Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2021-2022 - Laurea "Triennale" – Secondo semestre - Secondo anno

Università degli Studi di Napoli "Parthenope"

Stefano Perna

Color legend

New formulas, important considerations,
important formulas, important concepts

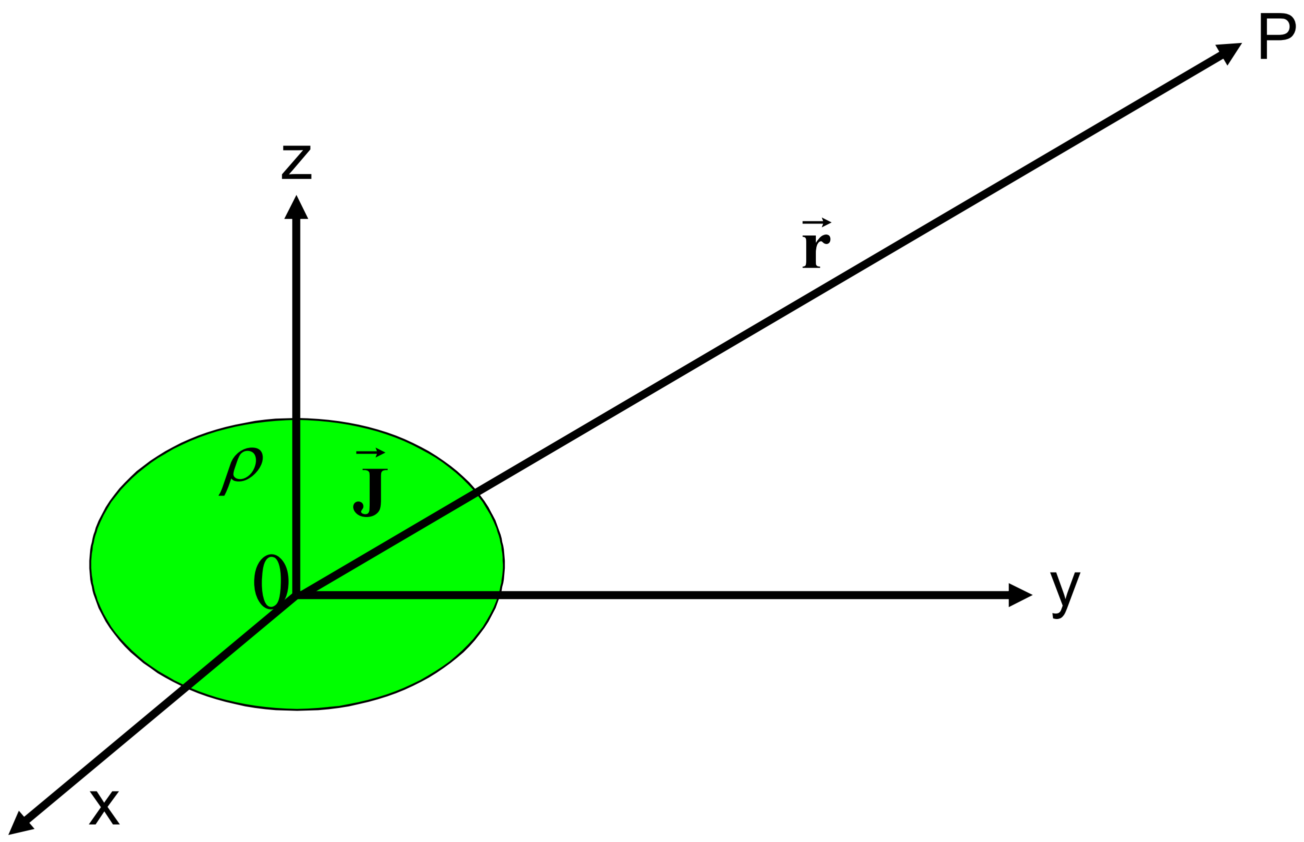
Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

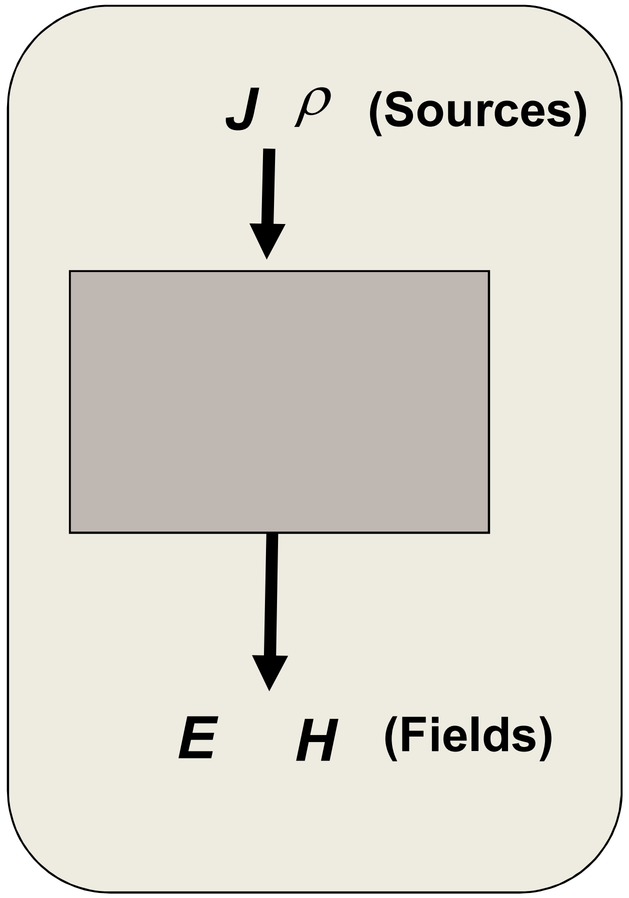
Radiation problem

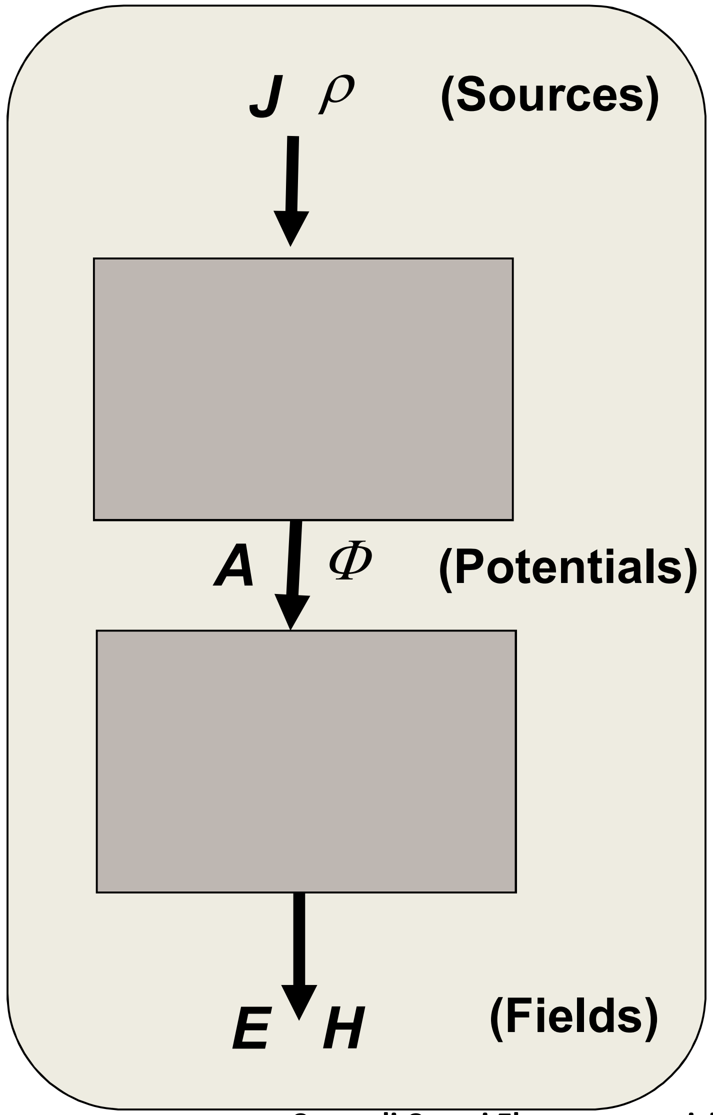


$$\vec{\mathbf{E}}\left(\vec{\mathbf{r}}\right)$$
$$\vec{\mathbf{H}}\left(\vec{\mathbf{r}}\right)$$

Radiation problem & potentials

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$



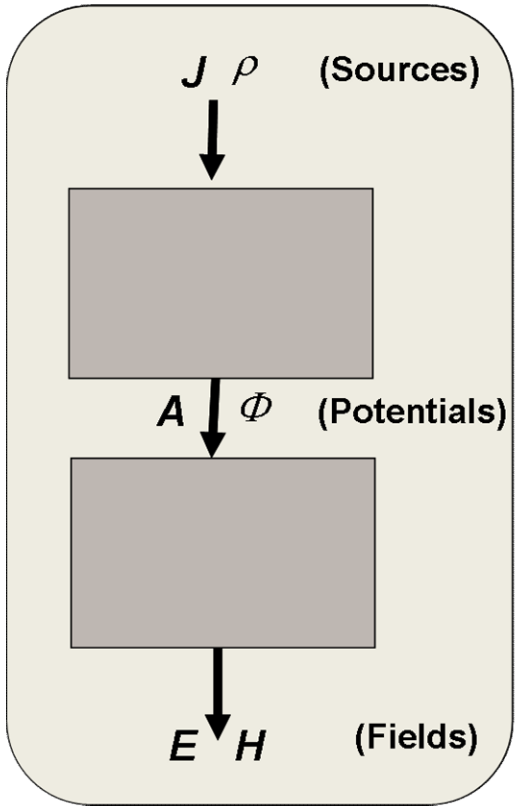


… mathematical tools that we will exploit today…

$$\nabla \times \mathbf{C} = \mathbf{0} \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$$

$$\nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$$

Potentials



J ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

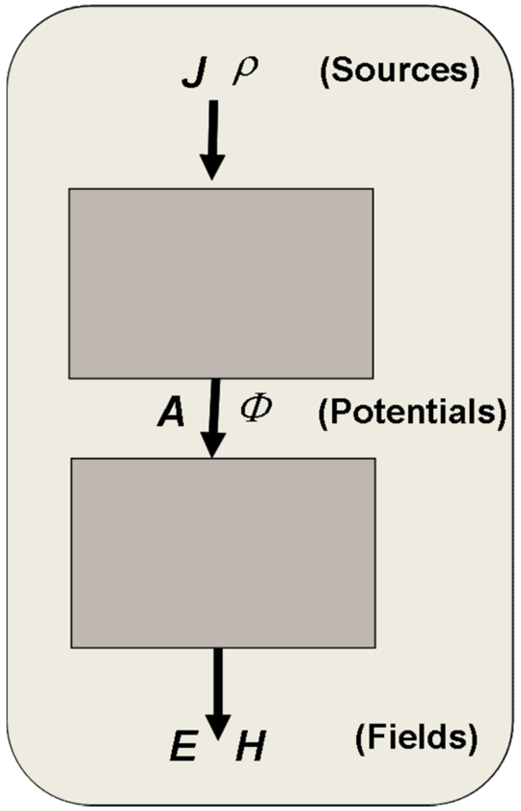
$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi\right)$$

A Φ

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

E **H**

Potentials



$\boldsymbol{J} \quad \rho$

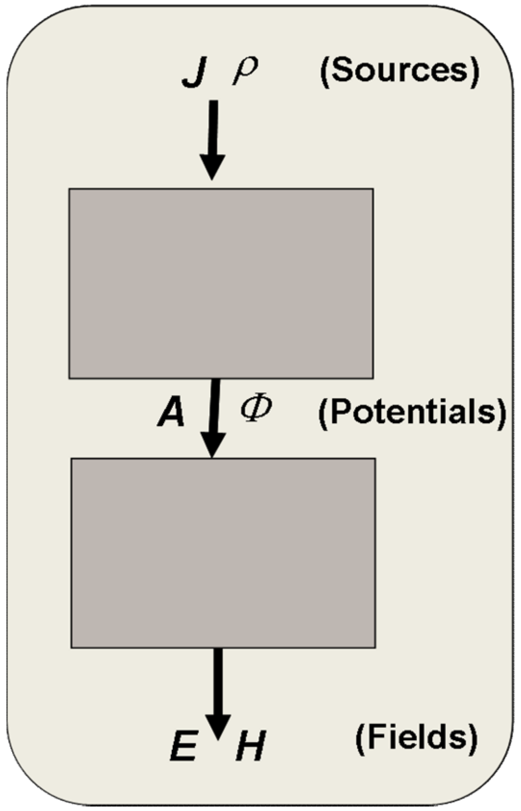
$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

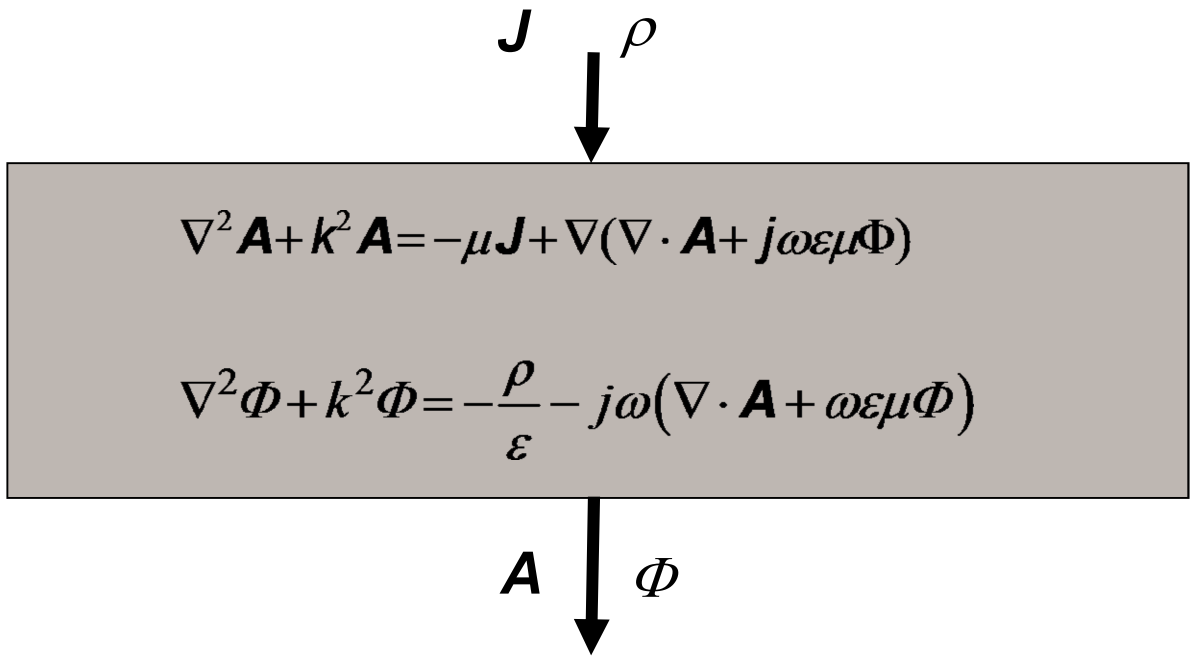
$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi\right)$$

$\boldsymbol{A} \quad \Phi$

Potentials

Amongst the infinite couples of potentials, is it possible to find a couple such that
$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$ **?**



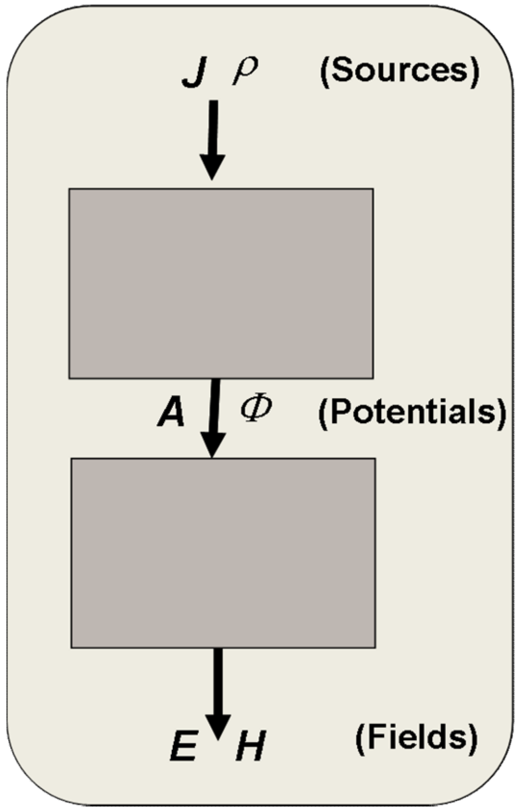


$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \boldsymbol{A} + \omega\varepsilon\mu\Phi\right)$$

Potentials

$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$$ *Lorentz gauge*



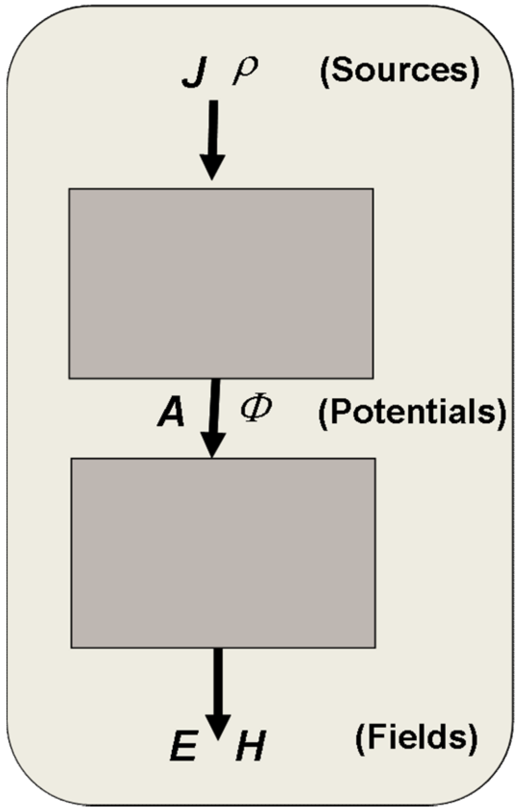
$\boldsymbol{J}$ ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$\boldsymbol{A}$ Φ

Potentials



$\mathbf{J}$ ρ

$$\begin{cases} \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} & \text{Vector Helmholtz equation} \\ \nabla^2 \Phi + k^2 \Phi = -\dfrac{\rho}{\varepsilon} & \text{Scalar Helmholtz equation} \end{cases}$$

$\mathbf{A}$ Φ

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega \mathbf{A} - \nabla \Phi$$

$\mathbf{E}$ $\mathbf{H}$

Potentials

$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$$

Lorentz gauge

Note that once ***A*** is calculated by solving the (vector) Helmholtz equation involving ***A*** and ***J***, subsequent calculation of Φ can be straightforwardly achieved by means of the *Lorentz gauge*

$$\Phi = -\frac{\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}$$

thus rendering unnecessary the solution of the (scalar) Helmholtz equation relevant to Φ

J ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon}$$ **Scalar Helmholtz equation**

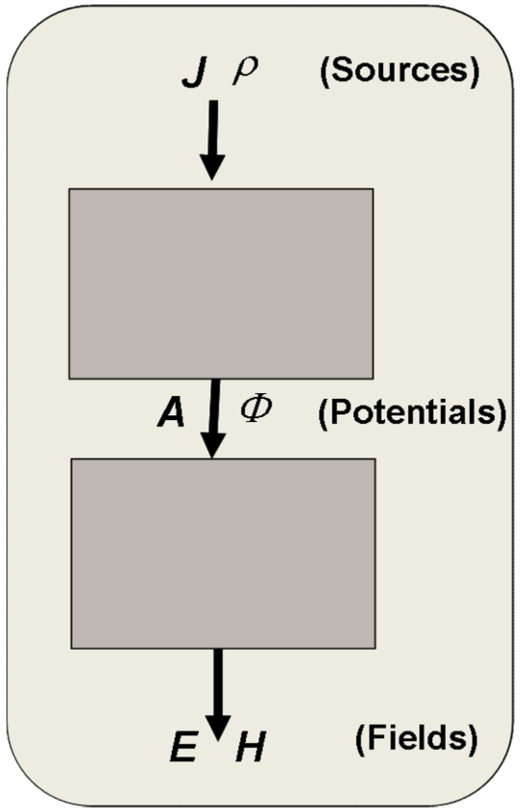
A Φ

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

$$\nabla\left(\frac{\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}\right)$$

E ***H***

Potentials



J ↓

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$ **Vector Helmholtz equation**

↓ **A**

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -j\omega \mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\varepsilon\mu}$$

↓ **E** **H**

Potentials

J

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

A

Mathematical tools

$$\mathbf{C} = \mathrm{C_x}(x,y,z)\hat{i}_x + \mathrm{C_y}(x,y,z)\hat{i}_y + \mathrm{C_z}(x,y,z)\hat{i}_z$$

$$\Phi = \Phi(x,y,z)$$

$$\nabla^2\Phi = \nabla\cdot(\nabla\Phi) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\mathbf{C} = \nabla^2\mathrm{C}_x\,\hat{i}_x + \nabla^2\mathrm{C}_y\,\hat{i}_y + \nabla^2\mathrm{C}_z\,\hat{i}_z$$

Potentials

J

↓

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$ **Vector Helmholtz equation**

↓

A

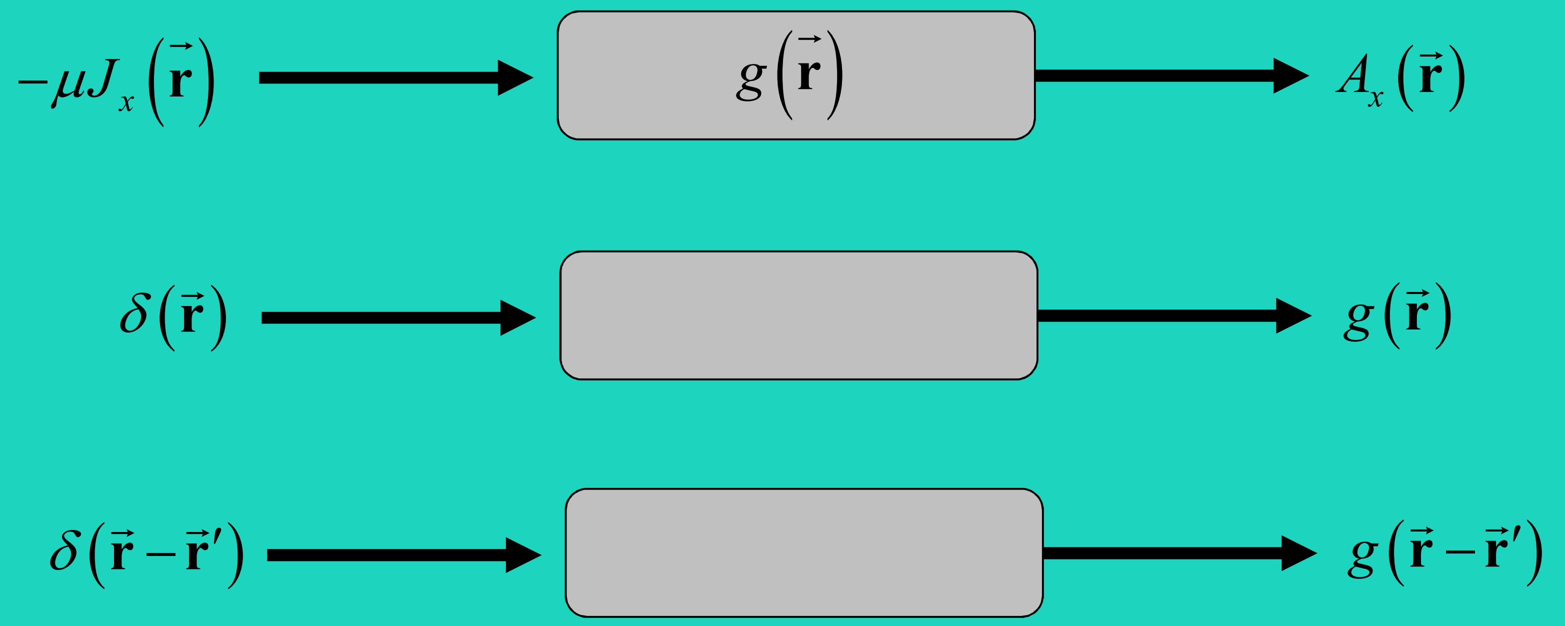
$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

Let us address the solution of the following scalar Helmholtz equation

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

Potentials

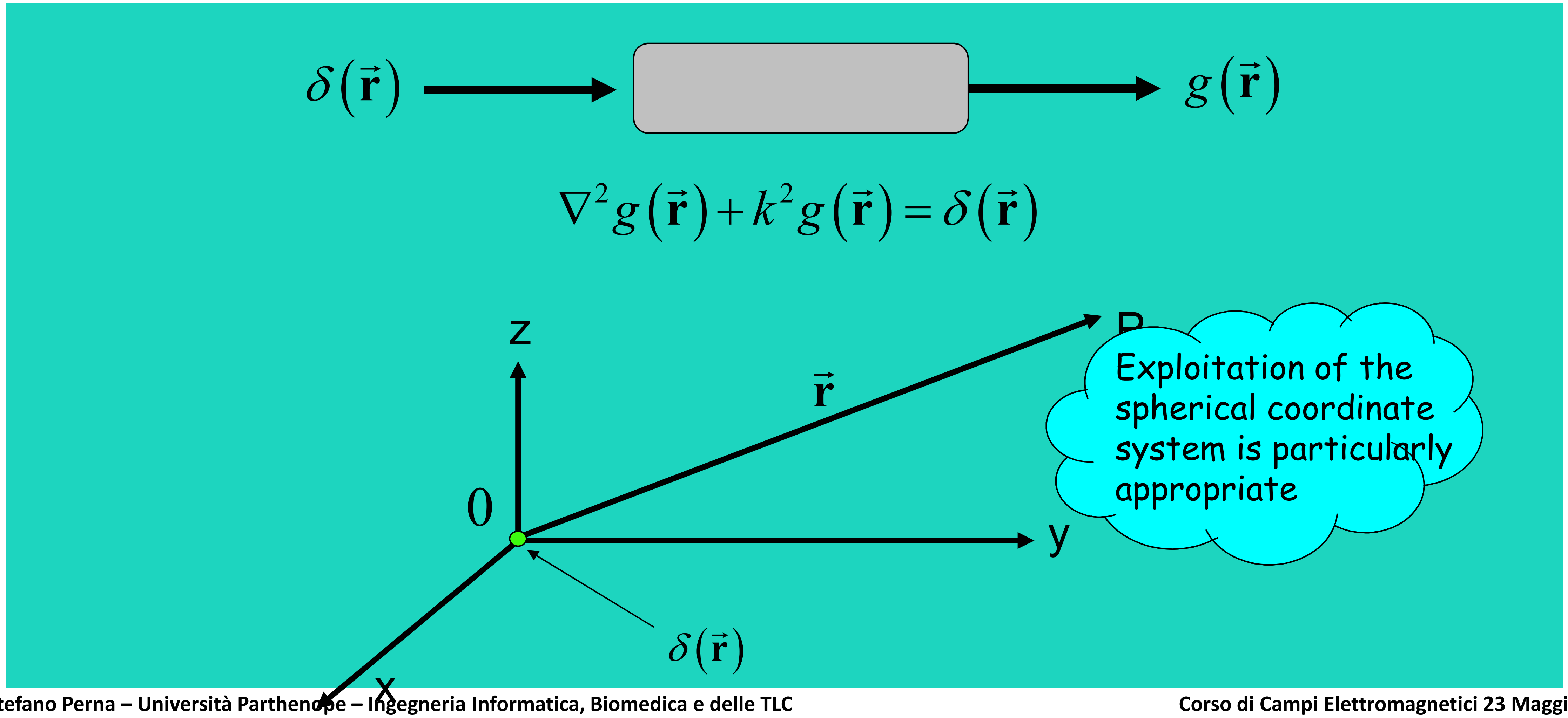
$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$



$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}') \, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \, d\vec{\mathbf{r}}'$$

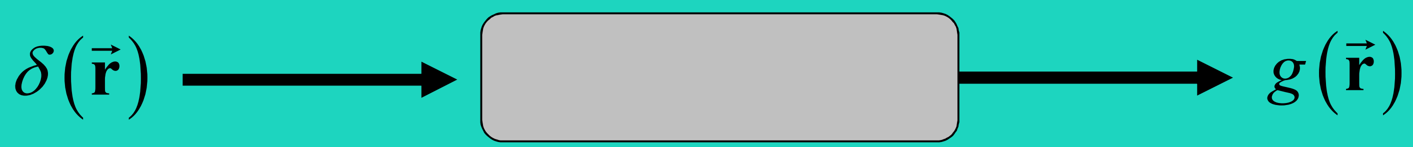
Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$



Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$



$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$

where, in principle, $g(\vec{\mathbf{r}}) = g(r, \vartheta, \varphi)$

However, due to symmetry considerations, the function $A_x(r,\vartheta,\varphi)$ turns out to be independent of ϑ and φ, that is,

$$g(\vec{\mathbf{r}}) = g(r)$$

Accordingly, in the whole three dimensional space the solution of the Helmholtz equation is:

$$g(r) = -\frac{1}{4\pi}\frac{e^{-jkr}}{r}$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$-\mu J_x(\vec{\mathbf{r}}) \longrightarrow \boxed{g(\vec{\mathbf{r}})} \longrightarrow A_x(\vec{\mathbf{r}})$$

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow g(\vec{\mathbf{r}}) = -\frac{1}{4\pi}\frac{e^{-jkr}}{r} = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}|}}{|\vec{\mathbf{r}}|}$$

$$\delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \longrightarrow \boxed{} \longrightarrow = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')$$

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\vec{\mathbf{r}}' = \frac{\mu}{4\pi}\int J_x(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}\, d\vec{\mathbf{r}}'$$

Potentials

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

c

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}') g(\vec{\mathbf{r}} - \vec{\mathbf{r}}') d\vec{\mathbf{r}}' = \frac{\mu}{4\pi} \int J_x(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

Potentials

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} \quad \Rightarrow \quad \mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int \mathbf{J}(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

$$\nabla^2 A_x + k^2 A_x = -\mu J_x \quad \Rightarrow \quad A_x(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int J_x(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

$$\nabla^2 A_y + k^2 A_y = -\mu J_y \quad \Rightarrow \quad A_y(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int J_y(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

$$\nabla^2 A_z + k^2 A_z = -\mu J_z \quad \Rightarrow \quad A_z(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int J_z(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}') g(\vec{\mathbf{r}} - \vec{\mathbf{r}}') d\vec{\mathbf{r}}' = \frac{\mu}{4\pi} \int J_x(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

Potentials

$\mathbf{J}(\mathbf{r})$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$ **Vector Helmholtz equation**

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

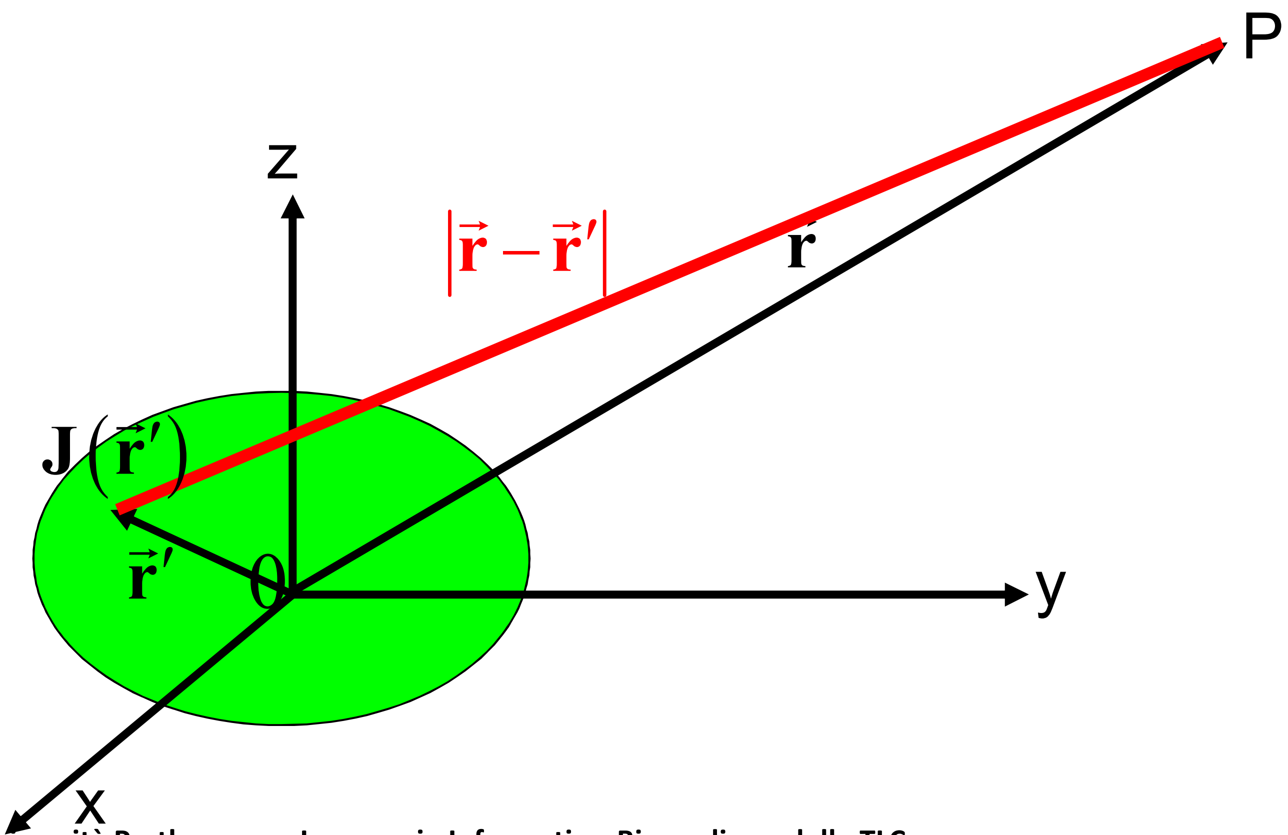
$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -j\omega \mathbf{A} + \frac{\nabla\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}$$

$\mathbf{E}(\mathbf{r})$

$\mathbf{H}(\mathbf{r})$

Potentials

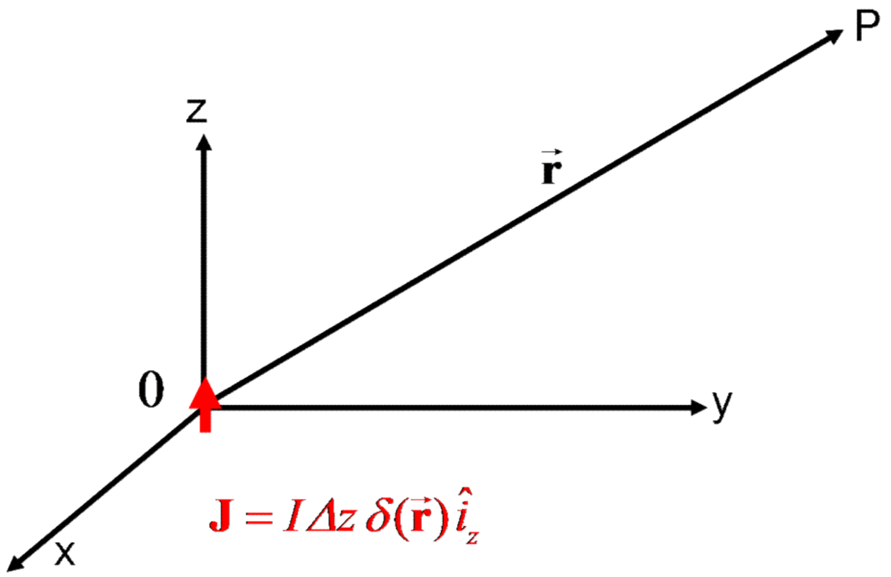
$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\vec{\mathbf{r}}'$$



Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

Elementary electrical dipole

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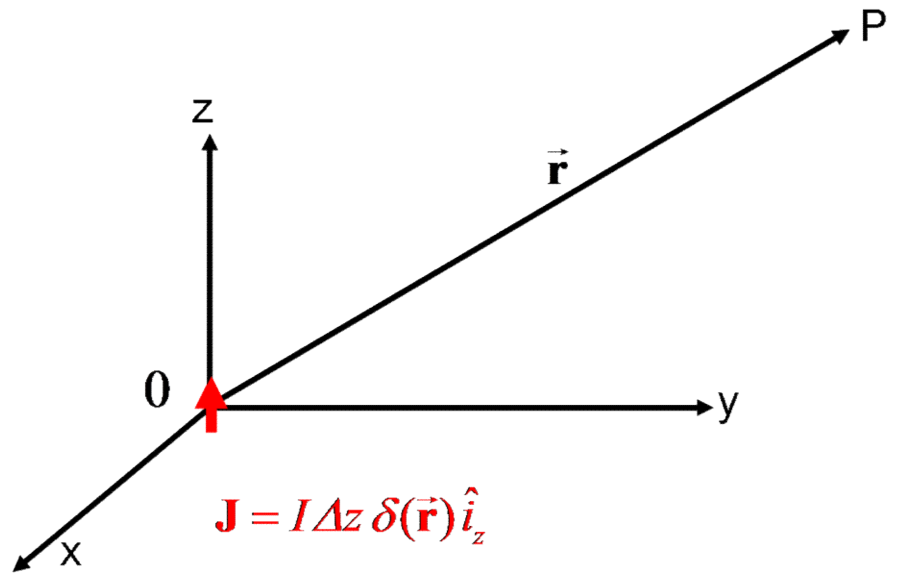

- **Why are we interested in such a radiating element?**
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

… memo …

$$\mathbf{A}(\vec{\mathbf{r}}) = \int -\mu \mathbf{J}(\vec{\mathbf{r}}') g(\vec{\mathbf{r}} - \vec{\mathbf{r}}') d\vec{\mathbf{r}}'$$

$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$

Mathematically, a δ-source radiating element is related to the radiation of any antenna!

Elementary electrical dipole

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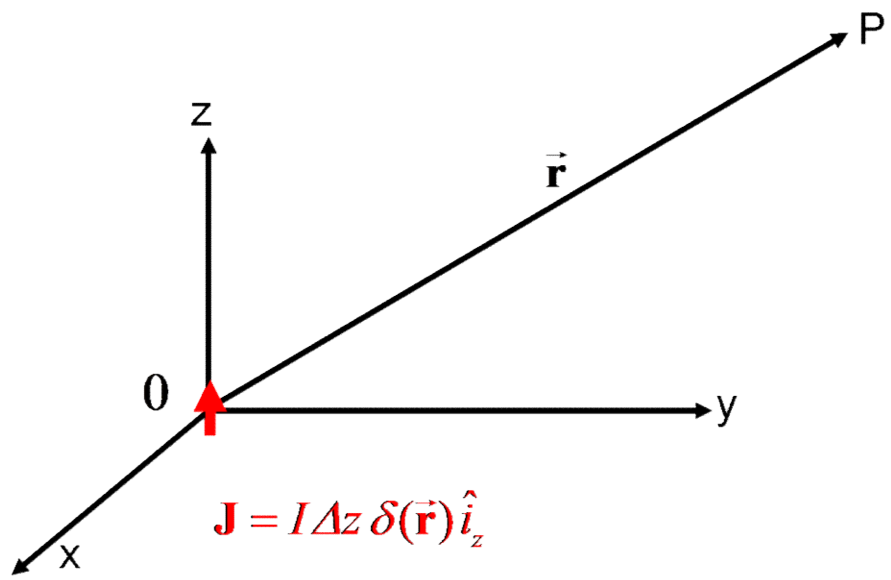

- Why are we interested in such a radiating element?
- **Why is such a radiating element referred to as elementary electrical dipole?**
- How can we physically approximate an elementary electrical dipole?

Elementary electrical dipole

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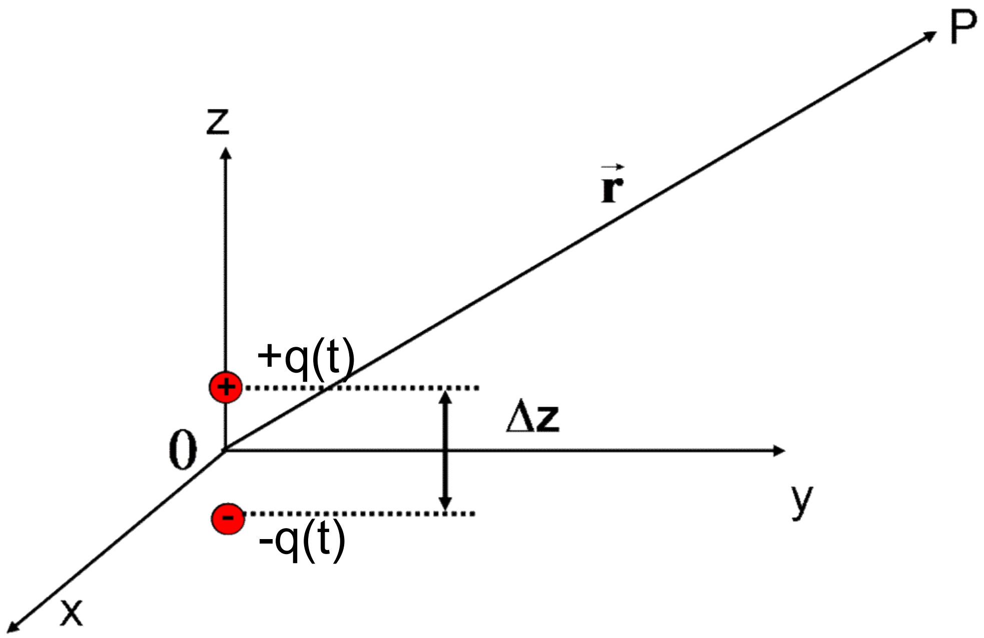

It can be easily shown that this electric current density is the same as that of an electrical dipole such that:

1) the two charges, of opposite sign, have equal time variation;

2) in the spectral domain, the relation between *I* and the time-varying charge *Q* is:

$$j\omega Q = I$$

Elementary electrical dipole

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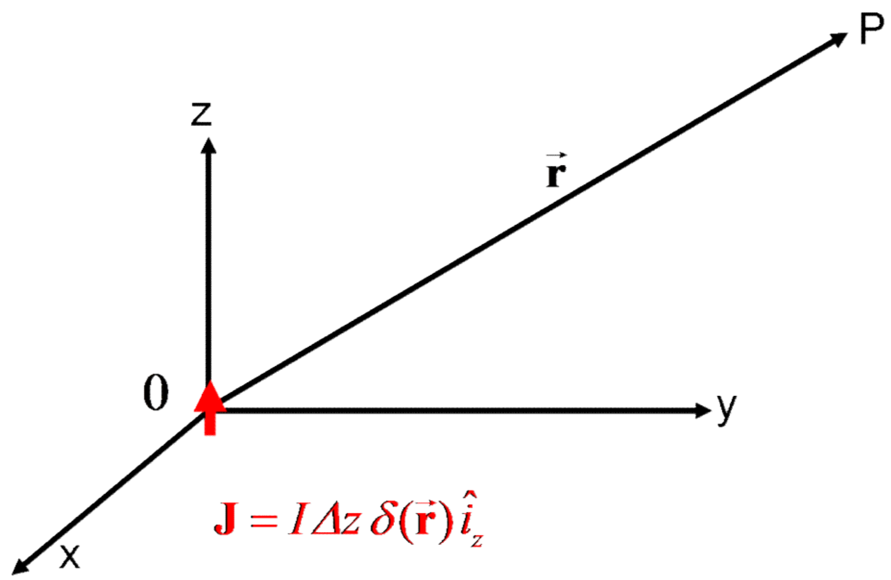


- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- **How can we physically approximate an elementary electrical dipole?**

Hertzian dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

- Of course in real life one cannot physically build up a δ-source radiating element but only an approximation.
- An approximation of the elementary dipole was used by Hertz in his experiments, in fact the elementary dipole is often called as Hertzian dipole.
- Note however that an Hertzian dipole is a dipole characterized by:

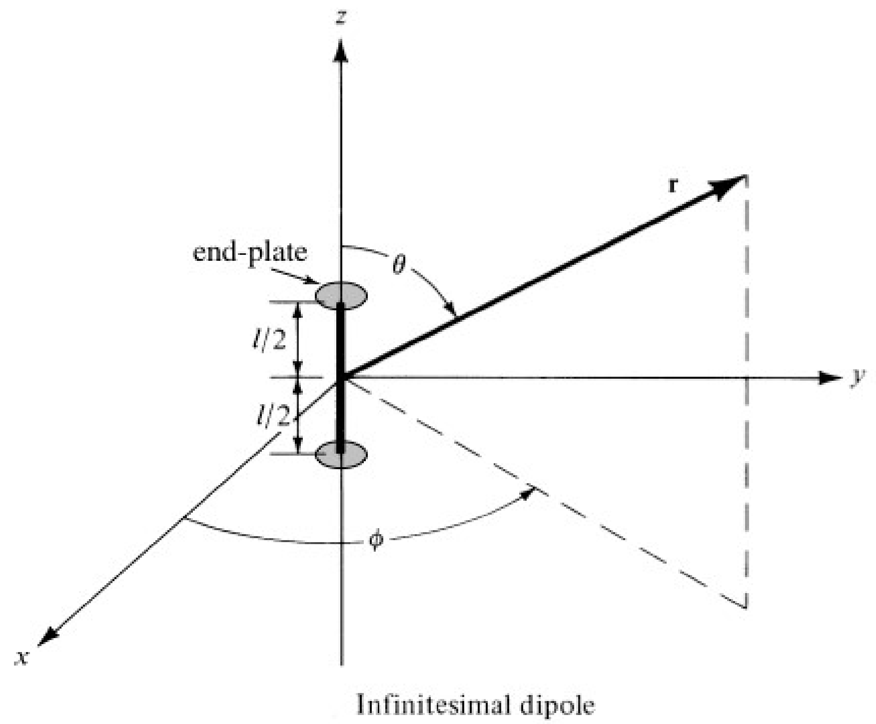
$$\mathbf{J} = I\delta(x)\delta(y)\,\mathrm{rect}\left[\frac{z}{\Delta z}\right]\hat{i}_z$$

$$when\ \ \Delta z \to 0\ \ \ then$$

$$\mathrm{rect}\left[\frac{z}{\Delta z}\right] \to \Delta z\,\delta(z)$$

Hertzian dipole

- The creation of the constant current distribution can be made by two large charge “tanks” at the two edges.
- Note that in practical case this model is meant to be suitable for electrical dipole smaller than λ/50.



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$\boldsymbol{J}$ ↓

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ **Vector Helmholtz equation**

↓ $\boldsymbol{A}$

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

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J ↓

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ **Vector Helmholtz equation**

↓ ***A***

$$\begin{cases} \nabla^2 A_x + k^2 A_x = 0 \\ \nabla^2 A_y + k^2 A_y = 0 \\ \nabla^2 A_z + k^2 A_z = -\mu J_z = -\mu I \Delta z\,\delta(x)\delta(y)\delta(z) = -\mu I \Delta z\,\delta(\vec{\mathbf{r}}) \end{cases}$$

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$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = -\mu I \Delta z\ \delta(\vec{\mathbf{r}})$$

$$-\mu I \Delta z\ \delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow g(\vec{\mathbf{r}}) = (-\mu I \Delta z) - \frac{1}{4\pi}\frac{e^{-jkr}}{r} = \frac{\mu}{4\pi} I \Delta z \frac{e^{-jkr}}{r}$$

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

↓ *J*

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ **Vector Helmholtz equation**

↓ $\mathbf{A} = \dfrac{\mu}{4\pi} I\Delta z \dfrac{e^{-jkr}}{r}\hat{i}_z$

$\boldsymbol{H} = \dfrac{1}{\mu}\nabla\times\boldsymbol{A}$ and $\boldsymbol{E} = -j\omega\boldsymbol{A} + \dfrac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu}$

↓ *E* *H*

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\hat{i}_z$$

The E.M. field radiated by the elementary electrical dipole

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_r(r,\vartheta)\hat{i}_r + \mathrm{E}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

Because of the problem symmetry there is no dependence on the azimuth angle φ.