot{y}(t) + ky(t) = u(t)$$

State space representation:

$$\dot{x}(t) = -\frac{k}{S}x(t) + \frac{1}{S}u(t)$$

$$y(t) = x(t)$$

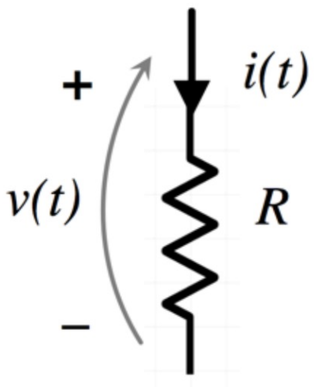
Example of first order LTI system: RC circuit

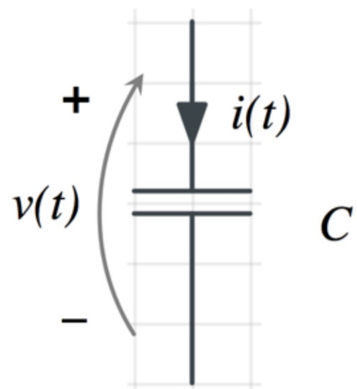


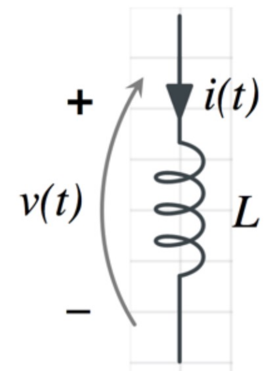
input: $\boldsymbol{u(t) = i(t)}$

output: $\boldsymbol{y(t) = v_c(t)}$

LTI systems – circuit elements





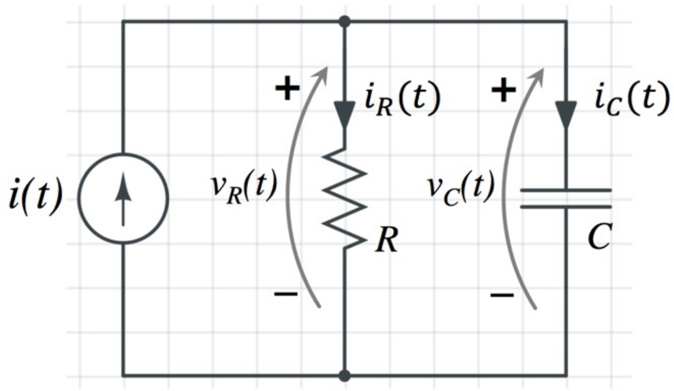


$$\mathbf{v}(t) = R\, i(t)$$

$$i(t) = C\,\frac{dv(t)}{dt}$$

$$v(t) = L\,\frac{di(t)}{dt}$$

Example of first order LTI system



$$u(t) = i(t), \quad y(t) = v_c(t)$$

$$i(t) = i_R(t) + i_c(t) = \frac{v_c(t)}{R} + C\frac{dv_c(t)}{dt}$$

Input-output representation:

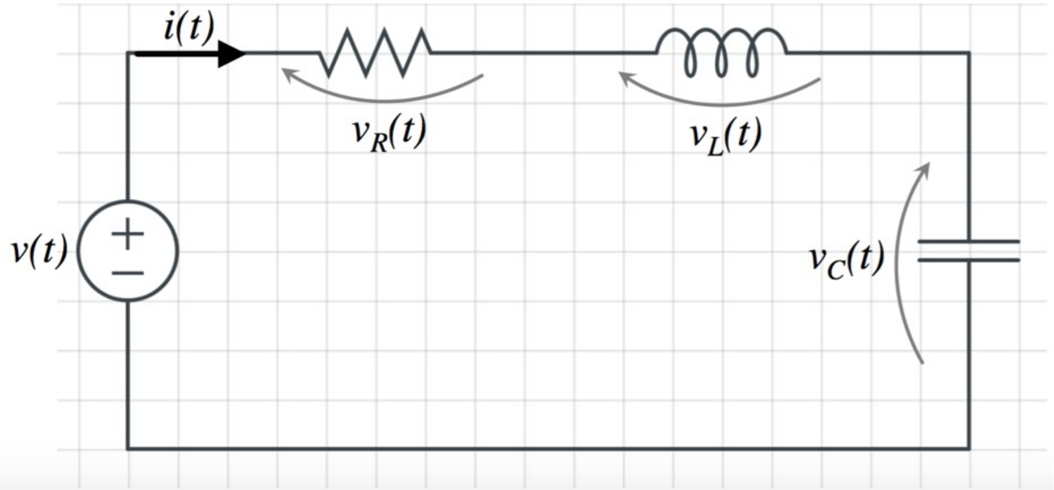
$$\Longrightarrow \quad C\dot{y}(t) + \frac{y(t)}{R} = u(t)$$

State space representation:

$$\Longrightarrow \quad \dot{x} = -\frac{1}{CR(t)}x + \frac{1}{C}u$$

$$y = x$$

Example of second-order LTI system: RLC circuit



$$u(t) = v(t), \quad y(t) = v_c(t)$$

Input-output representation:

$$LC\ddot{y}(t) + RC\dot{y}(t) + y(t) = u(t)$$

State space representation:

$$x_1(t) = v_c(t) \quad x_2(t) = i_L(t)$$

$$\dot{x}_1 = \frac{1}{C}x_2$$

$$\dot{x}_2 = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u$$

$$y = x_1$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/C \\ -1/L & -R/L \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/L \end{pmatrix} u$$

$$y = (1 \quad 0)x \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Lagrange Formula

- Let us consider a ***Linear Time Invariant*** (***LTI***) system in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \qquad (1)$$

The solution of the linear differential equation (1) defines the ***time evolution of the state variables*** and it is given by the Lagrange Formula

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B\,u(\tau)\,d\tau, \quad t \geq t_0 \qquad (2)$$

- The ***time evolution of the outputs*** turns out to be

$$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(t-\tau)}B\,u(\tau)\,d\tau + D\,u(t), \quad t \geq t_0 \qquad (3)$$

Lagrange Formula

- Taking into account that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)\,d\tau = f\big(t,b(t)\big)\frac{db(t)}{dt} - f\big(t,a(t)\big)\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{d}{dt} f(t,\tau)\,d\tau$$

- Lagrange formula (2) can be easily verified by derivation (assuming $t_0 = 0$)

$$\dot{x}(t) = \frac{d}{dt}(e^{At}x_0) + e^{A(t-t)}Bu(t) + \int_0^t \frac{d}{dt}\left[e^{A(t-\tau)}B\,u(\tau)\right]d\tau$$

$$= Ae^{At}x_0 + Bu(t) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau$$

$$= A\left[e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right] + Bu(t) = Ax(t) + Bu(t)$$

Free and forced evolution of LTI systems

- The ***time evolution of the state and output variables*** can be conceptually divided in two parts,

$$x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\textit{Free evolution}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}B\,u(\tau)\,d\tau}_{\textit{Forced evolution}}, \quad t \geq t_0$$

$$y(t) = \underbrace{Ce^{A(t-t_0)}x_0}_{\textit{Free evolution}} + \underbrace{C\int_{t_0}^{t} e^{A(t-\tau)}B\,u(\tau)\,d\tau + D\,u(t)}_{\textit{Forced evolution}}, \quad t \geq t_0$$

- The ***free evolution*** indicate the evolution of state and output vectors that would be obtained in the absence of input ($u(t) = 0$).

- The ***forced evolution*** indicate the evolution of state and output vectors that would be obtained in the presence of input and null initial conditions ($x_0 = 0$)

Free evolution: matrix 'A' diagonalizable

- The free evolution of an LTI system in the time domain is defined by the matrix exponential e^{At}. Generalizing the Taylor expansion of an exponential to the matrix case, we have

$$e^{M} = \sum_{i=0}^{\infty} \frac{1}{i!} M^{i} = I_n + M + \frac{M^2}{2!} + \cdots$$

- In case the matrix A has real and distinct eigenvalues, it is diagonalizable and e^{At} turns out to be

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} (A\,t)^{i} = U \sum_{i=0}^{\infty} \frac{1}{i!} (\Lambda t)^{i}\, U^{-1}$$

$$= U \operatorname{diag}\left\{e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}\right\} V$$

where $\boldsymbol{\lambda_1, \lambda_2 \dots \lambda_n}$ are the eigenvalues of the $\boldsymbol{A}$ matrix, $\boldsymbol{U}$ is eigenvector matrix and $\boldsymbol{V = U^{-1}}$ is the left eigenvector matrix.

Free evolution: matrix 'A' diagonalizable

- The free evolution of an LTI system when the matrix A is diagonalizable turns out to be:

$$
\begin{aligned}
e^{At}x_0 &= U\,\mathrm{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}\}V\mathrm{x}_0 \\
&= \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} \mathrm{x}_0 \\
&= \sum_{i=1}^{n} e^{\lambda_i t} u_i v_i^T x_0 \\
&= \sum_{i=1}^{n} e^{\lambda_i t} u_i c_i
\end{aligned}
$$

Aperiodic Modes

where the coefficient $c_i \in R^n$ are the projection of the initial state x_0 on the eigenvector u_i.

Aperiodic evolution modes (1/4)

- ***An aperiodic mode*** is an evolution mode of a linear system related to a real eigenvalue of the matrix A of multiplicity 1. It can be written in the form

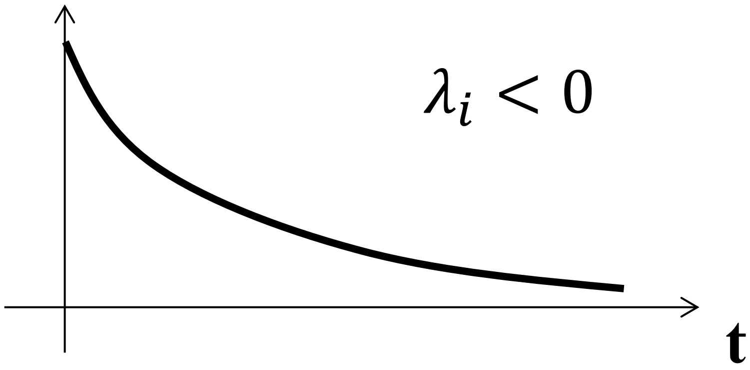
$$c_i e^{\lambda_i t} u_i$$

- It gives us the evolution of the state along the direction defined by the eigenvector u_i starting from an initial value c_i (projection of the initial state x_0 on the eigenvalue u_i).

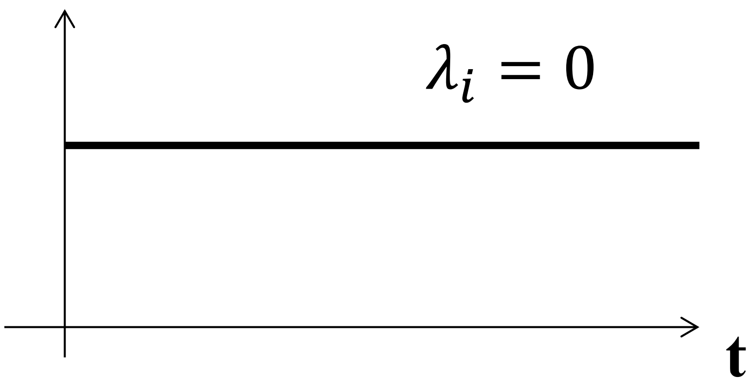
- Depending on the sign of the eigenvalue λ_i, an aperiod evolution modes can be
  - convergent ($\lambda_i < 0$)
  - constant ($\lambda_i = 0$)
  - divergent ($\lambda_i > 0$)

Aperiodic evolution modes (2/4)

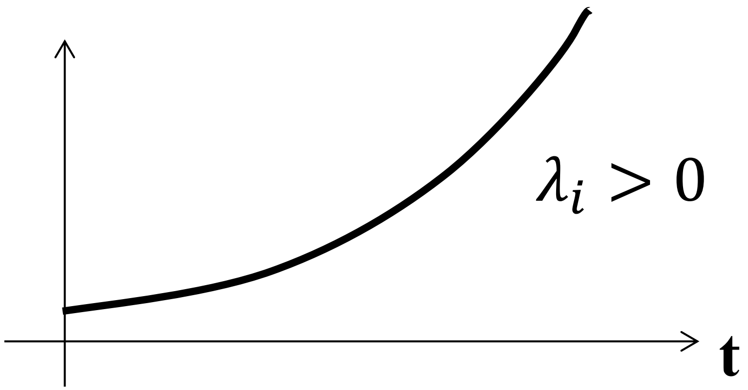
✧ ***Convergent aperiodic mode***



✧ ***Constant aperiodic mode***



✧ ***Divergent aperiodic mode***



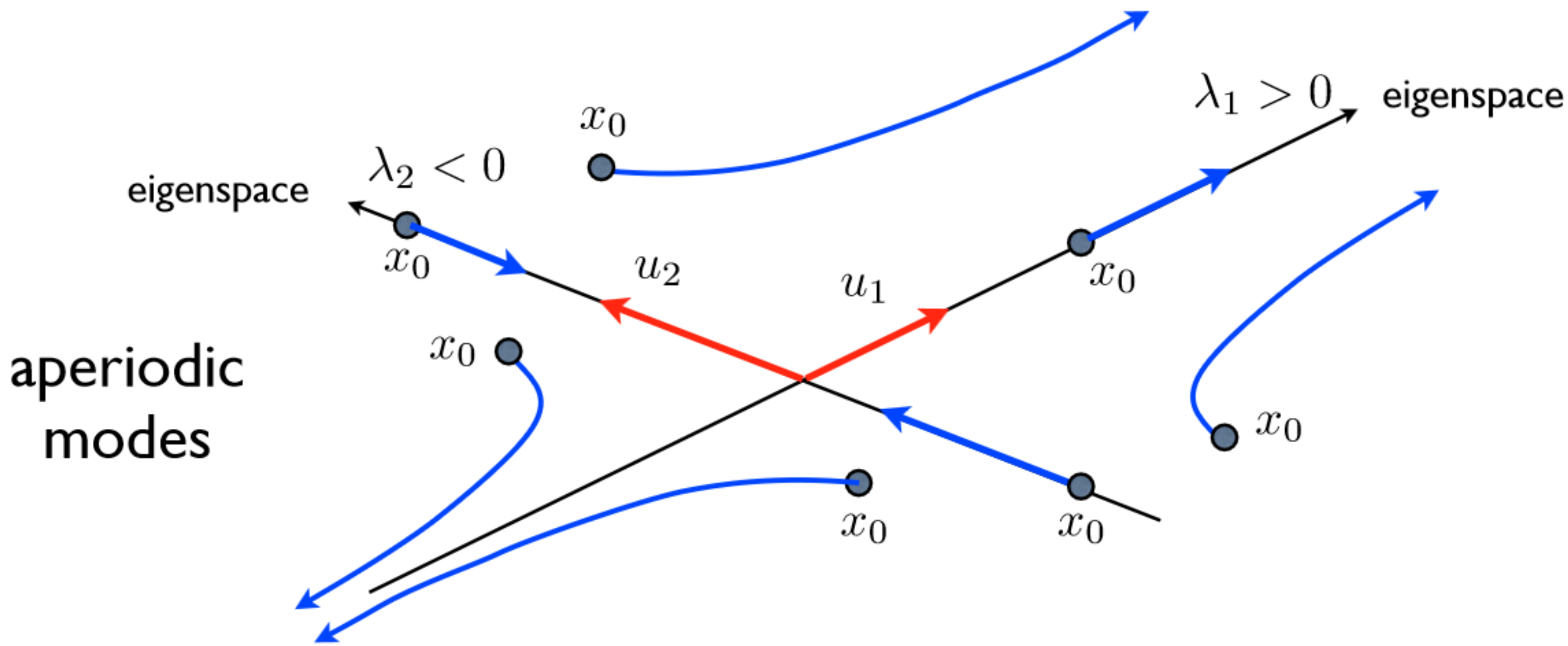
Aperiodic evolution modes (3/4)

CASE n=2 with $\lambda_1 > 0$ and $\lambda_2 < 0$

$$e^{At}x_0 = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2$$

Divergent aperiodic mode

Convergent aperiodic mode



Aperiodic evolution modes (4/4)

- When the evolution mode is convergent it is possible to introduce a new parameter said ***time constant of the mode*** defined as

$$\tau_i = -\frac{1}{\lambda_i}$$

- The time constant gives us an information about the time needed before the convergent mode will be extinguished.

- It is straightforward to verify that

  - ***After a time*** $\boldsymbol{\bar{t} = 3\tau}$ the magnitude of the mode will be reduced to the 5% of the initial value

  - ***After a time*** $\boldsymbol{\bar{t} = 4.6\tau}$ the magnitude of the mode will be reduced to the 1% of the initial value

Free evolution: matrix 'A' no diagonalizable

- If A is not diagonalizable the decompositions can be implemented using the ***Jordan form*** (see the book for details).

- When the matrix A has both ***'real distinct' eigenvalues*** $\boldsymbol{\lambda_1, \lambda_2 \dots \lambda_\mu}$ and ***'complex conjugate' eigenvalues*** $\boldsymbol{\alpha_1 \pm j\omega_1, \alpha_2 \pm j\omega_2 \dots \alpha_\nu \pm j\omega_\nu}$ of multiplicity one, the free evolution of an LTI system turns out to be:

$$e^{At}x_0 = \underbrace{\sum_{i=1}^{\mu} e^{\lambda_i t} u_i v_i^T x_0}_{\textit{Aperiodic Modes}} + \underbrace{\sum_{l=1}^{\nu} e^{\alpha_l t} \begin{pmatrix} u_{la} & u_{lb} \end{pmatrix} \begin{pmatrix} \cos(\omega_l t) & \sin(\omega_l t) \\ -\sin(\omega_l t) & \cos(\omega_l t) \end{pmatrix} \begin{pmatrix} v_{la}^T \\ v_{lb}^T \end{pmatrix} x_0}_{\textit{Pseudo-periodic Modes}}$$

where u_{la} and u_{lb} are the real and the imaginary part of the complex eigenvectors and v_{la} and v_{lb} are the real and the imaginary part of the complex left eigenvectors

Pseudo-periodic evolution modes (1/6)

- A pseudo-periodic mode is an evolution mode of a linear system related to a pair of complex conjugate eigenvalues of molteplicity 1. It can be written in the form

$$e^{\alpha_l t}\begin{pmatrix} u_{la} & u_{lb} \end{pmatrix}\begin{pmatrix} \cos(\omega_l t) & \sin(\omega_l t) \\ -\sin(\omega_l t) & \cos(\omega_l t) \end{pmatrix}\begin{pmatrix} v_{la}^T \\ v_{lb}^T \end{pmatrix} x_0$$

- Let us indicate with $c_{la} = v_{la}^T x_0$ and $c_{lb} = v_{lb}^T x_0$. Introducing a new set of variables related to the initial condition of the system:

$$m_l = \sqrt{c_{la}^2 + c_{lb}^2} \qquad\qquad \beta_l = arctg\left(\frac{c_{la}}{c_{lb}}\right)$$

the pseudo-periodic mode can be re-written as (see the book for details)

$$m_l e^{\alpha_l t}[u_{la}\sin(\omega_l t + \beta_l) + u_{lb}\cos(\omega_l t + \beta_l)\,]$$

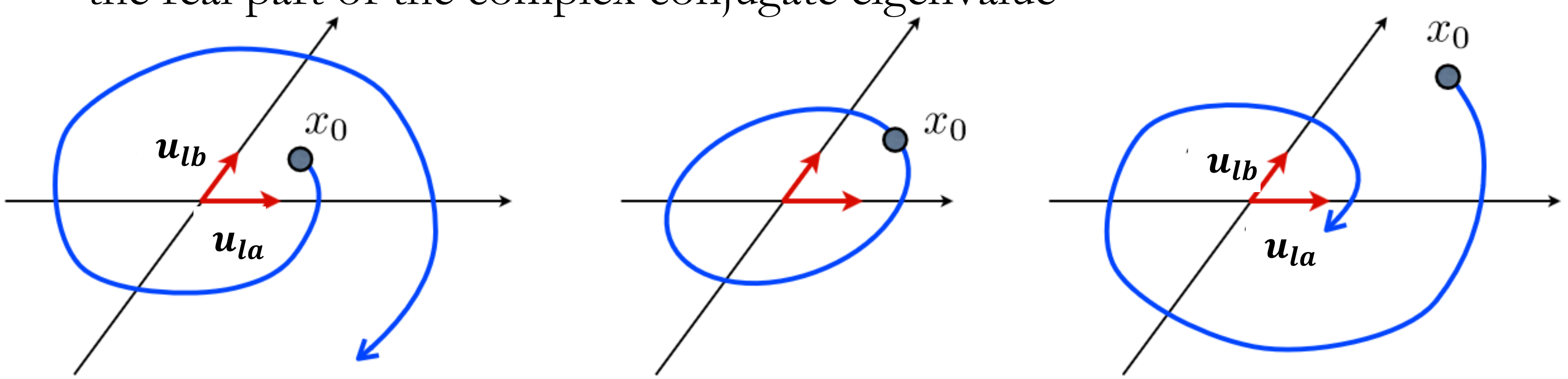
Pseudo-periodic evolution modes (2/6)

- Looking at a pseudo-periodic evolution mode in the form

$$m_l e^{\alpha_l t}[u_{la}\sin(\omega_l t + \beta_l) + u_{lb}\cos(\omega_l t + \beta_l)\,]$$

we note that:

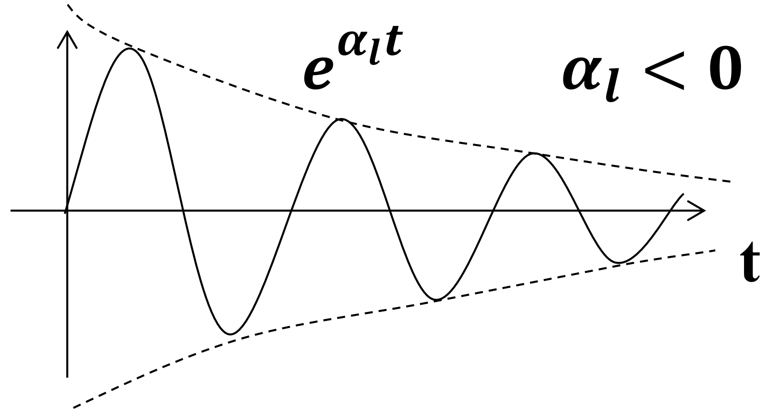
  - a pseudo-periodic evolution mode gives us the evolution of the state in the plane defined by the vector u_{la} and u_{lb}
  - a pseudo-periodic evolution mode defines spiral trajectories in the plane defined by the vector u_{la} and u_{lb}. The convergence of the mode depends on the real part of the complex conjugate eigenvalue



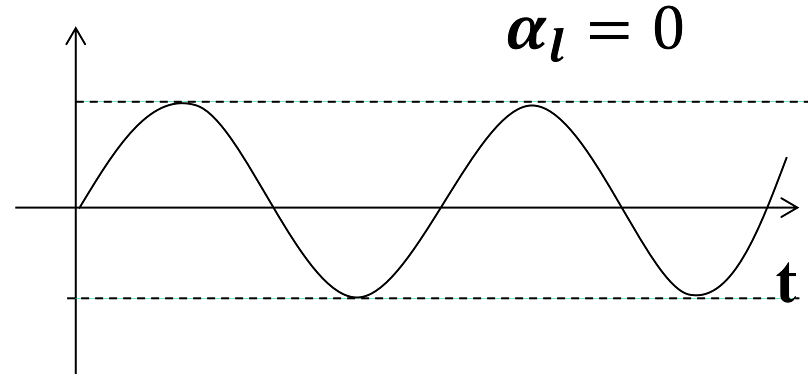
$\alpha_i > 0$ $\alpha_i = 0$ $\alpha_i < 0$

Pseudo-periodic evolution modes (3/6)

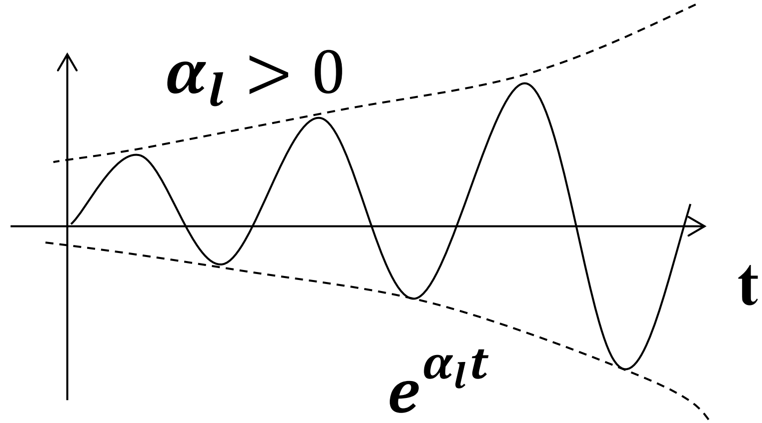
✧ ***Convergent pseudo-periodic mode***



✧ ***Constant pseudo-periodic mode***



✧ ***Divergent pseudo-periodic mode***



- For convergent pseudo-periodic mode, the ***time constant*** is defined as

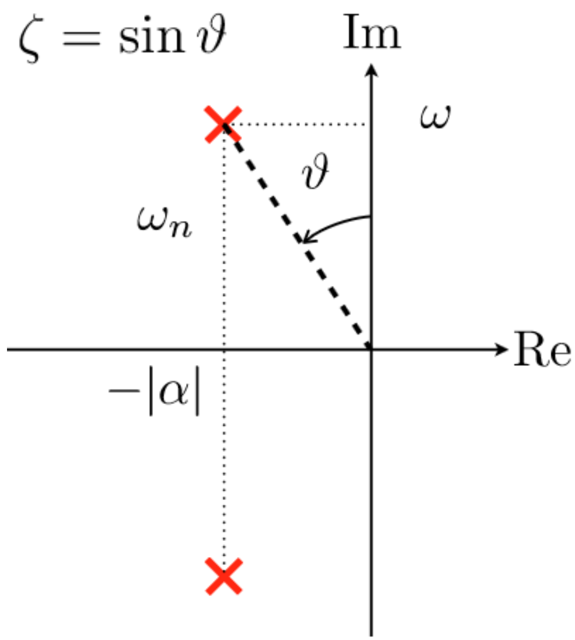
$$\tau_i = -\frac{1}{\alpha_i}$$

- Other important parameters for pseudo-periodic mode are the **natural frequency**

$$\omega_n = \sqrt{\alpha^2 + \omega^2}$$

and the **damping coefficient**

$$\zeta = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$



- The ***natural frequency*** $\boldsymbol{\omega_n}$ is the oscillation frequency of the pseudo-periodic mode when $\alpha = 0$.

- ***For convergent*** pseudo-periodic modes the ***damping coefficient ζ∈(0,1]*** while ***for divergent pseudo-periodic modes ζ∈[-1,0)***

- ***For convergent*** pseudo-periodic modes, the **damping coefficient** $\boldsymbol{\zeta}$ relates the oscillations of the pseudo-periodic mode to the time before the evolution will extinguish. For $\zeta \ll 1$

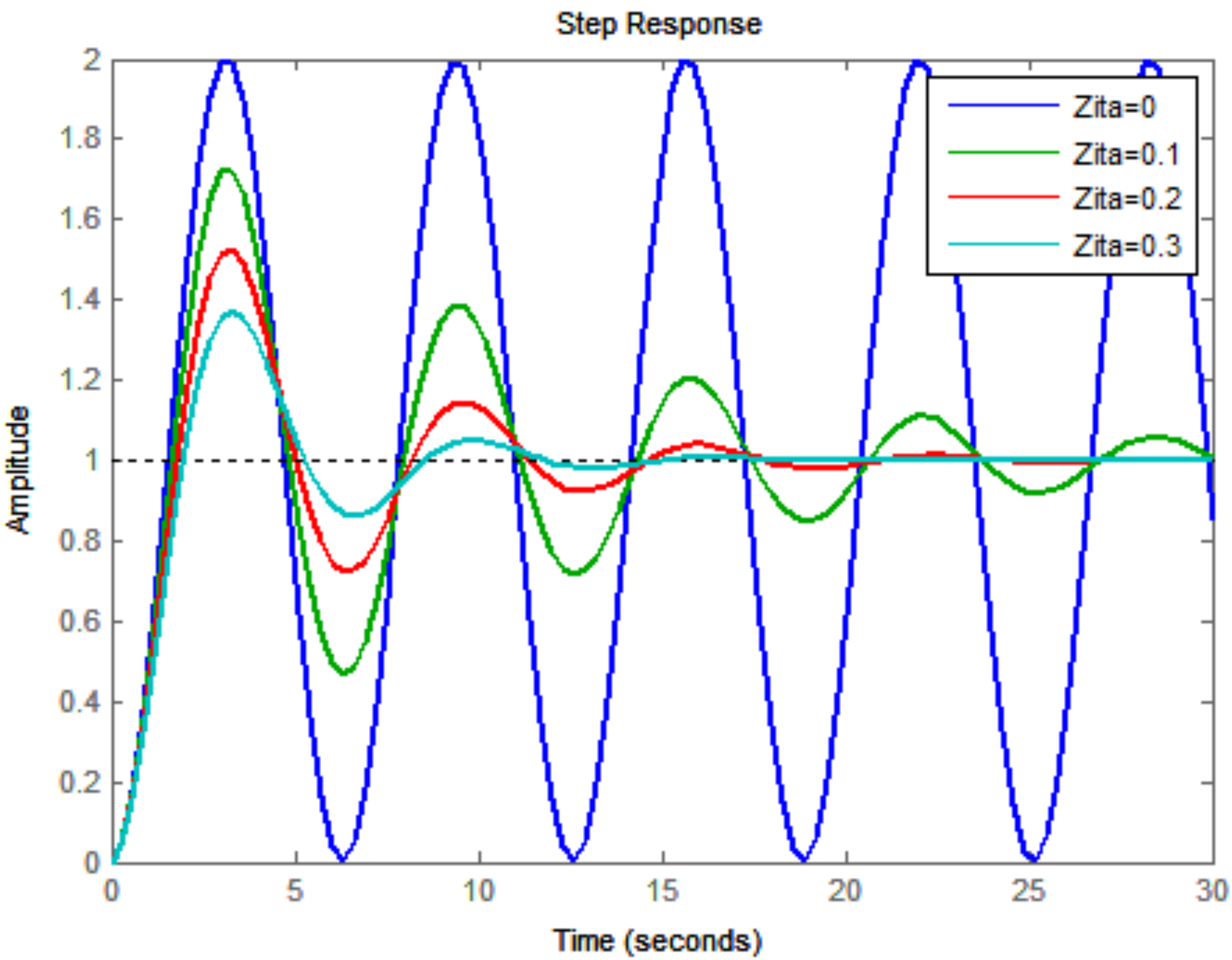
$$\zeta = -\frac{\alpha}{\omega_n} \cong -\frac{\alpha}{\omega} = \frac{T}{2\pi\tau} \ll 1$$

where T is the oscillation period. Indeed, the number of the oscillation before the mode will extinguish increases when $\boldsymbol{\zeta}$ becomes small.

$$\zeta = \frac{T}{2\pi\tau} \cong \frac{T}{6\tau} \quad \Longrightarrow \quad \frac{1}{2\zeta} \cong \frac{3\tau}{T}$$

# of oscillations before the mode will extinguish

Pseudo-periodic evolution modes (6/6)



Forced response in the time domain

- Let us consider the forced response of an LTI system in the output $(x_0 = 0)$

$$y(t) = C\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau + D\,u(t), \quad t \geq t_0$$

- The evaluation of the forced response in the time domain is demanding due to the presence of the convolution product.

- Only in some particular case, such as the ***step response*** $\boldsymbol{u(t) = \bar{u} \cdot 1(t)}$, it becomes straightforward

$$y(t) = C\int_0^t e^{A(t-\tau)}B\,\bar{u}\,d\tau + D\,\bar{u}$$

$$= \left[-CA^{-1}e^{A(t-\tau)}B\bar{u}\right]_0^t + D\,\bar{u}$$

$$= CA^{-1}e^{At}B\bar{u} + \left[-CA^{-1}B + D\right]\bar{u}$$

- In the other cases the forced response is evaluated in the ***Laplace domain***