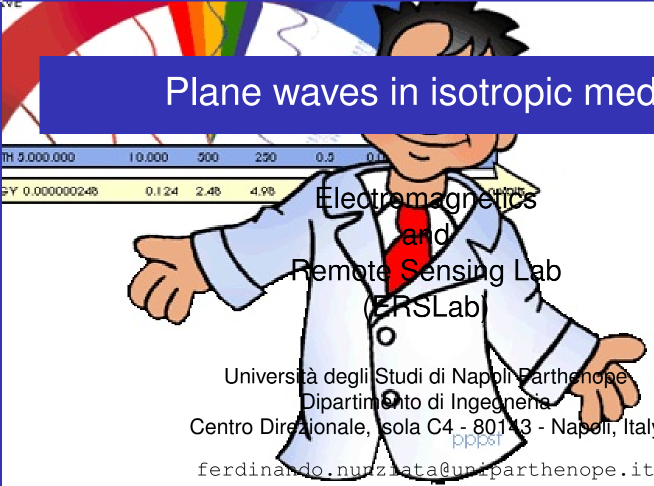
Plane waves in isotropic media



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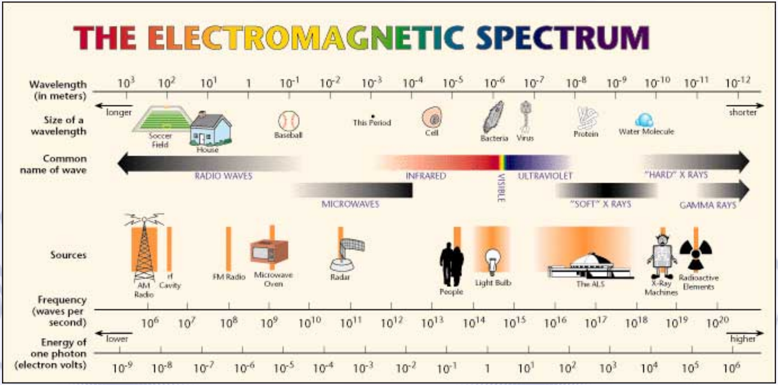
Outline

Propagation

"We have strong reason to conclude that light itself—including radiant heat and other radiation, if any—is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws."

—James C. Maxwell

Electromagnetic waves spectrum



Helmholtz wave equation

Hermann von Helmholtz

AKA Hermann Ludwig Ferdinand von Helmholtz

Born: 31-Aug-1821
Birthplace: Potsdam, Germany
Died: 8-Sep-1894
Location of death: Charlottenburg, Berlin, Germany
Cause of death: unspecified

Gender: Male
Race or Ethnicity: White
Sexual orientation: Straight
Occupation: Physicist

Nationality: Germany
Executive summary: Law of Conservation of Energy

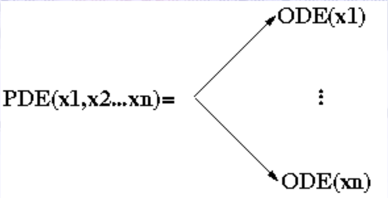
German philosopher and man of science, born on the 31st of August 1821 at Potsdam, near Berlin. His father, Ferdinand, was a teacher of philology and philosophy in the gymnasium, while his mother was a Hanoverian lady, a lineal descendant of the great Quaker William Penn. Delicate in early life, Helmholtz became by habit a student, and his father at the same time directed his thoughts to natural phenomena. He soon showed mathematical powers, but these were not fostered by the careful training mathematicians usually receive, and it may be said that in after years his attention was directed to the higher mathematics mainly by force of circumstances.

$$\boxed{\nabla^2 E - k^2 E = 0}$$

From a PDE to ODEs

One of the most used approaches to solve Partial Differential Equations (PDEs) in mathematical physics is the so-called method of *Separation of Variables* (SV).

It basically consists of breaking a given PDE in a set of *Ordinary Differential Equations* (ODEs), which can be solved separately from one another, by isolating each independent variable in a separate equation.



SV method

SV method is applicable only under restrictive assumptions:

- The PDE must be separable. The set S of solutions obtained by SV method needs to be a complete set of solutions. This means that S is dense enough to allow one writing any PDE solution as a linear combination of solutions belonging to S. A given PDE is typically separable only in few reference frames.
- The boundary conditions must be separable. Any differential equation must satisfy suitable boundary conditions (BC). BCs are themselves separable if the boundary is a coordinate surface (or a set of coordinate surfaces) in one of the reference frames where the PDE is separable.

Outline

The em problem

The PDE which governs both radiation and propagation phenomena is the so-called *Helmholtz* equation. It is a second-order elliptic PDE.

The following electromagnetic (em) problem is defined:

1. Domain: 3D space/ω.
2. Medium: linear, isotropic, homogeneous and lossy.
3. Sources: no imposed currents ($\mathbf{J}_o = 0$).
4. BCs: Sommerfield conditions for the field at infinity.

Uniqueness theorem ensures (ω exterior problem) that, once the above mentioned requirements are known, Maxwell's equations have a unique solution in the given domain.

The em problem

The em field satisfying Maxwell's equation under the previously stated requirements may be calculated solving Helmholtz equation for **E** or **H** through the SV method.

The Helmholtz equation to be solved is given by:

$$\nabla^2 \mathbf{E} - k_\varepsilon^2 \mathbf{E} = 0 \quad , \tag{1}$$

where:

$$k_\varepsilon^2 = -\omega^2 \mu \varepsilon_c = -\omega^2 \mu \left(\varepsilon - j\frac{\sigma}{\omega} \right) \quad . \tag{2}$$

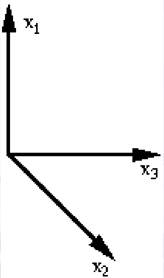
SV method: Helmholtz equation

3D Helmholtz equation (1) is separable only in a few number of coordinate systems which can be derived from the *orthogonal ellipsoidal coordinate* system:

1. Orthogonal Cartesian.
2. Circular cylindrical.
3. Elliptical cylindrical.
4. Parabolic cylindrical.
5. Rotation parabolic.
6. Paraboidal.
7. Spherical.
8. Prolate spheroidal.
9. Oblate spheroidal.
10. Conical.

SV method: Helmholtz equation

A Cartesian orthogonal coordinate system (x_1, x_2, x_3) is hereinafter adopted:



Eq.(1) can be written by components:

$$\nabla^2 E_i \equiv \frac{\partial^2 E_i}{\partial x_1^2} + \frac{\partial^2 E_i}{\partial x_2^2} + \frac{\partial^2 E_i}{\partial x_3^2} = k_\varepsilon^2 E_i \quad . \qquad (3)$$

The three scalar equations are independent of each other, hence, the linearity of the medium allows, without loss of generality, considering:

$$\mathbf{E} = E\hat{x}_1 \quad . \qquad (4)$$

SV method: Helmholtz equation

The SV method consists of making the following ansatz:

$$E(x_1, x_2, x_3) = f_1(x_1) f_2(x_2) f_3(x_3) \quad , \tag{5}$$

which leads to:

$$\frac{f_1''}{f_1} + \frac{f_2''}{f_2} + \frac{f_3''}{f_3} = k_\varepsilon^2 \quad , \tag{6}$$

where f_j'' denotes the second derivative of f_j.

SV method: Helmholtz equation

For a fixed ω, k_ε^2 is constant and, therefore, (6) can be satisfied if and only if:

$$\frac{f_i''}{f_i} = S_i^2 \quad i = 1, 2, 3 \quad , \tag{7}$$

with the following separation condition:

$$S_1^2 + S_2^2 + S_3^2 = k_\varepsilon^2 \quad . \tag{8}$$

SV method: Helmholtz equation

SV method leads to the following three ODEs:

$$\frac{f_i''}{f_i} = S_i^2 \quad i = 1, 2, 3 \quad , \tag{9}$$

whose general integral can be written as follows:

$$f_i = F_{1i} e^{-S_i x_i} + F_{2i} e^{S_i x_i} \quad i = 1, 2, 3 \quad . \tag{10}$$

where F_{1i} and F_{2i} are arbitrary complex constants.

- The separation equation (8) deals with S_i^2. It does not tell anything about $S_i = \pm\sqrt{S_i^2}$.

SV method: Helmholtz equation

Accordingly, there is no loss of generality in the following formula:

$$
\begin{aligned}
E(x_1, x_2, x_3) &= f_1(x_1) f_2(x_2) f_3(x_3) \\
&= E_o e^{-(S_1 x_1 + S_2 x_2 + S_3 x_3)} \\
&= E_o e^{-\mathbf{S} \cdot \mathbf{r}} \quad , \qquad (11)
\end{aligned}
$$

Propagation vector

$$
\mathbf{S} = \sum_i S_i \hat{x}_i \quad , \quad \mathbf{r} = \sum_i x_i \hat{x}_i
$$

are the propagation vector and the position vector, respectively.

It must be noted that $\mathbf{S} = \mathbf{a} + j\mathbf{k}$, where $\mathbf{a}$ is called attenuation vector and $\mathbf{k}$ phase vector, is such that:

$$\mathbf{S} \cdot \mathbf{S} = k_\varepsilon^2 = -\omega^2 \mu \varepsilon_c \quad .$$

Note that, since $\mathbf{S}$ is a complex vector: $\mathbf{S} \cdot \mathbf{S} \neq \mathbf{S} \cdot \mathbf{S}^* = |\mathbf{S}|^2$.

$$\mathbf{E} = \mathbf{E_o} e^{-\mathbf{S} \cdot \mathbf{r}} \quad , \tag{12}$$

where:

$$\mathbf{E_o} = E_o \hat{x}_1 \quad .$$

Electric and magnetic fields

From Maxwell's equations, it follows that:

$$\mathbf{H} = -\frac{\nabla \times \mathbf{E}_o e^{-\mathbf{S}\cdot\mathbf{r}}}{j\omega\mu} \quad . \tag{13}$$

By invoking the vector identity:

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A} \quad ,$$

where f and $\mathbf{A}$ are a scalar and a vector function of space coordinates, (13) becomes:

$$\begin{aligned} -\frac{E_o}{j\omega\mu}\nabla e^{-\mathbf{S}\cdot\mathbf{r}} \times \hat{x}_1 &= \frac{\mathbf{S} \times E_o e^{-\mathbf{S}\cdot\mathbf{r}} \hat{x}_1}{j\omega\mu} \\ &= \mathbf{H_o} e^{-\mathbf{S}\cdot\mathbf{r}} \end{aligned} \tag{14}$$

Remarks

From (14) it follows that:

$$\mathbf{H} = \frac{\mathbf{S} \times \mathbf{E}}{j\omega\mu} \quad . \tag{15}$$

- This term provides a relationship between **E** and **H** which further confirms that (15) is always true, despite the restrictive hypothesis of linear polarization previously made, see eq.(4).
- Under the (unnecessary) hypothesis that all the components of **E** share the same propagation vector **S**, the general solution for **E** is given by:

$$\mathbf{E} = \mathbf{E_o} e^{-\mathbf{S} \cdot \mathbf{r}} \quad . \tag{16}$$

Electric and magnetic fields

Plane waves

The general solution for the em field is given by:

$$\mathbf{E} = \mathbf{E_o} e^{-\mathbf{S}\cdot\mathbf{r}} \tag{17}$$

$$\mathbf{H} = \frac{\mathbf{S} \times \mathbf{E_o}}{j\omega\mu} e^{-\mathbf{S}\cdot\mathbf{r}} \tag{18}$$

These equations are referred to a generic reference frame which can be changed in a completely arbitrary way.

Remarks

The solution provided by (17)-(18) is physically untenable:

- In general, it does not satisfy Sommerfield conditions and, therefore, the uniqueness theorem (not even in the case of lossy medium).
- It carries on an infinite power.

Nevertheless, the solution (17)-(18) is:

- Perfectly legitimate as a mathematical solution of Maxwell's equations.
- A fundamental brick in building up a physically-consistent em field.

Remarks

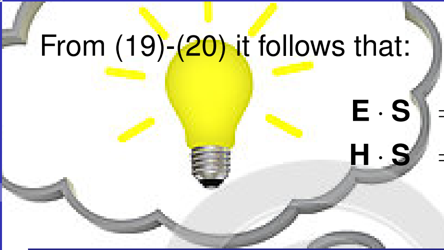
Inserting (18) in Maxwell's equation $\nabla \times \mathbf{H} = j\omega\varepsilon_c \mathbf{E}$:

$$\begin{aligned}
\mathbf{E} &= \frac{\nabla \times \mathbf{H}}{j\omega\varepsilon_c} = \frac{1}{j\omega\varepsilon_c} \nabla \times \frac{\left(\mathbf{S} \times \mathbf{E_o} e^{-\mathbf{S}\cdot\mathbf{r}}\right)}{j\omega\mu} \\
&= \frac{1}{j\omega\varepsilon_c} \frac{\mathbf{S}}{j\omega\mu} \times (\nabla \times \mathbf{E}) = \frac{1}{j\omega\varepsilon_c} \frac{\mathbf{S}}{j\omega\mu} \times -j\omega\mu\mathbf{H} \\
&= j\frac{\mathbf{S} \times \mathbf{H}}{\omega\varepsilon_c} = -\frac{\mathbf{S} \times \mathbf{E} \times \mathbf{S}}{\omega^2\varepsilon_c\mu}
\end{aligned} \tag{19}$$

In the same way:

$$\begin{aligned}
\mathbf{H} &= -\frac{\nabla \times \mathbf{E}}{j\omega\mu} = \frac{\nabla \times}{j\omega\mu} \frac{\mathbf{S} \times \mathbf{E} \times \mathbf{S}}{\omega^2\varepsilon_c\mu} = -\frac{j\omega\mu}{j\omega\mu} \frac{\mathbf{S} \times \mathbf{H} \times \mathbf{S}}{\omega^2\varepsilon_c\mu} \\
&= -\frac{\mathbf{S} \times \mathbf{H} \times \mathbf{S}}{\omega^2\varepsilon_c\mu}
\end{aligned} \tag{20}$$

Remarks



From (19)-(20) it follows that:

$$\begin{aligned} \mathbf{E} \cdot \mathbf{S} &= 0 \\ \mathbf{H} \cdot \mathbf{S} &= 0 \end{aligned} \tag{21}$$

Complex vectors

By similarities with vectors defined in a real space, one may ERRONEOUSLY think that (21) implies that **E**, **H** and **S** are mutually orthogonal.

This is actually true only for linearly polarized uniform plane waves!!

Plane waves

Inserting $\mathbf{S} = \mathbf{a} + j\mathbf{k}$ in (17)-(18) it follows that:

$$\mathbf{E} = \mathbf{E_o} e^{-(\mathbf{a}+j\mathbf{k})\cdot\mathbf{r}} = \mathbf{E_o} e^{-\mathbf{a}\cdot\mathbf{r}} e^{-j\mathbf{k}\cdot\mathbf{r}} \quad . \tag{22}$$

The following *loci* can be defined:

$\mathbf{a} \cdot \mathbf{r} = const$ - Equi-amplitude planes

- It implies $|\mathbf{E}| = const$ and $|\mathbf{H}| = const$.
- These *loci* are given by planes orthogonal to the attenuation vector and are generally called equi-amplitude or constant amplitude planes.

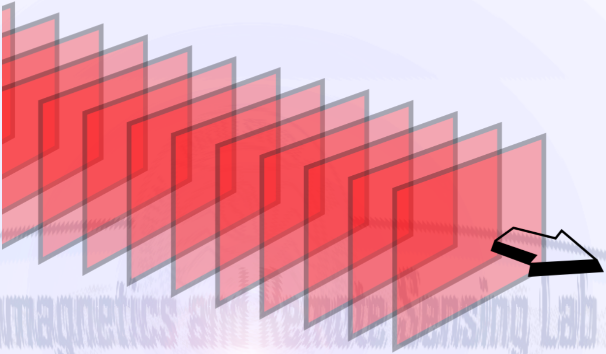
Plane waves

$\mathbf{k} \cdot \mathbf{r} = const$ - Equi-phase planes

- It implies $\angle E_i = const$ and $\angle H_i = const$.
- These *loci* are given by planes orthogonal to the phase vector and are generally called equi-phase or constant phase planes.

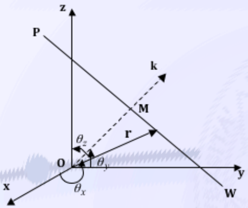
Since the equi-phase surfaces are generally called "wavefronts" and, in this case, they are planes; such solutions of Maxwell's equations are called: Plane waves.

Plane wave wavefronts



At the very root propagation is just the motion of wavefronts as the time goes!

Remarks on plane waves



The plane perpendicular to the vector **k** is seen from its side appearing as a line P-W. The dot product $\mathbf{k} \cdot \mathbf{r}$ is the projection of the radial vector **r** along the normal to the plane and will have the constant value *OM* for all points on the plane.

The equation $\mathbf{k} \cdot \mathbf{r} = \mathit{const}$ is the characteristic property of a plane perpendicular to the direction of propagation **k**.

Outline

Plane waves classification

Plane waves can be classified according to the relationship between the attenuation and phase vectors. It must be noted that:

$$k_\varepsilon = \alpha + j\beta = \sqrt{k_\varepsilon^2} = \sqrt{-\omega^2 \mu \varepsilon_c} \quad ,$$

belongs to the first quadrant of the complex plane. Therefore, $\beta > 0$ and $\alpha \geq 0$. The latter inequality is saturated when the medium is lossless.

$$\mathbf{S} \cdot \mathbf{S} = \mathbf{a}^2 - \mathbf{k}^2 + 2j\mathbf{a} \cdot \mathbf{k} = k_\varepsilon^2 = -\omega^2 \mu \left(\varepsilon - j\frac{\sigma}{\omega} \right) \quad .$$

Separating real and imaginary parts:

$$\mathbf{a}^2 - \mathbf{k}^2 = -\omega^2 \mu \varepsilon \tag{23}$$

$$2\mathbf{a} \cdot \mathbf{k} = \omega \mu \sigma \tag{24}$$

Plane waves classification

From (23) it follows that $|\mathbf{k}|^2 > |\mathbf{a}|^2$ and, therefore:

$$|\mathbf{k}| > 0 \quad , \tag{25}$$

Traveling solution

According to (25), the solutions of Maxwell's equations can never have a constant phase in the region where they are defined.

Plane waves classification

The meaning of (24) depends on the fact that the medium is lossless ($\sigma = 0$) or lossy ($\sigma \neq 0$).

- $\sigma = 0 \Longrightarrow \mathbf{a} \cdot \mathbf{k} = 0$. This is satisfied in either of the two following cases:

  1. $\mathbf{a} = 0$.
     This implies that $|\mathbf{E}| = const$ and $|\mathbf{H}| = const$ hold for the whole 3D space. Therefore, any plane is a equi-amplitude plane. Generally, a convention is adopted which makes equi-amplitude planes coincident with the equi-phase ones.
     Such a wave is called uniform plane wave.

Plane waves classification

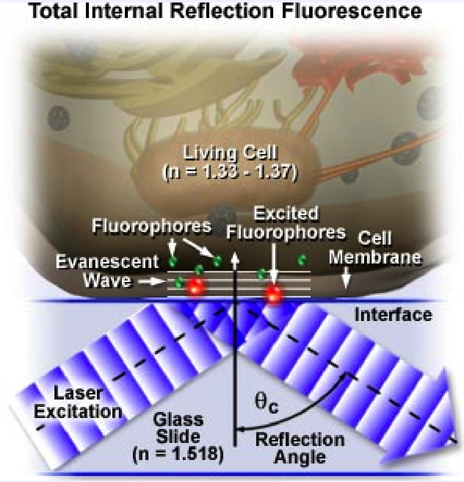
2. **a** $\neq 0$ and **a** $\perp$ **k**.
   It follows that:

$$|\mathbf{k}| > |\mathbf{a}| \quad ,$$

Evanescent wave

Therefore, equi-phase planes are orthogonal to equi-amplitude ones. This implies that this wave attenuates while propagating in a lossless medium.
Such a wave is called evanescent plane wave.

Uniform vs evanescent plane wave



http://www.olympusmicro.com/primer/techniques/fluorescence/tirf/tirfintro.html

Plane waves classification

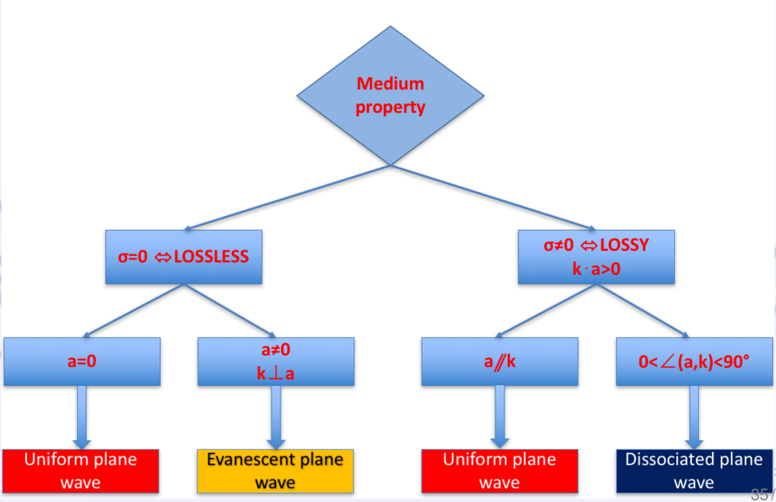
- $\sigma \neq 0 \Longrightarrow \mathbf{a} \cdot \mathbf{k} > 0$. This means that:
  - $|\mathbf{a}| \neq 0$. The wave attenuates while propagating in a lossy media.
  - The angle between **a** and **k** is smaller than $\pi/2$.

Such a wave is called dissociated plane wave.

It must be explicitly pointed out that, in the special case where **a** and **k** are parallel, the wave is still called uniform plane wave.

In a nutshell

The classification depends on medium parameters only



Remarks on orthogonality

It can be shown that **E**, **H** and **S** are mutually orthogonal if and only if the following conditions are satisfied:

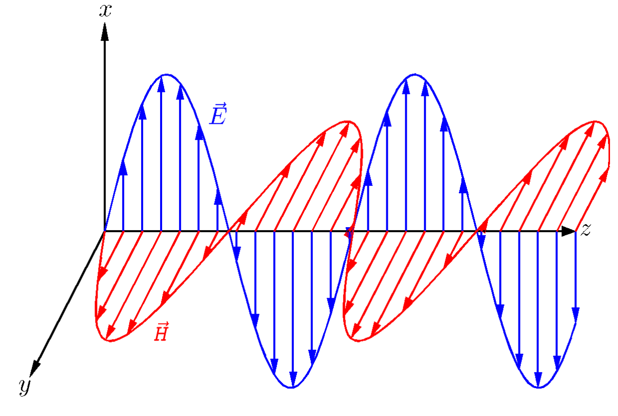
1. The wave is linearly polarized.
2. **a** and **k** are parallel (including also the special case $\mathbf{a} = 0$).

This means that, both in a lossless and in a lossy medium, the three above mentioned vectors are mutually orthogonal only for:

linearly polarized uniform plane waves

Note that orthogonality between the complex vectors **E**, **H** and **S** should not be confused with orthogonality between instantaneous time-harmonic vectors. The latter are of course mutually orthogonal!!!

Linearly polarized uniform plane waves



Outline

Traveling waves

To analyze the physical features that characterize the plane wave solution, eq.(17) is transformed into the correspondent time-domain solution:

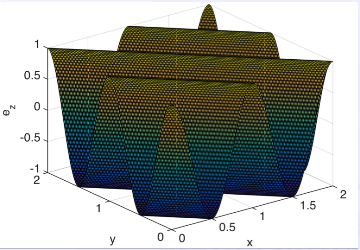
$$\mathbf{e}(\mathbf{r},t) = \Re(\mathbf{E}_o e^{-\mathbf{S}.\mathbf{r}} e^{j\omega t}) = \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \qquad (26)$$

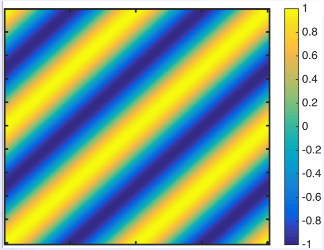
where, without any loss in generality, $\mathbf{E}_o$ is assumed to be a real constant.

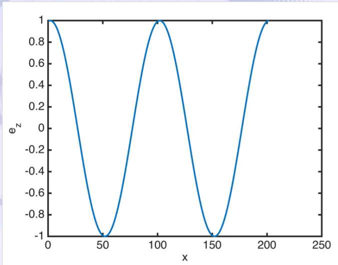
Traveling wave

$\mathbf{e}(\cdot)$ varies sinusoidally in time and (neglecting the exponential decay factor) in space.
A wave of this kind is called a Traveling Wave

Do it yourself - Plane wave lossless case

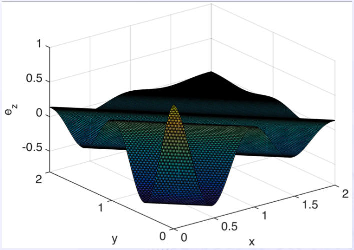


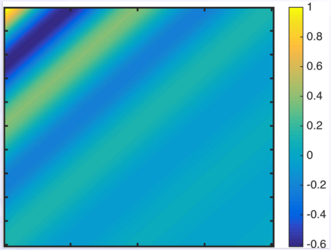


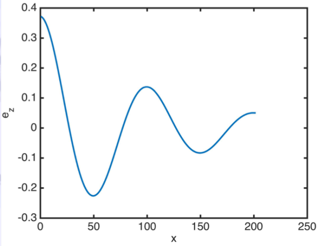


Eq. (26); $x = y = 0 : 0.01 : 2$; $\lambda = 1$; $t = 0$

Do it yourself - Plane wave lossy case

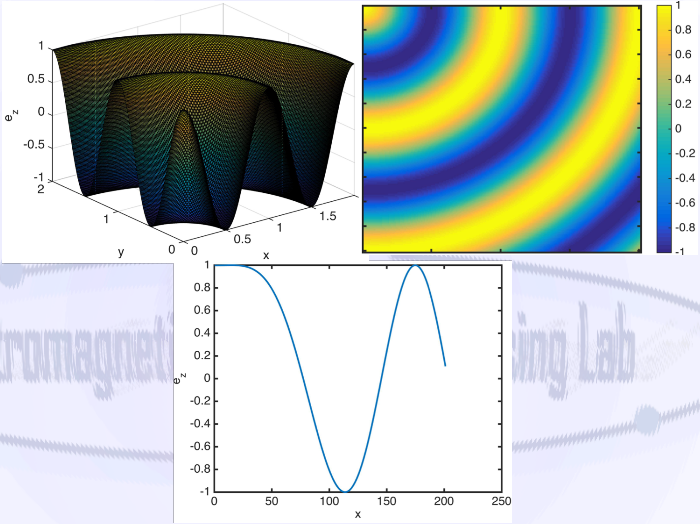






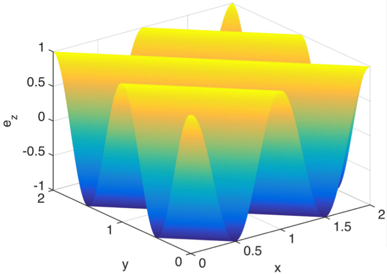
Eq. (26); $x = y = 0 : 0.01 : 2$; $\lambda = 1$; $t = 0$; $\alpha = 1$

Do it yourself - Spherical wave lossless case

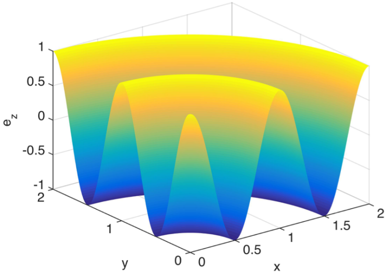


Spherical wave: e^{-jkr} with $r = \sqrt{x^2 + y^2}$

Do it yourself - 2D time evolution



Plane wave



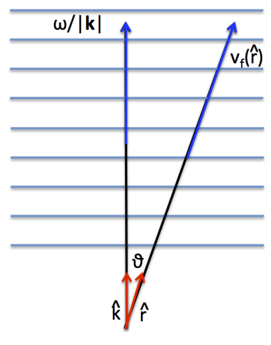
Spherical wave

$x = y = -1 : 0.01 : 1$, $\lambda = 1\text{m}$, $t = \mathit{linspace}(0, 60e - 9, 100)$

Outline

Phase velocity

Eq.(26) is a traveling wave and the factor $cos(\mathbf{k}\cdot\mathbf{r}-\omega t)$ describes an ondulatory motion.



The ondulatory motion can be analyzed by looking at points with constant phase:

$$d(\mathbf{k}\cdot\mathbf{r}-\omega t)=0$$

$$\mathbf{k}\cdot\hat{r}dr-\omega dt$$

$$v_f(\hat{r})=\frac{dr}{dt}=\frac{\omega}{\mathbf{k}\cdot\hat{r}}=\frac{\omega}{|\mathbf{k}|cos\vartheta} \quad (27)$$

Phase velocity

- The minimum value of (27) is obtained when $\hat{k} = \hat{r}$:

$$v_f = \frac{\omega}{|\mathbf{k}|}$$

- In a lossless medium two cases must be analyzed:
  - $\mathbf{a} = 0$ - Uniform plane wave. According to (23):

  $$|\mathbf{k}| = \omega\sqrt{\mu\epsilon} = \beta = \frac{2\pi}{\lambda}, \quad v_f = \frac{1}{\sqrt{\mu\epsilon}}$$

  Note that in the vacuum $v_f = c$.
  - $\mathbf{a} \neq 0$ - Evanescent wave. According to (23):

  $$v_f < \frac{1}{\sqrt{\mu\epsilon}}$$

  The evanescent wave in a lossless medium is also called “slow wave”
- Lossy medium. Since $\mathbf{a} \neq 0$, $v_f < \frac{1}{\sqrt{\mu\epsilon}}$.

Outline

Poynting vector

- The Poynting vector is defined as:

$$\mathbf{P} = \frac{\mathbf{E} \times \mathbf{H}^*}{2} \tag{28}$$

- Hence, replacing **E** and **H** with the traveling wave solution (17-18) and considering that $\mathbf{S} + \mathbf{S}^* = 2\mathbf{a}$, one obtains:

$$\mathbf{P} = \frac{\mathbf{E}_o \times (\mathbf{S}^* \times \mathbf{E}_o^*) e^{-2\mathbf{a}\cdot\mathbf{r}}}{2j\omega\mu} \tag{29}$$

- Invoking the vector identity: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$:

Poynting vector

$$\mathbf{P} = \left(\frac{|\mathbf{E}_o|^2}{2j\omega\mu} \mathbf{S}^* + \frac{\mathbf{E}_o \cdot \mathbf{S}^*}{2j\omega\mu} \mathbf{E}_o^* \right) e^{-2\mathbf{a}\cdot\mathbf{r}} \tag{30}$$

Poynting vector

Plane wave solution is physically untenable

P depends on space coordinates only through the exponential factor $e^{-2\mathbf{a}\cdot\mathbf{r}}$:

- This implies that the flux of **P** through any plane in space is infinite.

This is physically untenable.

To determine the direction of **P** it is convenient to analyze separately the cases of uniform, evanescent and dissociated waves.

Poynting vector: TEM wave in a lossless medium

- A uniform plane wave calls for $\mathbf{S} = j\mathbf{k} = j\beta\hat{k} = j\omega\sqrt{\mu\epsilon}\hat{k}$ that implies: $\mathbf{E}_o \perp \mathbf{k}$. Hence, eq.(30) becomes:

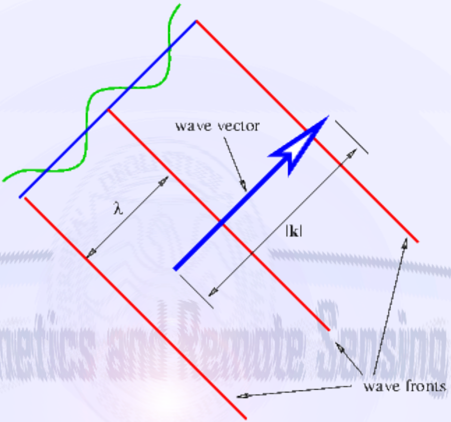
$$\begin{aligned} \mathbf{P} &= \frac{|\mathbf{E}_o|^2}{2\omega\mu}\beta\hat{k} \qquad (31) \\ &= \frac{|\mathbf{E}_o|^2}{2}\sqrt{\frac{\mu}{\epsilon}}\hat{k} \\ &= \frac{|\mathbf{E}_o|^2}{2\eta}\hat{k} \end{aligned}$$

- where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ is the intrinsic wave impedance and $\hat{k}$ is aka direction of wave normal.

Poynting vector

The complex power carried by a uniform plane wave is real, hence it consists of active power only.

Uniform plane wave: TEM wave



The wave fronts are constant phase surfaces separated by one wavelength λ. The wave vector **k** is normal to the wave fronts and its length is the wavenumber β. Note that, since $\eta \approx 377\Omega$, The electric and magnetic field components are in phase.

For Further Reading I

C.G. Someda. *Electromagnetic Waves*
CRC press - Taylor & Francis, Boca Raton, FL, 2006.

For further reading

'O tell me, when along the line
From my full heart the message flows,
What currents are induced in thine?
One click from thee will end my woes'.
Through many an Ohm the Weber flew,
And clicked the answer back to me,
'I am thy Farad, staunch and true,
Charged to a Volt with love for thee'.