Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2021-2022 - Laurea “Triennale” – Secondo semestre - Secondo anno

Università degli Studi di Napoli “Parthenope”

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Color legend

New formulas, important considerations,
important formulas, important concepts

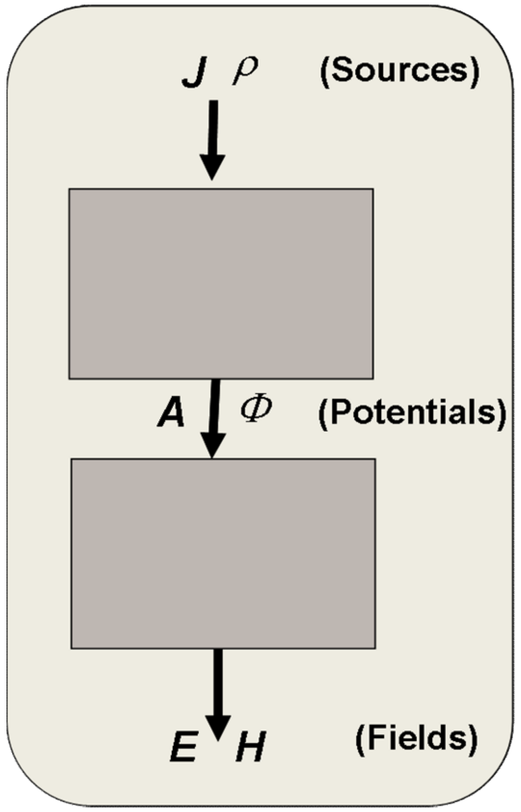
Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

Potentials



J

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$ **Vector Helmholtz equation**

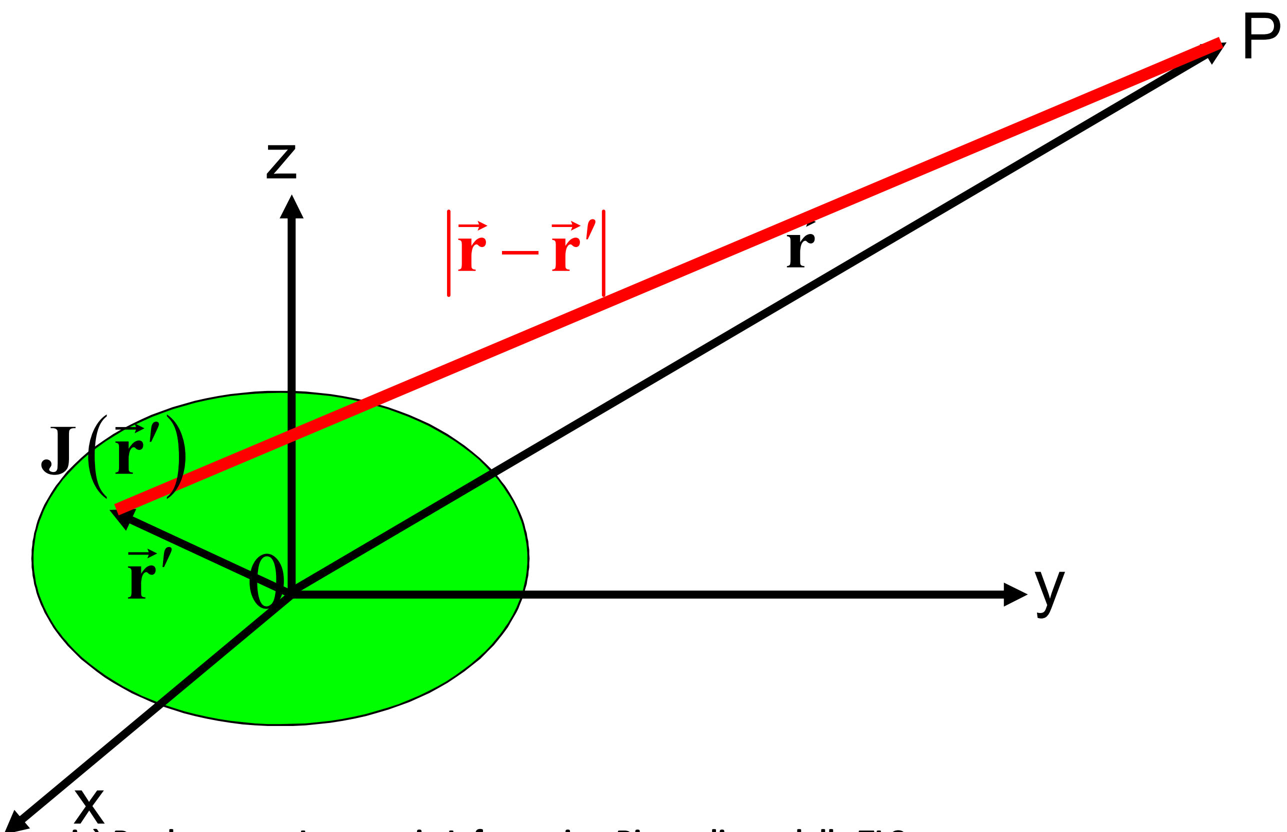
A

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -j\omega \mathbf{A} + \frac{\nabla\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}$$

E **H**

Potentials

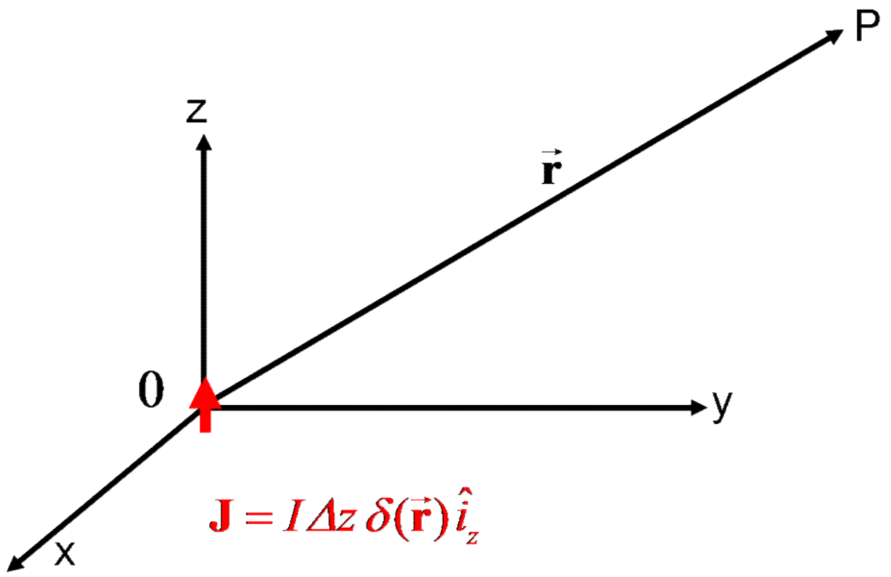
$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\,\vec{\mathbf{r}}'$$



Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

↓ *J*

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ Vector Helmholtz equation

↓ $\mathbf{A} = \frac{\mu}{4\pi} I\Delta z \frac{e^{-jkr}}{r}\hat{i}_z$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu}$$

↓ *E* *H*

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\hat{i}_z$$

The E.M. field radiated by the elementary electrical dipole

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_r(r,\vartheta)\hat{i}_r + \mathrm{E}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

Because of the problem symmetry there is no dependence on the azimuth angle φ.

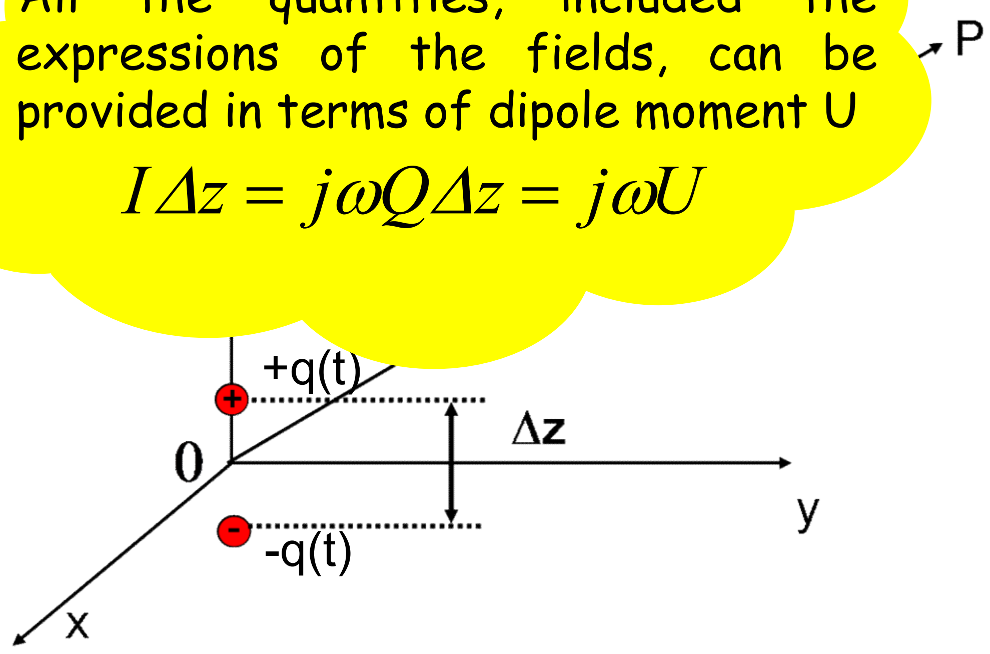
Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\hat{i}_z$$

All the quantities, included the expressions of the fields, can be provided in terms of dipole moment U

$$I\Delta z = j\omega Q\Delta z = j\omega U$$



It can be easily shown that this electric current density is the same as that of an electrical dipole such that:

1) the two charges, of opposite sign, have equal time variation;

2) in the spectral domain, the relation between *I* and the time-varying charge *Q* is:

$$j\omega Q = I$$

Elementary electrical dipole

The E.M. field radiated by the elementary electrical dipole

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$j\omega Q = I$$

Elementary electrical dipole: static case

The E.M. field radiated by the elementary electrical dipole

$$\begin{cases} E_r = \zeta \dfrac{j\omega Q \Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{j\omega Q \Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{j\omega Q \Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$j\omega Q = I$$

… for $\omega=0$ simplifies as

Elementary electrical dipole: static case

The E.M. field radiated by the elementary electrical dipole

$$
\begin{cases}
E_r = \zeta \dfrac{jQ\Delta z}{2\pi}\left(\cancel{\dfrac{\omega}{r^2}} + \dfrac{\omega}{j\beta r^3}\right)\cos\vartheta \boxed{\exp(-j\beta r)} \\[2ex]
E_\vartheta = \zeta \dfrac{jQ\Delta z}{4\pi}\left(\cancel{\dfrac{j\omega\beta}{r}} + \cancel{\dfrac{\omega}{r^2}} + \dfrac{\omega}{j\beta r^3}\right)\sin\vartheta \boxed{\exp(-j\beta r)} \\[2ex]
H_\varphi = \dfrac{jQ\Delta z}{4\pi}\left(\cancel{\dfrac{j\omega\beta}{r}} + \cancel{\dfrac{\omega}{r^2}}\right)\sin\vartheta \boxed{\exp(-j\beta r)}
\end{cases}
$$

$\exp(-j\beta r) \rightarrow 1$

$j\omega Q = I$

… for ω=0 simplifies as

$$
\begin{cases}
E_r = \dfrac{\zeta}{\sqrt{\mu\varepsilon}} \dfrac{Q\Delta z}{2\pi} \dfrac{1}{r^3}\cos\vartheta \\[2ex]
E_\vartheta = \dfrac{\zeta}{\sqrt{\mu\varepsilon}} \dfrac{Q\Delta z}{4\pi} \dfrac{1}{r^3}\sin\vartheta \\[2ex]
H_\varphi = 0
\end{cases}
$$

$\beta = \omega\sqrt{\mu\varepsilon} \rightarrow 0$

$\Rightarrow \omega\beta = \omega^2\sqrt{\mu\varepsilon} \rightarrow 0$

$\Rightarrow \omega/\beta = 1/\sqrt{\mu\varepsilon}$

Elementary electrical dipole: static case

The E.M. field radiated by the elementary electrical dipole

$$\begin{cases} E_r = \zeta \dfrac{jQ\Delta z}{2\pi}\left(\dfrac{\omega}{r^2} + \dfrac{\omega}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{jQ\Delta z}{4\pi}\left(\dfrac{j\omega\beta}{r} + \dfrac{\omega}{r^2} + \dfrac{\omega}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{jQ\Delta z}{4\pi}\left(\dfrac{j\omega\beta}{r} + \dfrac{\omega}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$\exp(-j\beta r) \rightarrow 1$

$$j\omega Q = I$$

… for ω=0 simplifies as

$$\begin{cases} E_r = \dfrac{\zeta}{\sqrt{\mu\varepsilon}}\dfrac{Q\Delta z}{2\pi}\dfrac{1}{r^3}\cos\vartheta \\ E_\vartheta = \dfrac{\zeta}{\sqrt{\mu\varepsilon}}\dfrac{Q\Delta z}{4\pi}\dfrac{1}{r^3}\sin\vartheta \\ H_\varphi = 0 \end{cases}$$

$$\zeta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\Rightarrow \frac{\zeta}{\sqrt{\mu\varepsilon}} = \frac{1}{\varepsilon}$$

Elementary electrical dipole: static case

The E.M. field radiated by the elementary electrical dipole

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\begin{cases}
E_r = \zeta \dfrac{jQ\Delta z}{2\pi}\left(\cancel{\dfrac{\omega}{r^2}} + \dfrac{\omega}{j\beta r^3}\right)\cos\vartheta \boxed{\exp(-j\beta r)} \\
E_\vartheta = \zeta \dfrac{jQ\Delta z}{4\pi}\left(\cancel{\dfrac{j\omega\beta}{r}} + \cancel{\dfrac{\omega}{r^2}} + \dfrac{\omega}{j\beta r^3}\right)\sin\vartheta \boxed{\exp(-j\beta r)} \\
H_\varphi = \dfrac{jQ\Delta z}{4\pi}\left(\cancel{\dfrac{j\omega\beta}{r}} + \cancel{\dfrac{\omega}{r^2}}\right)\sin\vartheta \boxed{\exp(-j\beta r)}
\end{cases}
$$

$\exp(-j\beta r) \rightarrow 1$

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... for ω=0 simplifies as

$$
\begin{cases}
E_r = \dfrac{\zeta}{\sqrt{\mu\varepsilon}} \dfrac{Q\Delta z}{2\pi} \dfrac{1}{r^3}\cos\vartheta = \dfrac{Q\Delta z}{2\pi} \dfrac{1}{\varepsilon r^3}\cos\vartheta \\
E_\vartheta = \dfrac{\zeta}{\sqrt{\mu\varepsilon}} \dfrac{Q\Delta z}{4\pi} \dfrac{1}{r^3}\sin\vartheta = \dfrac{Q\Delta z}{4\pi} \dfrac{1}{\varepsilon r^3}\sin\vartheta \\
H_\varphi = 0
\end{cases}
$$

$$\zeta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\Rightarrow \quad \frac{\zeta}{\sqrt{\mu\varepsilon}} = \frac{1}{\varepsilon}$$

Elementary electrical dipole: far field

The E.M. field radiated by the elementary electrical dipole

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) & = \zeta \dfrac{I\Delta z}{2\pi}\dfrac{j\beta}{r}\left(\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) & = \zeta \dfrac{I\Delta z}{4\pi}\dfrac{j\beta}{r}\left(1+\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) & = \dfrac{I\Delta z}{4\pi}\dfrac{j\beta}{r}\left(1+\dfrac{1}{j\beta r}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right) = \frac{j\beta}{r}\left(1+\frac{r}{j\beta}\frac{1}{r^2}+\frac{r}{j\beta}\frac{1}{j\beta r^3}\right) = \frac{j\beta}{r}\left(1+\frac{1}{j\beta r}+\frac{1}{[j\beta r]^2}\right)$$

Elementary electrical dipole: far field

The E.M. field radiated by the elementary electrical dipole

$$
\begin{cases}
E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) & = \zeta \dfrac{I\Delta z}{2\pi}\dfrac{j\beta}{r}\left(\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\cos\vartheta \exp(-j\beta r) \\
E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) & = \zeta \dfrac{I\Delta z}{4\pi}\dfrac{j\beta}{r}\left(1+\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\sin\vartheta \exp(-j\beta r) \\
H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) & = \dfrac{I\Delta z}{4\pi}\dfrac{j\beta}{r}\left(1+\dfrac{1}{j\beta r}\right)\sin\vartheta \exp(-j\beta r)
\end{cases}
$$

Elementary electrical dipole: far field

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$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$$

... for $\beta r >> 1$ simplifies as

$$\begin{cases} E_r = 0 \\ E_\vartheta = j\zeta \dfrac{I\Delta z}{r}\dfrac{\beta}{4\pi}\sin\vartheta \exp(-j\beta r) = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_\varphi = j\dfrac{I\Delta z}{r}\dfrac{\beta}{4\pi}\sin\vartheta \exp(-j\beta r) = \dfrac{E_\vartheta}{\zeta} \end{cases}$$

$$\frac{\beta}{4\pi} = \frac{2\pi}{\lambda}\frac{1}{4\pi} = \frac{1}{2\lambda}$$

$\beta = \omega\sqrt{\mu\varepsilon}$

$2\pi f = \omega$

$c = 1/\sqrt{\mu\varepsilon}$

$\lambda = c/f$

Elementary electrical dipole: far field

The E.M. field radiated by the elementary electrical dipole

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) = \zeta \dfrac{I\Delta z}{2\pi}\dfrac{j\beta}{r}\left(\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) = \zeta \dfrac{I\Delta z}{4\pi}\dfrac{j\beta}{r}\left(1+\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) = \dfrac{I\Delta z}{4\pi}\dfrac{j\beta}{r}\left(1+\dfrac{1}{j\beta r}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

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$$\beta r >> 1 \Rightarrow \frac{2\pi}{\lambda} r >> 1 \Rightarrow r >> \lambda$$

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$2\pi f = \omega$$

$$c = 1/\sqrt{\mu\varepsilon}$$

$$\lambda = c/f$$

Elementary electrical dipole: far field

The E.M. field radiated by the elementary electrical dipole

$$
\begin{cases}
E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\
E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\
H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r)
\end{cases}
$$

… for r>>λ simplifies as

$$
\begin{cases}
E_r = 0 \\
E_\vartheta = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\
H_\varphi = \dfrac{E_\vartheta}{\zeta}
\end{cases}
$$

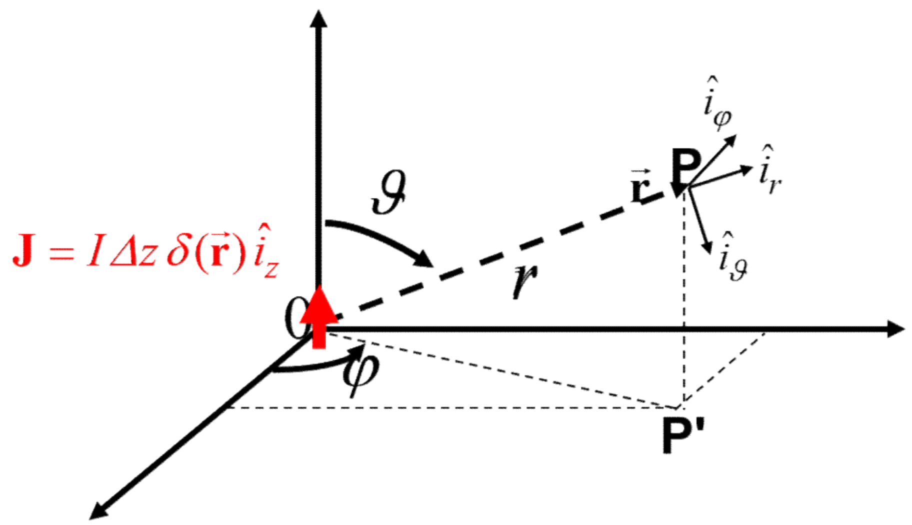
Elementary electrical dipole: far field

In the far-field case (r>>λ) the elementary electrical dipole behaves as follows

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_{\vartheta}(r,\vartheta)\hat{i}_{\vartheta}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_{\varphi}(r,\vartheta)\hat{i}_{\varphi}$$

$$\begin{cases} E_{\vartheta} = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_{\varphi} = \dfrac{E_{\vartheta}}{\zeta} \end{cases}$$



- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction
- |E| and |H| exhibit the decaying factor 1/r
- |E| and |H| are proportional through ζ

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$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\begin{cases} E_\vartheta = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{E_\vartheta}{\zeta} \end{cases}$$

$$\zeta\vec{\mathbf{H}} = \zeta \mathrm{H}_\varphi \hat{i}_\varphi = E_\vartheta \hat{i}_\varphi$$

$$\hat{i}_r \times \vec{\mathbf{E}} = \hat{i}_r \times E_\vartheta \hat{i}_\vartheta = E_\vartheta \hat{i}_\varphi$$

$$\Rightarrow \quad \zeta\mathbf{H} = \hat{i}_r \times \mathbf{E}$$

$$\hat{i}_\varphi = \hat{i}_r \times \hat{i}_\vartheta$$
$$\hat{i}_\vartheta = \hat{i}_\varphi \times \hat{i}_r$$
$$\hat{i}_r = \hat{i}_\vartheta \times \hat{i}_\varphi$$

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$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\begin{cases} E_\vartheta = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{E_\vartheta}{\zeta} \end{cases}$$

$$\vec{\mathbf{S}} = \frac{1}{2}\vec{\mathbf{E}}\times\vec{\mathbf{H}}^* = \frac{1}{2}E_\vartheta\hat{i}_\vartheta \times \left(H_\varphi \hat{i}_\varphi\right)^* = \frac{1}{2}E_\vartheta H_\varphi^* \hat{i}_r$$

$$\hat{i}_\varphi = \hat{i}_r \times \hat{i}_\vartheta$$
$$\hat{i}_\vartheta = \hat{i}_\varphi \times \hat{i}_r$$
$$\hat{i}_r = \hat{i}_\vartheta \times \hat{i}_\varphi$$

$$\frac{1}{2}E_\vartheta H_\varphi^* \hat{i}_r = \frac{1}{2\zeta}E_\vartheta E_\vartheta^* \hat{i}_r = \frac{1}{2\zeta}\left|E_\vartheta\right|^2 \hat{i}_r = \frac{1}{2\zeta}\left|\vec{\mathbf{E}}\right|^2 \hat{i}_r$$

$$\frac{1}{2}E_\vartheta H_\varphi^* \hat{i}_r = \frac{1}{2}\zeta H_\varphi H_\varphi^* \hat{i}_r = \frac{\zeta}{2}\left|H_\varphi\right|^2 \hat{i}_r = \frac{\zeta}{2}\left|\vec{\mathbf{H}}\right|^2 \hat{i}_r$$

- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction
- |E| and |H| exhibit the decaying factor 1/r
- |E| and |H| are proportional through ζ

$$\zeta\mathbf{H} = \hat{i}_r \times \mathbf{E}$$

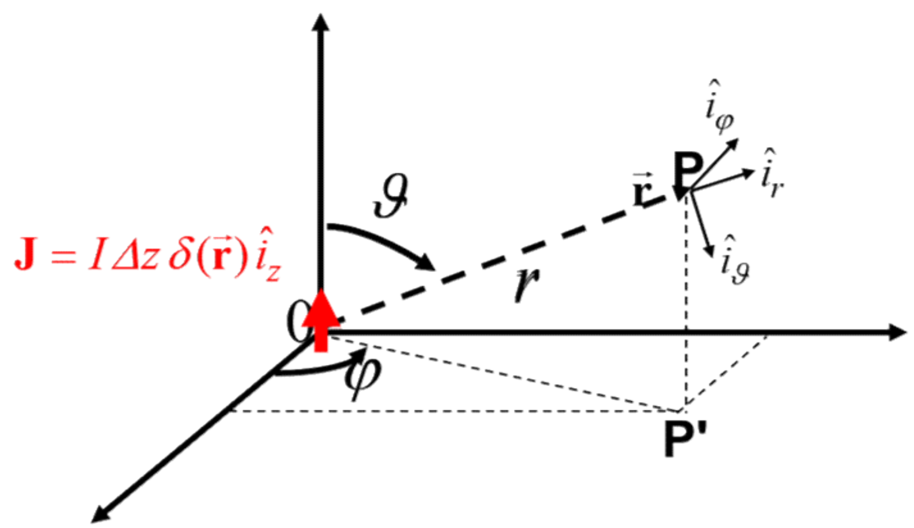
$$\vec{\mathbf{S}} = \frac{1}{2\zeta}\left|\vec{\mathbf{E}}\right|^2 \hat{i}_r = \frac{\zeta}{2}\left|\vec{\mathbf{H}}\right|^2 \hat{i}_r$$

Elementary electrical dipole: far field

In the far-field case (r>>λ) the elementary electrical dipole behaves as follows

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_{\vartheta}(r,\vartheta)\hat{i}_{\vartheta} \qquad \begin{cases} E_{\vartheta} = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_{\varphi} = \dfrac{E_{\vartheta}}{\zeta} \end{cases}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_{\varphi}(r,\vartheta)\hat{i}_{\varphi}$$



- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction
- |E| and |H| exhibit the decaying factor 1/r
- |E| and |H| are proportional through ζ

$$\zeta\mathbf{H} = \hat{i}_r \times \mathbf{E}$$

$$\vec{\mathbf{S}} = \frac{1}{2\zeta}\left|\vec{\mathbf{E}}\right|^2 \hat{i}_r = \frac{\zeta}{2}\left|\vec{\mathbf{H}}\right|^2 \hat{i}_r$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$P = \frac{1}{2}\oiint_S \left[\mathbf{E}\times\mathbf{H}^*\right]\cdot \hat{i}_r dS \;= \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta E_\vartheta H_\varphi^*$$

$$dS = r^2 \sin\vartheta d\vartheta d\varphi$$

$$\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot \hat{i}_r = \left[\left(E_\vartheta \hat{i}_\vartheta + E_r \hat{i}_r\right)\times\left(H_\varphi^* \hat{i}_\varphi\right)\right]\cdot \hat{i}_r \;= E_\vartheta H_\varphi^* \hat{i}_r \cdot \hat{i}_r \;= E_\vartheta H_\varphi^*$$

$$\hat{i}_\varphi = \hat{i}_r \times \hat{i}_\vartheta$$
$$\hat{i}_\vartheta = \hat{i}_\varphi \times \hat{i}_r$$
$$\hat{i}_r = \hat{i}_\vartheta \times \hat{i}_\varphi$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$P = \frac{1}{2}\oint\!\!\!\oint_S \left[\mathbf{E}\times\mathbf{H}^*\right]\cdot \hat{i}_r dS = \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta E_\vartheta H_\varphi^*$$

$$= \frac{1}{2}\left[\zeta\frac{|I|^2\Delta z^2}{(4\pi)^2}\right]\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta \left[\sin^2\vartheta\right]\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^*$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta\exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta\exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$

$$P=\frac{1}{2}\oiint_S\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{i}_r dS=\frac{1}{2}\int_0^{2\pi}d\varphi\int_0^{\pi}d\vartheta r^2\sin\vartheta E_\vartheta H_\varphi^* =\frac{1}{2}\left[\zeta\frac{|I|^2\Delta z^2}{(4\pi)^2}\right]r^2\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^* 2\pi\frac{4}{3}$$

$$=\frac{1}{2}\left[\zeta\frac{|I|^2\Delta z^2}{(4\pi)^2}\right]\int_0^{2\pi}d\varphi\int_0^{\pi}d\vartheta r^2\sin\vartheta\left[\sin^2\vartheta\right]\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^*$$

$$=\frac{1}{2}\left[\zeta\frac{|I|^2\Delta z^2}{(4\pi)^2}\right]r^2\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^*\int_0^{2\pi}d\varphi\int_0^{\pi}d\vartheta\sin^3\vartheta$$

$$\int_0^{2\pi}d\varphi=2\pi$$

$$\int_0^{\pi}d\vartheta\sin^3\vartheta=\frac{4}{3}$$

Elementary electrical dipole: power flux

$$
\begin{cases}
E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\[2ex]
E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\[2ex]
H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r)
\end{cases}
$$

$$
P=\frac{1}{2}\oiint_S \left[\mathbf{E}\times\mathbf{H}^*\right]\cdot \hat{i}_r dS = \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta E_\vartheta H_\varphi^* = \frac{1}{2}\left[\zeta \frac{|I|^2 \Delta z^2}{(4\pi)^2}\right] r^2 \left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^* 2\pi\frac{4}{3}
$$

$$
=\frac{1}{2}\left[\zeta \frac{|I|^2 \Delta z^2}{(4\pi)^2}\right] r^2 \frac{\beta^2}{r^2}\left[1-j\frac{1}{(\beta r)^3}\right] 2\pi\frac{4}{3} = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \left[1-j\frac{1}{(\beta r)^3}\right]|I|^2
$$

$$
\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{-j\beta}{r}+\frac{1}{r^2}\right) = \left(\frac{\beta}{r}\right)^2 + \cancel{\frac{j\beta}{r^3}} - \cancel{\frac{j\beta}{r^3}} + \cancel{\frac{1}{r^4}} - \cancel{\frac{1}{r^4}} - j\frac{1}{\beta r^5} = \left(\frac{\beta}{r}\right)^2 - j\frac{1}{\beta r^5} = \frac{\beta^2}{r^2}\left[1-j\frac{1}{(\beta r)^3}\right]
$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I \Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I \Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I \Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 |I|^2$$

$$P_2 = -\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

$$P = \frac{1}{2}\oiint_S \left[\mathbf{E} \times \mathbf{H}^*\right] \cdot \hat{i}_r dS$$

$$= \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \left[1 - j\frac{1}{(\beta r)^3}\right]|I|^2$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

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$$P_2 = -\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

- Note that in the far-field case only the first active power term exists and it does not depend on r
- Note that the real part of the power, in lossless medium, is independent of *r*, therefore if one consider two different spherical surfaces one gets the same result. Only the so-called radiative terms contribute.
- The reactive part depends on *r*. Its sign is negative showing that there is an excess of stored **electric** energy in the neighbor of the electrical dipole (see Poynting's theorem)

$$\int_0^{\pi} d\vartheta \sin^3\vartheta = \frac{4}{3}$$

$$\int_0^{\pi} d\vartheta \sin^3\vartheta = \int_0^{\pi} d\vartheta \sin\vartheta \left(1-\cos^2\vartheta\right) = \int_{-1}^{1} dx \left(1-x^2\right) = \left[x-\frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3}$$

$$x = \cos\vartheta$$

$$dx = -d\vartheta \sin\vartheta$$