Geometric Optic and eikonal equation

Electromagnetics
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Outline

Geometric optic

Meaning

Geometric optics, or ray optics, refers to a high frequency approximation of Maxwell's equation that in terms of rays describes waves propagation.

In addition, the ray in geometric optics is an abstraction useful for approximating the paths along which the propagation of em waves takes place under certain circumstances.

The scalar approximation

Scalar problems

In a simple medium the vector nature of the em field does not play any role. In fact, the components of the electric field satisfy the scalar Helmholtz equation.

- This is not always the case.
- A scalar problem is always very attractive that is why, often, approximations are made to simplify the vector problem into a scalar one.
- Inhomogeneous media are a notable example of media for which the scalar approach cannot be obtained rigorously but only by making approximations.

Outline

The scalar approximation

Inhomogeneous medium

An inhomogeneous medium is such that when at least one of its constitutive parameters is a function of spatial coordinates, e.g.; $\epsilon \leftarrow \epsilon(\mathbf{r})$. Hereinafter, the spatial dependence is understood but omitted for brevity.

- When moving from Maxwell's equations to the Helmholtz equation, one needs to deal with the following term:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= \omega^2 \epsilon \mu \mathbf{E} \\ &= k_o^2 n^2 \mathbf{E} \end{aligned} \tag{1}$$

- with $n^2 = \epsilon$

The scalar approximation

- According to the divergence Maxwell's equation:

$$\nabla \cdot \epsilon \mathbf{E} = 0 \tag{2}$$

- Invoking vector identities:

$$\nabla \cdot \epsilon \mathbf{E} = \nabla \epsilon \cdot \mathbf{E} + \epsilon \nabla \cdot \mathbf{E} = 0 \tag{3}$$

- which means:

$$\nabla \cdot \mathbf{E} = -\frac{\nabla \epsilon \cdot \mathbf{E}}{\epsilon} \tag{4}$$

The scalar approximation

- The vector Helmholtz equation in the most general case reads as follows:

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = k_o^2 n^2 \mathbf{E} \tag{5}$$

The vector Helmholtz equation

The behavior of this equation must be discussed in case of:

- Homogeneous medium.
- Inhomogeneous medium.

The scalar approximation

- A homogeneous medium is such that ϵ (and, therefore, n), does not depend on the spatial coordinates.
- This means that (4) vanishes since $\nabla\epsilon = 0$.
- Hence, the vector Helmholtz equation (5) reads as follows:

$$\nabla^2 \mathbf{E} + k_o^2 n^2 \mathbf{E} = 0 \tag{6}$$

- Each field component satisfies the scalar Helmholtz equation:

$$\nabla^2 \psi + k_o^2 n^2 \psi = 0 \tag{7}$$

- The uniform plane wave solution is achieved:

$$\psi(\mathbf{r}) = \psi_o e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{8}$$

The scalar approximation

- An inhomogeneous medium is such that its constitutive parameters, e.g.; ϵ (and, therefore, n), depend on the spatial coordinates.
- This means that (4) does not vanish since $\nabla\epsilon \neq 0$.
- Hence, the vector Helmholtz equation (5) reads as follows:

$$\nabla^2 \mathbf{E} + k_o^2 n^2 \mathbf{E} + 2\nabla\left(\frac{\nabla n \cdot \mathbf{E}}{n}\right) = 0 \tag{9}$$

with $\nabla(n^2) = 2n\nabla n$

The last term is non-zero and it describes the coupling between the 3 field components of **E** and the resulting polarization effects.

It cannot be broken down into scalar problems in an exact way

Outline

The scalar approximation

The scalar approximation

In case of slow varying media:

$$\left|\frac{\nabla n}{n}\right| \ll \lambda. \tag{10}$$

we drop the last term in eq.(9).

- This is a reasonable approximation in applications where polarization is not measurable.
- Under this hypothesis, the field Cartesian components must satisfy the scalar Helmholtz equation:

$$\nabla^2\psi + k_o^2 n^2 \psi = 0 \tag{11}$$

Outline

Local plane waves

- Let us suppose that any solution of eq.(11) - in an inhomogeneous medium - can be expressed as:

$$\psi(\mathbf{r}) = C(\mathbf{r})e^{-jk_o S(\mathbf{r})} \tag{12}$$

- $k_o = \omega\sqrt{\mu_o \epsilon_o}$ is the intrinsic phase constant of free space;
- $C(\mathbf{r})$ and $S(\mathbf{r})$ are two real-valued functions of spatial coordinates that are termed as amplitude and phase functions, respectively.

Local plane waves

A a generic fixed point $P = O + \mathbf{r}_o$, eq.(12) looks like a plane wave. That is why, the solutions described by eq.(12) consist of expanding the field into locally plane waves.

HOMOGENEOUS MEDIUM ← *Scalar number* — $\varepsilon = \sqrt{n}$ — *Function of space coordinates* → **INHOMOGENEOUS MEDIUM**

$\varepsilon = \varepsilon(\boldsymbol{r}) \Rightarrow n = n(\boldsymbol{r})$

$$\nabla\varepsilon \cdot \boldsymbol{E} + \varepsilon \nabla \cdot \boldsymbol{E} = 0$$

$\nabla\varepsilon = 0 \Rightarrow \nabla \cdot \boldsymbol{E} = 0$

$\nabla \cdot \boldsymbol{E} = \dfrac{\nabla\varepsilon \cdot \boldsymbol{E}}{\varepsilon}$

HELMHOLTZ EQUATION

$$\nabla^2 \boldsymbol{E} + {k_0}^2 n^2 \boldsymbol{E} + 2\left(\frac{\nabla n \cdot \boldsymbol{E}}{n}\right) = 0$$

$$\nabla^2 \boldsymbol{E} + {k_0}^2 n^2 \boldsymbol{E} = 0$$

$$\nabla^2 \boldsymbol{E} + {k_0}^2 n^2(\boldsymbol{r}) \boldsymbol{E} + 2\left(\frac{\nabla n \cdot \boldsymbol{E}}{n}\right) = 0$$

$$\left|\frac{\nabla n}{n}\right| \ll \lambda$$

Slowly varying inhomogeneous medium

$$\nabla^2 \psi + {k_0}^2 n^2 \psi = 0$$
$$\psi(\boldsymbol{r}) = E_i(\boldsymbol{r}) = \psi_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$$

UNIFORM PLANE WAVE

YES

$$\nabla^2 \psi + k_0 n^2 2\psi = 0$$
$$\psi(\boldsymbol{r}) = E_i(\boldsymbol{r}) = A(\boldsymbol{r}) e^{-jk_0 s(\boldsymbol{r})}$$

LOCALLY UNIFORM PLANE WAVE

NO

The Helmholtz equation cannot break down into scalar equations

Outline

Eikonal equation

- To analyze the properties of the local plane wave solution, eq.(12) can be substituted into the scalar Helmholtz equation (11) and invoking the vector identity:

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \nabla f \cdot \mathbf{A} \tag{13}$$

- one obtains (noting that $\nabla^2\psi = \nabla \cdot \nabla\psi$):

$$\nabla\psi = \nabla(Ce^{-jk_oS}) = \nabla Ce^{-jk_oS} - jk_oC\nabla Se^{-jk_oS} \tag{14}$$

$$\nabla \cdot \left(\nabla Ce^{-jk_oS}\right) = e^{-jk_oS}\nabla^2 C - jk_o\nabla S \cdot \nabla C \tag{15}$$

$$\nabla \cdot \left(-jk_oC\nabla Se^{-jk_oS}\right) = \nabla \cdot \left(\nabla S\left(-jk_oCe^{-jk_oS}\right)\right) =$$

$$-jk_oCe^{-jk_oS}\nabla^2 S + \nabla\left(-jk_oCe^{-jk_oS}\right) \cdot \nabla S =$$

$$-jk_oCe^{-jk_oS}\nabla^2 S - jk_o\nabla Ce^{-jk_oS} \cdot \nabla S - k_o^2 C\nabla Se^{-jk_oS} \cdot \nabla S \tag{16}$$

Eikonal equation

- Dividing by $e^{-jk_o S} \neq 0$ one obtains:

$$\nabla^2 C - jk_o \nabla S \cdot \nabla C - jk_o C \nabla^2 S - jk_o \nabla C \cdot \nabla S - k_o^2 C (\nabla S)^2 =$$

$$\nabla^2 C - jk_o \left(2\nabla S \cdot \nabla C + C \nabla^2 S \right) - k_o^2 C (\nabla S)^2 = k_\epsilon^2 C \tag{17}$$

- Since C and S are two real functions, the equality (17) consists of equating to zero separately the real and the imaginary parts.
- Let us start from the real part:

$$(\nabla S)^2 = -\frac{k_\epsilon^2}{k_o^2} + \frac{1}{k_o^2} \frac{\nabla^2 C}{C} \tag{18}$$

Eikonal equation

- Since $n = \sqrt{\frac{\epsilon\mu}{\epsilon_o\mu_o}}$ is the spatial dependent refractive index, eq.(18) can be rewritten as follows:

$$(\nabla S)^2 = n^2 + \frac{1}{k_o^2}\frac{\nabla^2 C}{C} \tag{19}$$

High frequency approximation - eikonal equation

Note that, for very high frequencies (and, thus, small wavelengths):

$$|\nabla^2 C| \ll k_o^2 |C| \tag{20}$$

hence, eq.(19) can be approximated as follows:

$$(\nabla S)^2 = n^2 \tag{21}$$

This equation is known as eikonal equation

Outline

Ray paths

High frequency approximation

Note that the inequality provided by eq.(20) is achieved when two conditions are met:

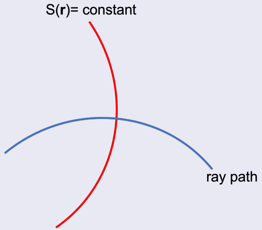
- High frequency (or, equivalently, high k_o).
- The relative change in the amplitude function C over space should be small.

This is why rough features and sharp boundaries always pose problems in geometrical optics predictions.

Ray paths

- Solving eq.(21) consists of finding the constant-phase surfaces $S(\mathbf{r})$=const of the wave (12).
- At any point $R = O + \mathbf{r}$ each of these waves calls for a phase vector that is orthogonal to the constant-phase surface that passes through R.

The lines that envelope the local phase vectors are called ray paths

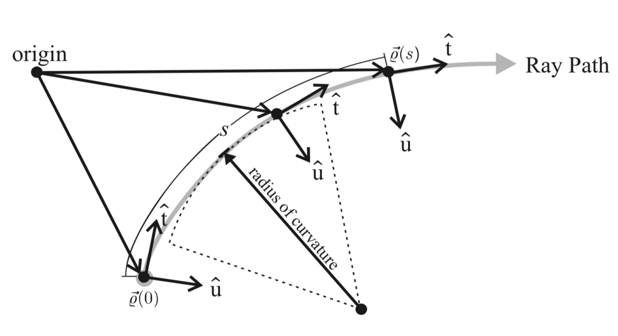


Outline

Ray paths

Solving the Eikonal Equation with Rays

A ray is an intrinsically one-dimensional construct, but may bend in an arbitrary manner through three-dimensional space.



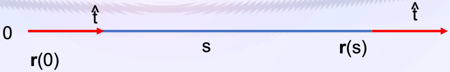
Outline

Straight ray

- A straight ray can be described by a vector function $\mathbf{r} = (x(s), y(s), z(s))$ whose components are the Cartesian coordinates of points along the ray; while s is a unit of length measured from the starting point of the ray.
- For a straight ray that starts in $\mathbf{r}(0)$ its position as function of distance can be written as follows:

$$\mathbf{r}(s) = \mathbf{r}(0) + s\hat{t} \tag{22}$$

with $\hat{t}$ being a unit vector that points in the direction of ray travel.



Outline

Curved ray

- To describe a ray that curves, $\hat{t}$ needs to vary with position s along the ray.

Rays that curve

- The ray is the collection of positions **r** that follow the general system of equations:

$$\frac{d\mathbf{r}(s)}{ds} = \hat{t} \qquad \frac{d^2\mathbf{r}(s)}{ds^2} = \frac{d\hat{t}}{ds} = \rho\hat{u} \tag{23}$$

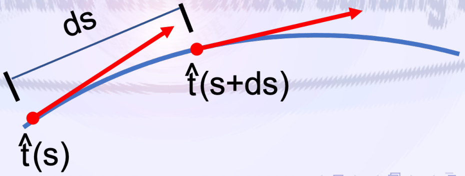
- In addition, one can define the direction-derivative operator along the direction of a ray and orthogonal everywhere to the $S(\mathbf{r}) = const$ surface:

$$\frac{d}{ds} = \hat{t} \cdot \nabla \tag{24}$$

First derivative term

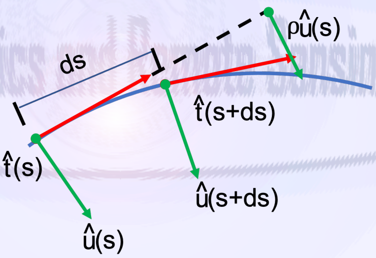
First derivative

- The first derivative of **r** with respect to ray length must, by definition, point along this unit vector and have unitary length.
- This means that the first derivative is allowed to change direction as *s* increases but its magnitude must be always unitary to be a natural measure of length in space.



Second derivative

- The second derivative of **r** with respect to ray length must point in a direction $\hat{u}$ that is perpendicular to the direction of ray travel $\hat{t}$.
- The magnitude of this second derivative is related to the curvature ρ of the ray at a given point



Ray behavior wrt n

- To link the ray behavior to n, let's start from the eikonal equation (21). If the rays are to trace waves, the direction of ray travel $\hat{t}$ must point along the gradient of the eikonal function S:

$$\nabla S = n\frac{d\mathbf{r}}{ds} = n\hat{t} \tag{25}$$

- By deriving both sides with respect to s:

$$\begin{aligned} \frac{d}{ds}\left(n\frac{d\mathbf{r}}{ds}\right) &= \frac{d}{ds}\left(n\hat{t}\right) \\ \frac{d}{ds}n\hat{t} + n\frac{d^2\mathbf{r}}{ds^2} &= \frac{d}{ds}n\hat{t} \\ \left(\hat{t}\cdot\nabla n\right)\hat{t} + n\rho\hat{u} &= \nabla n \end{aligned} \tag{26}$$

Ray behavior wrt n

- Eq.(27) can be rearranged as follows:

$$n\rho\hat{u} = \nabla n - \left(\hat{t} \cdot \nabla n\right)\hat{t} \tag{27}$$

to show that:

the curvature ρ of the ray is proportional to the portion of ∇n that projects transverse to the direction of ray travel $\hat{t}$

- A ray will always curve into the direction of the ∇n.
- If the spatial change of n is perfectly aligned with the direction of ray travel $\hat{t}$, then the curvature is 0 and the ray travels in a straight line.
- Maximum curvature of the ray is achieved when the gradient of n is perpendicular to the direction of travel $\hat{t}$.

Curvature and bending direction

- The ray curvature ρ is given by:

$$\rho = \frac{1}{n} \| \nabla n - \left(\hat{t} \cdot \nabla n \right) \hat{t} \| \tag{28}$$

- The bending direction $\hat{u}$ is given by:

$$\hat{u} = \hat{t} \times \left(\nabla n \times \hat{t} \right) \tag{29}$$

Straight rays

The above equations show that waves are traced by straight rays in homogeneous media.

In such media, the gradient of n is always the zero-vector.

For Further Reading I

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