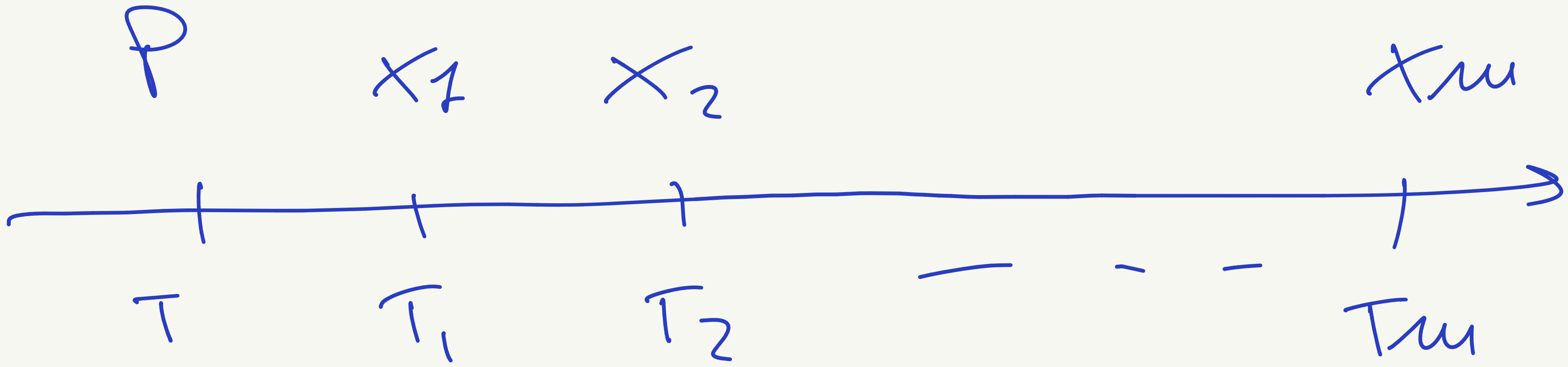
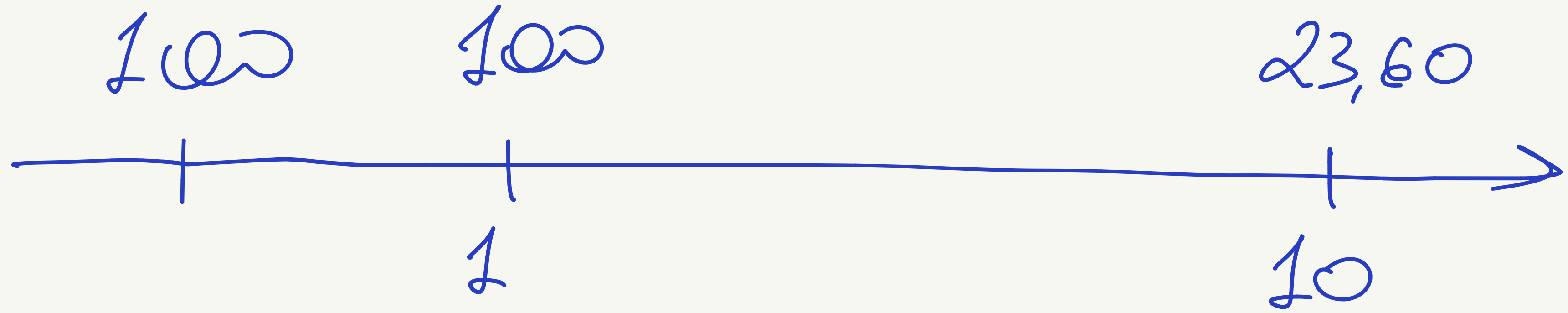


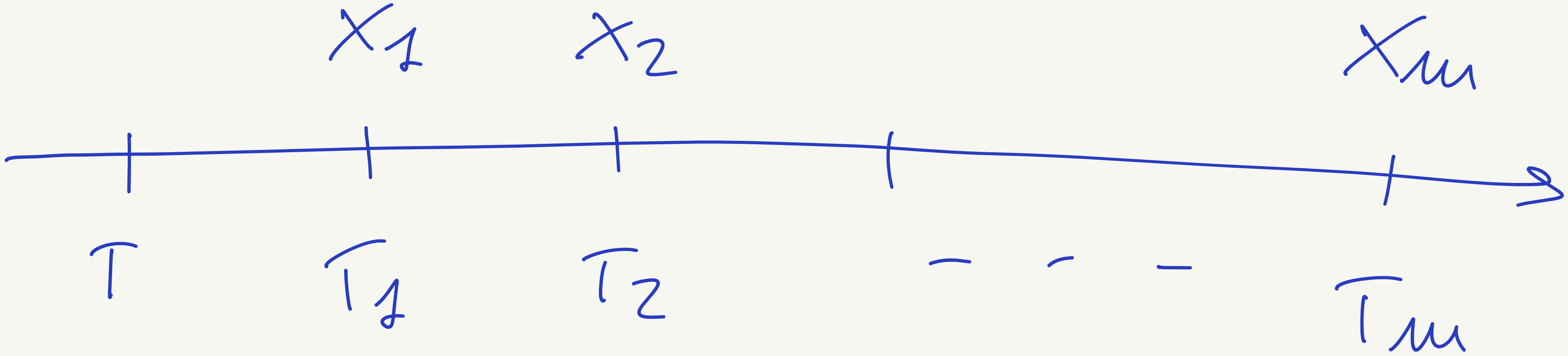
TITOLO

- scadenzario
- parte
- struttura

10%



i due titoli hanno la stessa vita a scadenza, ma il primo è più "rischioso" del secondo



$$\{\nu(T, T_k) \quad k = 1, \dots, m\}$$

$$\mathrm{dur}(T, \underline{x}) = \frac{\sum_{k=1}^{m} (T_k - T)\, x_k\, \nu(T, T_k)}{\sum_{k=1}^{m} x_k\, \nu(T, T_k)} =$$

$$= \frac{(T_1 - T)\, x_1\, \nu(T, T_1) + \dots + (T_m - T)\, x_m\, \nu(T, T_m)}{\sum_{k=1}^{m} x_k\, \nu(T, T_k)} =$$

$$\frac{(T_1-T)\,X_1\,\gamma(T,T_1)}{\sum_{k=1}^{m} X_k\,\gamma(T,T_k)} + \cdots + \frac{(T_m-T)\,X_m\,\gamma(T,T_m)}{\sum_{k=1}^{m} X_k\,\gamma(T,T_k)} =$$

$$= (T_1-T)\,\frac{X_1\,\gamma(T,T_1)}{\sum_{k=1}^{m} X_k\,\gamma(T,T_k)} + \cdots + (T_m-T)\,\frac{X_m\,\gamma(T,T_m)}{\sum_{k=1}^{m} X_k\,\gamma(T,T_k)}$$

$$\boxed{p_k = \frac{X_k\,\gamma(T,T_k)}{\sum_{k=1}^{m} X_k\,\gamma(T,T_k)}}$$

PESI

$$\boxed{\begin{aligned} dur(T,\underline{x}) &= (T_1-T)\,p_1 + \cdots + (T_m-T)\,p_m \\ &= \sum_{k=1}^{m} (T_k-T)\,p_k \end{aligned}}$$

$J = 1\%$

t	0	1	2	3	4	5
		1	1	1	1	101

SOTTO LA PARI

$J = 6\%$

t	0	1	2	3	4	5
		6	6	6	6	106

SOPRA LA PARI

dur

4,48

$i(0,1) = 4{,}9958\%$ $\quad$ $i(0,2) = 4.8646\%$

$i(0,3) = 4{,}7336\%$ $\quad$ $i(0,4) = 4{,}6028\%$

$i(0,5) = 4{,}4721\%$

T_k	x_k	$i(0,T_k)$	$v(0,T_k)$	$x_k\, v(0,T_k)$	$T_k\, x_k\, v(0,T_k)$
0					
1	6	4,9958%	0,9524	5,7144	5,7144
2	6	4,8646%	0,9094	5,4564	10,9128
3	6	4,7336%	0,8704	5,2224	15,6672
4	6	4.6028%	0,8353	5,0118	20,0472
5	106	4.4721%	0,8035	85,171	425,855

$$\frac{\sum (T_k - T)\, x_k\, v(T,T_k)}{\sum x_k\, v(T,T_k)}$$

$$dur(T, x) = \frac{\sum_{k=1}^{m} (T_k - T)\, x_k\, \gamma(T, T_k)}{\sum_{k=1}^{m} x_k\, \gamma(T, T_k)}$$

struttura piatta - flat duration

$$dur(T, x) = \frac{\sum_{k=1}^{m} (T_k - T)\, x_k\, (1+i)^{-(T_k - T)}}{\sum_{k=1}^{m} x_k\, (1+i)^{-(T_k - T)}}$$

$$\mathrm{dur}(T, x) = \frac{(T_m - T)\, x_m\, v(T, T_m)}{x_m\, v(T, T_m)} = (T_m - T)$$

La duration di un titolo a cedola nulla coincide con la vita a scadenza

Vogliamo ora studiare quanto il prezzo di un titolo è sensibile a variazioni del tasso di interesse

$$dur(0, \underline{x}) = \frac{\sum_{k=1}^{m} T_k x_k (1+i)^{-T_k}}{\sum_{k=1}^{m} x_k (1+i)^{-T_k}}$$

tasso di interesse

$$dur(0, \underline{x}) = \frac{\sum_{k=1}^{m} T_k x_k e^{-\delta T_k}}{\sum_{k=1}^{m} x_k e^{-\delta T_k}}$$

intensità istantanea

Calcoliamo la derivata prima rispetto al tasso di interesse

$$V(i) = \sum_{k=1}^{m} x_k (1+i)^{-T_k} \qquad V(\delta) = \sum_{k=1}^{m} x_k e^{-\delta T_k}$$

$$V(i) > 0 \qquad V(0) = \sum x_k \qquad V'(i) = \sum_{k=1}^{m} (-T_k) x_k (1+i)^{-T_k-1}$$

$$\lim_{i \to \infty} V(i) = 0$$

$$\boxed{= -\sum_{k=1}^{m} T_k x_k (1+i)^{-T_k-1}} \quad \text{derivata}$$

Otteniamo:

$$\frac{V'(i)}{V(i)} = \frac{-\sum_{k=1}^{m} T_k x_k (1+i)^{-T_k-1}}{\sum_{k=1}^{m} x_k (1+i)^{-T_k}} = \frac{-\sum_{k=1}^{m} T_k x_k (1+i)^{-T_k}(1+i)^{-1}}{\sum_{k=1}^{m} x_k (1+i)^{-T_k}}$$

$$= -\frac{1}{(1+i)} \frac{\sum_{k=1}^{m} T_k x_k (1+i)^{-T_k}}{\sum_{k=1}^{m} x_k (1+i)^{-T_k}} = -\frac{1}{(1+i)} D(0, \underline{x})$$

modified duration

Rispetto a δ otteniamo

$$V'(\delta) = \sum_{k=1}^{m} (-T_k)\, x_k\, e^{-\delta T_k} = -\sum_{k=1}^{m} T_k\, x_k\, e^{-\delta T_k}$$

da cui

$$\frac{V'(\delta)}{V(\delta)} = \frac{-\sum_{k=1}^{m} T_k\, x_k\, e^{-\delta T_k}}{\sum_{k=1}^{m} x_k\, e^{-\delta T_k}} = -\operatorname{dur}(0, \underline{x})$$

$$\frac{V'(i)}{V(i)} = -\frac{1}{(1+i)} D(0,x)$$

$$\frac{1}{1+i} \simeq 1$$

$$D(0,x) = -\frac{V'(i)}{V(i)} \simeq \boxed{\frac{\Delta V}{V \Delta i}}$$

rapporto incrementale

$$D(0,x) \simeq -\frac{\Delta V}{V \Delta i}$$

$$\Delta i = 1\% = 0{,}01 = \frac{1}{100}$$

$$\frac{1}{\Delta i} = 100$$

$$D(0,x) = -100 \frac{\Delta V}{V}$$

K	x_k	$i(0,T_k)$	$x_k v(0,T_k)$	$T_k x_k v(0,T_k)$
0				
0.5	12	11.25%	11,3771	5,6885
1	12	11.5%	10,7623	10,7623
1.5	12	12.05%	10,1173	15,1759
2	112	12.7%	88,18	176,36
			120,4367	207,9867

$$dur(0,x) = \frac{207{,}9867}{120{,}4367} = 1{,}7269$$

$$dur(T, \underline{x}) = \frac{\sum_{k=1}^{m} (T_k - T)\, x_k\, \gamma(T, T_k)}{\sum_{k=1}^{m} x_k\, \gamma(T, T_k)} =$$

Duration di portafoglio

n Titoli $\qquad T = \{T_1, T_2, \ldots, T_m\}$

$$\underline{x}_j = \{x_{j1}, x_{j2}, \ldots, x_{jm}\} \qquad j = 1, \ldots, n$$

$$\begin{array}{c c} & \xrightarrow{\text{m colonne date}} \\ \text{n righe Titoli} \downarrow & \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix} \end{array}$$

α_j del $j$esimo titolo (quantità del j-esimo titolo)

$\underline{x} = \{x_1, x_2, \dots, x_m\}$ (flusso del portafoglio)

$$x_k = \sum_{j=1}^{n} \alpha_j x_{jk}$$

fisso la data t_k

$$V(t, \underline{x}) = \sum_{k=1}^{m} x_k \, v(t, t_k) = \sum_{k=1}^{m} \left(\sum_{j=1}^{n} \alpha_j x_{jk} \right) v(t, t_k)$$

$$= \sum_{j=1}^{n} \alpha_j \underbrace{\boxed{\sum_{k=1}^{m} x_{jk} \, v(t, t_k)}}_{V(t, \underline{x}_j)} = \sum_{j=1}^{n} \alpha_j V(t, \underline{x}_j)$$

$$\mathrm{dur}(T, \underline{z}) = \frac{\sum_{k=1}^{m} (T_k - T)\, z_k\, \gamma(T, T_k)}{\sum_{k=1}^{m} z_k\, \gamma(T, T_k)} \rightarrow V(T, \underline{z}) \qquad = (\otimes)$$

$$\mathrm{dur}(T, \underline{x}_j) = \frac{\sum_{k=1}^{m} (T_k - T)\, x_{jk}\, \gamma(T, T_k)}{\sum_{k=1}^{m} x_{jk}\, \gamma(T, T_k)} \quad V(T, \underline{x}_j)$$

$$(\otimes) = \frac{\sum_{k=1}^{m} (T_k - T) \sum_{j=1}^{n} \alpha_j x_{jk}\, \gamma(T, T_k)}{V(T, z)} =$$

$$= \frac{\sum_{j=1}^{n} \alpha_j \sum_{k=1}^{m} (T_k - T)\, x_{jk}\, \gamma(T, T_k)}{V(T, z)} =$$

$$\sum_{j=1}^{n} \frac{\alpha_j \sum_{k=1}^{m} (T_k - T)\, x_{jk}\, \gamma(T, T_k)}{V(T, z)} =$$

$$= \sum_{j=1}^{n} \alpha_j \underbrace{\boxed{\frac{\sum_{k=1}^{m} (T_k - T)\, x_{jk}\, \gamma(T, T_k)}{V(T, x_j)}}}_{dur(T, \underline{x}_j)} \; \frac{V(T, x_j)}{V(T, z)} =$$

$$= \sum_{j=1}^{n} \boxed{\frac{\alpha_j V(T, \underline{x}_j)}{V(T, z)}}\, dur(T, \underline{x}_j)$$

durazione di PORTAFOGLIO

peso del j-esimo titolo rispetto al portafoglio

BOT 0.5 100

BTP 1.5 100 TAN 4%

$i(0, 0.5) = 3{,}4\%$ $i(0, 1) = 3{,}6\%$ $i(0, 1.5) =$
$3{,}7\%$

2 BOT + 1 BTP