Polarization

Electromagnetics
and
Remote Sensing Lab
(ERSLab)

Università degli Studi di Napoli Parthenope
Dipartimento di Ingegneria
Centro Direzionale, isola C4 - 80143 - Napoli, Italy

ferdinando.nunziata@uniparthenope.it

Outline

Outline

Polarization plane

Polarization

Given a point in space, the state of polarization of an electromagnetic field is given by the temporal evolution of the electric field vector.

- Lest us consider a uniform plane wave propagating in the $\hat{z}$ direction in the phasor domain:

$$\mathbf{E}(z) = \mathbf{E}_o e^{-j\beta z} \quad (1)$$

- with $\mathbf{E}_o$ being a complex vector.
- In the time domain:

$$\mathbf{e}(z,t) = \Re\left(\mathbf{E}_o e^{-j\beta z} e^{j\omega t}\right) = \quad (2)$$
$$\Re(\mathbf{A} + j\mathbf{B}) cos(\omega t - \beta z) + jsin(\omega t - \beta z)$$

Polarization plane

$$\mathbf{e}(z,t) = \mathbf{A}cos(\omega t - \beta z) - \mathbf{B}cos(\omega t - \beta z) \quad (3)$$

- The two vectors **A** and **B** identify a plane in the 3D space that is termed as polarization plane
- The polarization plane is always orthogonal to the direction of propagation.
- In a planar (2D) wave all the polarization planes are parallel to each other.

Polarization

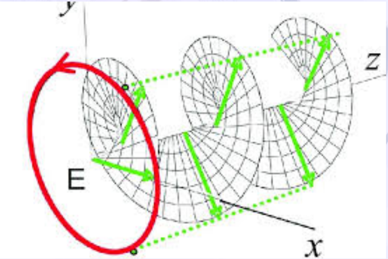
The polarization of a uniform plane wave describes the locus traced by the tip of the **E** vector (in the plane orthogonal to the direction of propagation) at a given point in space as a function of time

Polarization plane

- Since the *sin* and *cos* functions are limited, the above-mentioned locus will be a closed curve.

Polarization

It can be demonstrated that, in the most general case, the locus described by the tip of the **E** field is an ellipse. Therefore, the wave is said to be elliptically polarized.



Outline

Special cases

Polarization

The polarization ellipse can degenerate into special geometric loci according to the link between the orthogonal field components.

- Since the electric field includes only two components, eq.(1) can be rewritten as follows:

$$\mathbf{E}(z) = (E_{ox}\hat{x} + E_{oy}\hat{y})\, e^{-j\beta z} \quad (4)$$

- The amplitude of the $\hat{x}$ and $\hat{y}$ component is complex:

$$\begin{aligned} E_{ox} &= a_x \\ E_{oy} &= a_y e^{j\delta} \end{aligned} \quad (5)$$

- $\delta = \delta_y - \delta_x$ is the phase difference between the field components.

Special cases

- Hence, eq.(4) can be written as follows

$$\mathbf{E}(z) = \left(a_x \hat{x} + a_y e^{j\delta} \hat{y} \right) e^{-j\beta z} \tag{6}$$

- which, in the time domain, results in:

$$\begin{aligned} \mathbf{e}(z,t) = & \Re\left(\mathbf{E}(z) e^{j\omega t}\right) = \hat{x} e_x(z,t) + \hat{y} e_y(z,t) \\ = & \hat{x} a_x cos(\omega t - \beta z) + \hat{y} a_y cos(\omega t - \beta z + \delta) \end{aligned} \tag{7}$$

Shape of the polarization ellipse

The shape of the polarization ellipse depends on: a_x, a_y and δ.

Special cases

To fully characterize the shape of the polarization ellipse the following metrics are introduced:

- Magnitude of the field $\mathbf{e}(z,t)$:

$$|\mathbf{e}(z,t)| = \sqrt{a_x^2 cos^2(\omega t - \beta z) + a_y^2 cos^2(\omega t - \beta z + \delta)} \tag{8}$$

- Direction of the field $\mathbf{e}(z,t)$:

$$\tau(z,t) = tan^{-1}\left(\frac{e_y(z,t)}{e_x(z,t)}\right) \tag{9}$$

Special cases

In general, both the magnitude and the direction are functions of space and time. Special cases may apply.

Outline

Linear polarization

Shape of the polarization ellipse

When the polarization ellipse degenerates into a straight line a linear polarization is in place, i.e., for a fixed z, if the tip of $\mathbf{e}(z,t)$ traces a straight line segment as a function of time. This happens when $e_x(z,t)$ and $e_y(z,t)$ are in phase (i.e., $\delta = 0$) or out of phase ($\delta = \pi$).

- Under the in-phase condition, eq.(8) becomes:

$$\mathbf{e}(z,t) = (\hat{x}a_x + \hat{y}a_y)\cos(\omega t - \beta z) \tag{10}$$

- Under the out-of-phase condition, eq.(8) becomes:

$$\mathbf{e}(z,t) = (\hat{x}a_x - \hat{y}a_y)\cos(\omega t - \beta z) \tag{11}$$

Linear polarization: $\delta = \pi$

- The magnitude (8) becomes:

$$|\mathbf{e}(z,t)| = \sqrt{a_x^2 + a_y^2} cos(\omega t - \beta z) \qquad (12)$$

it depends on both z and t. This implies that, for a fixed z, the magnitude varies according to t

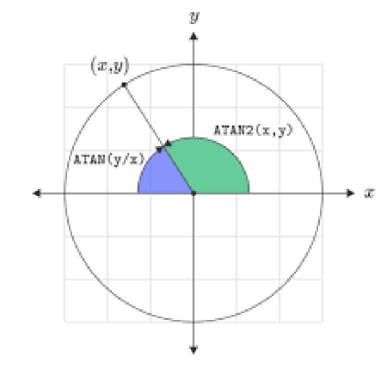
- The direction(9) becomes:

$$\tau = tan^{-1}\left(\frac{-a_y}{a_x}\right) \qquad (13)$$

it is independent of on both z and t. This implies that, for a fixed z, the direction of $\mathbf{e}(z,t)$ maintains a fixed angle τ with the $x-$axis.

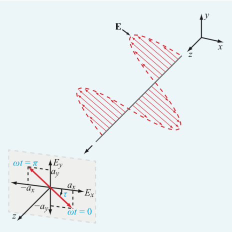
Linear polarization: $\delta = \pi$

- The angle τ follows the rule of the *atan* function



Linear polarization: $\delta = \pi$

- Since a negative $y-$component applies, the strait line is oriented at τ degree off the $x-$axis an lies in the II and IV quadrants.



Linear polarization

$\mathbf{e}(z, t)$ oscillates back and forth along a straight line that is oriented at τ degree off the $x-$direction.

Linear polarization: $\delta = 0$

- The magnitude (8) becomes:

$$|\mathbf{e}(z,t)| = \sqrt{a_x^2 + a_y^2} cos(\omega t - \beta z) \tag{14}$$

it does not change with respect to the out-of-phase case

- The direction(9) becomes:

$$\tau = tan^{-1}\left(\frac{a_y}{a_x}\right) \tag{15}$$

it is still independent of both z and t but now the angle is such that the straight line spans the I and the III quadrants

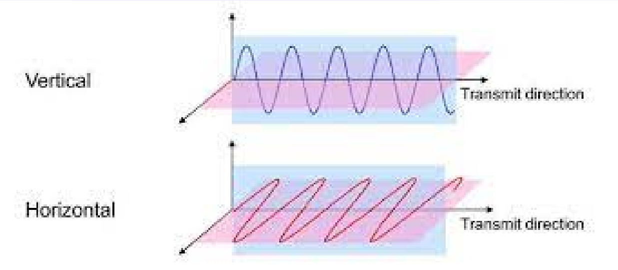
Linear polarization: Special cases

Linear polarization

There is an infinite number of linear polarizations that apply at variance of τ. However, two special cases are very often used operationally:

- $a_x = 0$: Vertical polarization. The straight line is aligned with the $y-$axis.
- $a_y = 0$: Horizontal polarization. The straight line is aligned with the $x-$axis.

Linear polarization: Special cases



Outline

Circular polarization

Shape of the polarization ellipse

When the polarization ellipse degenerates into a circle, circular polarization is in place, i.e., for a fixed z the tip of $\mathbf{e}(z,t)$ traces out a circle as a function of time. This happens when $e_x(z,t)$ and $e_y(z,t)$ are equal and their phase difference is $\delta = \pm\frac{\pi}{2}$.

- When $a_x = a_y = a$ and $\delta = \frac{\pi}{2}$, eq.(8) becomes:

$$\mathbf{E}(z) = a\left(\hat{x} + \hat{y}e^{j\frac{\pi}{2}}\right)e^{-j\beta z} \tag{16}$$

$$\mathbf{e}(z,t) = a\left(\hat{x}cos(\omega t - \beta z) - \hat{y}sin(\omega t - \beta z)\right) \tag{17}$$

Circular polarization: $\delta = \frac{\pi}{2}$

- The magnitude (8) becomes:

$$|\mathbf{e}(z,t)| = \sqrt{a^2 cos^2(\omega t - \beta z) + a^2 sin^2(\omega t - \beta z)} = a \tag{18}$$

it does not depend on both z and t. This implies that, for a fixed z, the magnitude does not vary according to t

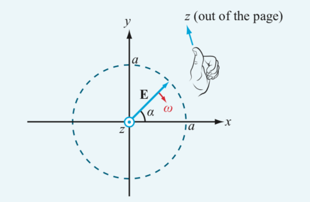
- The direction(9) becomes:

$$\tau = tan^{-1}\left(\frac{-sin(\omega t - \beta z)}{cos(\omega t - \beta z)}\right) = -(\omega t - \beta z) \tag{19}$$

it depends of on both z and t. This implies that, for a fixed z, the direction of $\mathbf{e}(z,t)$ describes an angle $\tau(t)$ with the $x-$axis that varies with time.

Circular polarization: $\delta = \frac{\pi}{2}$

- Eq.(19) implies that the inclination angle τ decreases as time increases. Hence, since the magnitude is constant (20), the tip of $\mathbf{e}(z,t)$ traces out a circle in the polarization plane.



Left hand circular (LHC) polarization

The direction the $\mathbf{e}(z,t)$ field rotates is such that when the thumb of the LEFT hand is pointing towards the direction of propagation ($\hat{z}$), the other fingers curl in the direction of the rotating field.

Circular polarization: $\delta = -\frac{\pi}{2}$

ERSLab

F. Nunziata

- The magnitude (8) becomes:

$$|\mathbf{e}(z,t)| = \sqrt{a^2 cos^2(\omega t - \beta z) + a^2 sin^2(\omega t - \beta z)} = a \tag{20}$$

it does not depend on both z and t as for the LHC case.

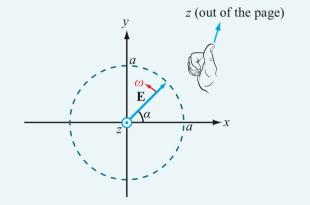
- The direction(9) becomes:

$$\tau = tan^{-1}\left(\frac{sin(\omega t - \beta z)}{cos(\omega t - \beta z)}\right) = (\omega t - \beta z) \tag{21}$$

it depends of on both z and t as in the LHC case but the angle $\tau(t)$ increases with increasing time.

Circular polarization: $\delta = -\frac{\pi}{2}$

- Eq.(21) implies that the inclination angle τ increases as time increases. Hence, since the magnitude is constant (20), the tip of $\mathbf{e}(z,t)$ traces out a circle in the polarization plane.



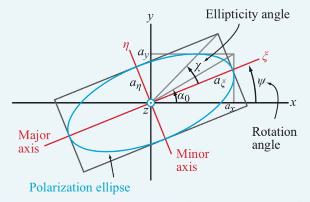
Right hand circular (RHC) polarization

The direction the $\mathbf{e}(z,t)$ field rotates is such that when the thumb of the RIGHT hand is pointing towards the direction of propagation ($\hat{z}$), the other fingers curl in the direction of the rotating field.

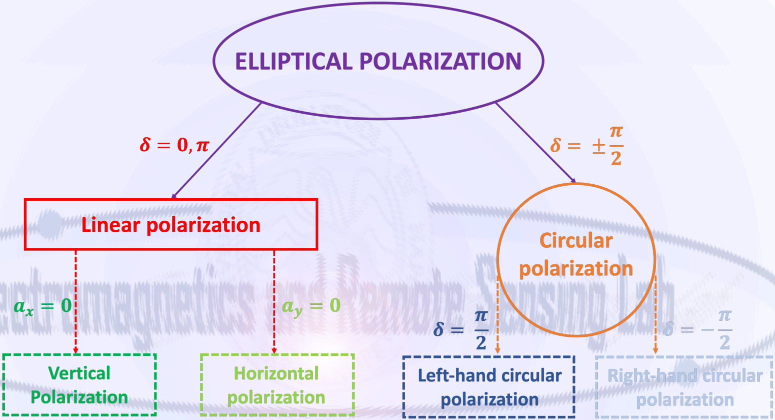
Elliptical polarization

Elliptical polarization

Plane waves that are not linearly or circularly polarized are elliptically polarized, i.e.; the tip of $\mathbf{e}(z,t)$ traces out an ellipse in the plane perpendicular to the direction of propagation. The shape of the ellipse and the field's handedness (left-hand or right-hand) are determined by the values of the ratio (a_x/a_y) and the phase difference δ.

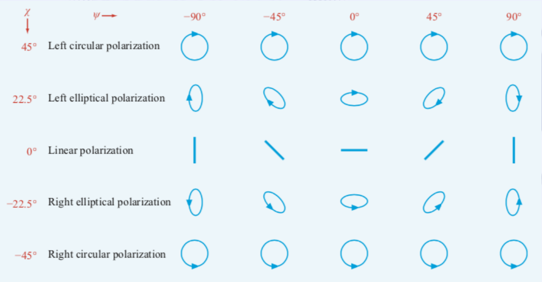


Elliptical polarization - In a nutshell



Elliptical polarization - In a nutshell

- The polarization ellipse (and its degenerating loci) can be fully described using alternative parameters:
  - Orientation angle $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$: it accounts for the inclination of the major axis wrt the reference direction (in this case $\hat{x}$).
  - Ellipticity $-\frac{\pi}{4} < \psi < \frac{\pi}{4}$: it accounts for the shape and the handedness of the ellipse.



Partial polarization

Polarized waves

A wave whose electric field oscillates in a particular way is termed as polarized.

More specifically, the wave at given point is said to be polarized if and only if the tip of the electric field vector traces out an ellipse with increasing time.

- A sufficient condition for a wave to be fully polarized is that the wave is monocromatic.
- The necessary condition for a wave to be fully polarized is that its orthogonal field components have a deterministic phase and amplitude relationship.

Partial polarization

Partially polarized waves

Actual electromagnetic waves, in general, present random fluctuations and can be represented, for instance, by statistical ensemble of realizations.

- In particular, a field can even be unpolarized, i.e. the end point of the electric vector moves in an irregular way, for increasing time.
- However, fully polarized and unpolarized waves are two extreme cases and real random fields will in general have component of both. They are called partially polarized and their polarization properties are studied using the coherence matrix, aka the polarization matrix.

Outline

Polarization matrix

- Let $\mathbf{E} = (E_x, E_y)^T$ be a uniform plane wave propagating along the z-direction of an xyz orthogonal coordinate system at a frequency ω, the polarization matrix (22) is given by:

$$\mathbf{W} = \begin{pmatrix} W_{xx} & W_{xy} \\ W_{yx} & W_{yy} \end{pmatrix} = \begin{pmatrix} \langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle \\ \langle E_y^* E_x \rangle & \langle E_y^* E_y \rangle \end{pmatrix} \quad , \quad (22)$$

- where the random variables E_i, with $i = (x, y)$, represent the components of the field fluctuating in a plane perpendicular to the propagation direction z.

Polarization matrix

- The trace of the matrix:

$$\mathrm{tr}(\mathbf{W}) = W_{xx} + W_{yy} \quad , \tag{23}$$

is a real number and stands for the total power of the wave. The off-diagonal elements W_{xy} and W_{yx} are complex conjugates of each other:

$$W_{xy} = W_{yx}^{*} \quad , \tag{24}$$

- thus, the polarization matrix is Hermitian and contains four real independent parameters.
- in addition, since the determinant is non-negative:

$$\det(\mathbf{W}) = W_{xx}W_{yy} - W_{xy}W_{yx} \geq 0 \quad , \tag{25}$$

the polarization matrix is non-negative definite.

Polarization matrix

- The off-diagonal elements represent the correlation prevailing between the mutually orthogonal components of the electric field in the plane $z = z_o$, they can be normalized as follows:

$$\mu_{xy} = |\mu_{xy}| e^{j\beta_{xy}} = \frac{W_{xy}}{\sqrt{W_{xx}}\sqrt{W_{yy}}} \quad . \tag{26}$$

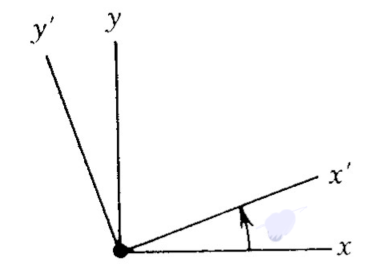
- Considering (24) and (25):

$$0 \leq |\mu_{xy}| \leq 1 \quad . \tag{27}$$

- Since μ_{xy} can be considered to be a measure of the degree of correlation between the x- and y-components of the electric field, it is called (spectral) correlation coefficient.

Polarization matrix

- The polarization matrix has been defined with respect to an arbitrary (x, y) coordinate system in a plane perpendicular to the direction of propagation of the field.
- If we define a new $(x^{'}, y^{'})$ basis, related to the x- and y-axis by a rotation about z through an angle ϕ.



- The spectral correlation (26) may change; while the determinant (25) and the trace (23) are invariant.

Outline

Unpolarized field

A random uniform plane for which:

$$\mu_{xy} = 0 \tag{28}$$

is termed as unpolarized independently of the particular choice of the x- and y-axis.

- This implies, according to (25-26) that:

$$W_{xy} = W_{yx} = 0 \quad \text{and} \quad W_{xx} = W_{yy} \quad . \tag{29}$$

- Hence, when (28) holds, considering $I_o = W_{xx} + W_{yy}$, the polarization matrix is proportional to the unit matrix:

$$W = \frac{I_o}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{30}$$

The field has the same intensity for every direction which is orthogonal to the propagation direction of the field.

Outline

Fully polarized field

A random uniform plane for which:

the x- and y-components are completely correlated:

$$|\mu_{xy}| = 1 \quad . \tag{31}$$

is fully polarized.

- Eq.(31) implies that $|W_{xy}| = \sqrt{W_{xx}}\sqrt{W_{yy}}$ which, at once, results in $\det(W) = 0$. Note that, since the determinant is an invariant, this latter condition will be verified for each x, y pair of directions orthogonal to the propagation direction.
- Accordingly, if the field components are completely correlated along any pair of mutually orthogonal directions, they are correlated for all such pairs of directions

Fully polarized field

- Eq (31), together with (24), implies that the polarization matrix can be written as:

$$W_2 = \begin{pmatrix} W_{xx} & \sqrt{W_{xx}}\sqrt{W_{yy}}e^{j\alpha} \\ \sqrt{W_{xx}}\sqrt{W_{yy}}e^{-j\alpha} & W_{yy} \end{pmatrix} , \quad (32)$$

- where α is a real factor.

Fully polarized

A field characterized by the polarization matrix (32) is called fully polarized. This terminology arises from the fact that a deterministic monochromatic wave, which is necessarily fully polarized in the conventional sense may be regarded as a deterministic analogue of a wave of this kind.

Outline

Degree of polarization

- The polarization matrix **W** can be uniquely decomposed into a sum of two matrices, one corresponding to an unpolarized field and the other to a fully polarized one:

$$\mathbf{W} = \mathbf{W}^U + \mathbf{W}^P \quad , \tag{33}$$

The degree of polarization

can be defined as the ratio between the intensity of the polarized part and the total intensity of the field

$$P = \frac{\mathrm{tr}(\mathbf{W}^P)}{\mathrm{tr}(\mathbf{W})} = \sqrt{1 - \frac{4\det(\mathbf{W})}{\mathrm{tr}^2(\mathbf{W})}} \quad . \tag{34}$$

Degree of polarization

- Since trace and determinant are invariant with respect unitary transformations P is independent on the transverse reference frame.
- In addition,

$$0 \leq P \leq 1 \quad . \tag{35}$$

- When P=0, it follows from (34) that:

$$(W_{xx} - W_{yy})^2 + 4W_{xy}W_{yx} = 0 \quad , \tag{36}$$

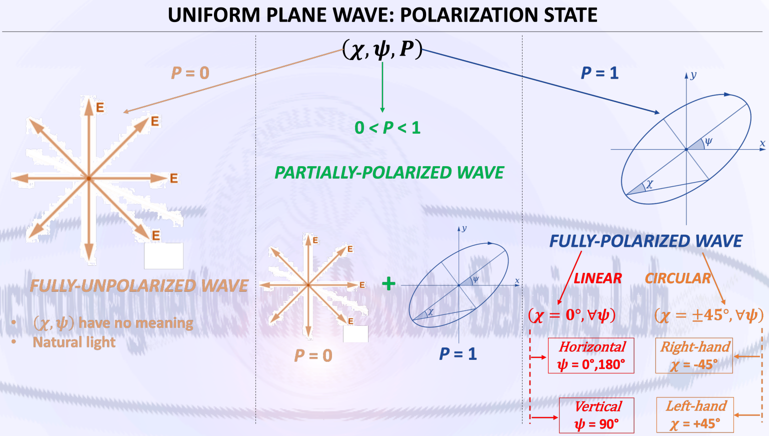
which, taking into account (24), can be satisfied only if:

$$\begin{aligned} W_{xx} &= W_{yy} \quad , \\ W_{xy} &= W_{yx}^* \quad , \end{aligned} \tag{37}$$

which are exactly the requirements for a field to be unpolarized.

- When P=1 the determinant of the polarization matrix vanishes and the field is fully polarized.

Polarization in a nutshell



For further reading

- F.T. Ulaby and D.G. Long, "Microwave Radar and Radiometric Sensing," *The University of Michigan Press*, 2014.
- R.S. Cloude, "Polarisation applications in remote sensing," *Oxford*, 2010.
- J. Ellis and A. Dogariu, "On the degree of polarization of random electromagnetic fields," *Optics Communications*, 257-265, 2005.