# Metallic waveguides
# Examples

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# Outline

# Metallic waveguides



# Introduction

A metallic waveguide is a structure that guides electromagnetic waves and it is particularly effective at frequencies above 1GHz since it exhibits very low losses.


According to the shape of the cross-section:

- Rectangular
- Circular
- Elliptical
- ...

# Helmholtz equation



The homogeneous Helmholtz equation to be satisfied by the longitudinal field components $\mathcal{F} = \{\mathcal{E}_z(x,y), \mathcal{H}_z(x,y)\}$ in a cylindrical structure filled with a homogeneous medium is:

$$\nabla_t^2 \mathcal{F} = -\chi_n^2 \mathcal{F} \quad , \tag{1}$$

where $\chi$ is the eigenvalue:

$$\chi_n^2 = (\gamma_n^2 - k_\varepsilon^2) \neq 0 \quad . \tag{2}$$

## Separability

Eq.(1), with real and positive eigenvalues, can be separated only in four coordinate systems that include the rectangular cartesian and the circular polar.

In the follows the eigenvalue problem will be solved in the rectangular and circular cases.

# Rectangular waveguides

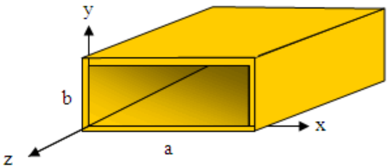


Rectangular waveguides, as opposed to circular and elliptical waveguides, are by far the dominant configuration for for compact systems like radar and inside equipment shelters.

- That is probably due to the generally greater rigidity of rectangular structures because the wall thickness can be easily made thicker than with circular.
- It is also easier to route and mount in close quarters, and attaching penetrating objects like probes and switches is much simpler.
- Most rectangular waveguide calculations can be performed on any calculator that has trig function keys.

# Coordinate system

Rectangular waveguides are by far the dominant configuration for the installed base of waveguides for compact systems like radar and inside equipment shelters.




- Let us consider a rectangular cartesian system.
- The origin is in a corner of the cross-section.
- $x$ axis is along the longer side, whose length is $a$.
- The length of the shorter side is $b$

# Outline

# Separation of variables



- The differential operator $\nabla_t^2$ can be written as:

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3}$$

- Hence, eq.(1):

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \chi_n^2 F = 0 \tag{4}$$

- Let $F(x, y) = X(x)Y(y)$, eq.(4) becomes (dividing by $XY \neq 0$):

$$\frac{X''}{X} + \frac{Y''}{Y} = -\chi_n^2 \tag{5}$$

# Separation of variables



The left-hand side of eq.(5) consists of two terms each depending on a different coordinate. Hence, the equation can be satisfied if and only if the two terms are separately constant:

$$\frac{X''}{X} = -k_x^2 \tag{6}$$

$$\frac{Y''}{Y} = -k_y^2 \tag{7}$$

with the separation equation:

$$k_x^2 + k_y^2 = \chi_n^2 \tag{8}$$

The two ODEs admit the following solutions:

$$\begin{aligned} X(x) &= a_1 e^{-jk_x x} + a_2 e^{jk_x x} \quad \text{or} \quad b_1 \cos k_x x + j b_2 \sin k_x x \\ Y(y) &= c_1 e^{-jk_y y} + c_2 e^{jk_y y} \quad \text{or} \quad d_1 \cos k_y y + j d_2 \sin k_y y \end{aligned} \tag{9}$$

# Outline

# BCs - TE



The boundary conditions on the metallic walls are different for TE and TM modes. When dealing with TE, the following BCs apply for the separated variables:

$$\frac{dX}{dx} = 0 \quad \text{for } x = 0, x = a$$
$$\frac{dY}{dy} = 0 \quad \text{for } y = 0, y = b \tag{10}$$

Hence, differentiating the trigonometric form of (9) and imposing BCs (let us focus on the X component):

$$\left.\frac{dX}{dx}\right|_{x=0} = 0 \quad \rightarrow \quad b_2 = 0 \tag{11}$$

$$\left.\frac{dX}{dx}\right|_{x=a} = 0 \quad \rightarrow \quad \sin k_x a = 0 \tag{12}$$

# TE

Applying the same rationale to the $Y$ component one obtains $d_2 = 0$ and $\sin k_y b = 0$. Hence:

$$\begin{aligned} k_x &= \frac{m\pi}{a} \\ k_y &= \frac{n\pi}{b} \end{aligned} \tag{13}$$

with $n$ and $m$ being two integers.

## Longitudinal field component

As a matter of fact, the longitudinal field component $\mathcal{H}_z$ related to the TE$_{m,n}$ mode is given by:

$$\mathcal{H}_z = H \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{14}$$

Note that $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ can be straightforwardly obtained from $\mathcal{H}_z$.

# TE - In a nutshell

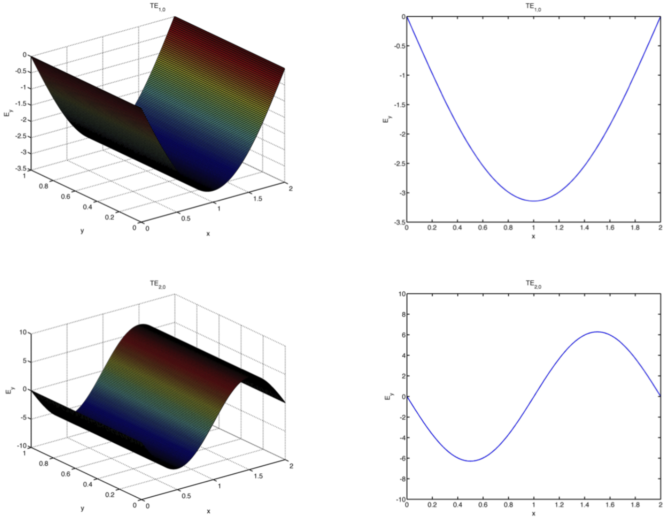


Field components of the TE modes in a rectangular waveguide

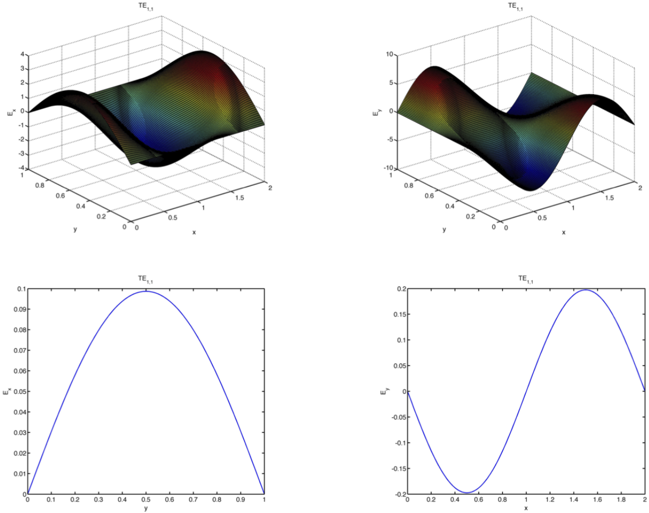
$$
\begin{aligned}
\mathcal{E}_x &= E_o \frac{n\pi}{b} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
\mathcal{E}_y &= -E_o \frac{m\pi}{a} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \\
\mathcal{E}_z &= 0 \\
\mathcal{H}_x &= -\frac{\mathcal{E}_y}{\eta_{TE}} \\
\mathcal{H}_y &= \frac{\mathcal{E}_x}{\eta_{TE}} \\
\mathcal{H}_z &= \frac{\chi^2}{j\omega\mu} E_o \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}
\end{aligned} \tag{15}
$$

# TE - In a nutshell: $TE_{1,0}$ and $TE_{2,0}$ modes

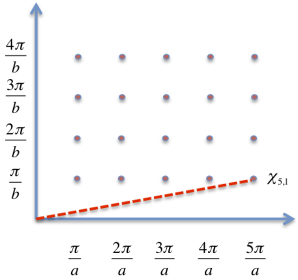




# TE - In a nutshell: $TE_{1,1}$ mode



# TE




- The eigenvalue $\chi^2_{m,n}$ is given by:

$$\chi^2_{m,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad (16)$$

- The cut-off frequency $\omega_{cm,n}$ is given by:

$$\omega_{cm,n} = \frac{\chi_{m,n}}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}} \quad (17)$$

Note that to have a non-vanishing transverse field, *m* and *n* cannot vanish simultaneously!

# BCs - TM

When dealing with TM, the following BCs apply for the separated variables:

$$\begin{aligned} X &= 0 \quad \text{for } x = 0, x = a \\ Y &= 0 \quad \text{for } y = 0, y = b \end{aligned} \tag{18}$$

## Longitudinal field component

Hence (13) holds again and the longitudinal field component $\mathcal{E}_z$ related to the TM$_{m,n}$ mode is given by:

$$\mathcal{E}_z = E \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{19}$$

# Degenerate modes

The eigenvalue of (19) is still given by (16)

- Neither $m$ nor $n$ can vanish.
- Since both TE and TM modes share the same eigenvalues, this implies that $\mathrm{TE}_{m,n}$ and $\mathrm{TM}_{m,n}$ modes are degenerate for $m \cdot n \neq 0$.

## Degenerate modes

Any linear combination of TE and TM modes is still a mode since it satisfies BCs, depends exponentially on the longitudinal coordinate and it is characterized by the eigenvalue (16).

# Fundamental mode



- Despite TM case, which needs $m \cdot n \neq 0$, the TE case allows $m = 0$ or $n = 0$. Hence, those TE modes are not degenerate with TM modes.
- As a matter of fact, the smallest value that is allowed for $\chi^2_{m,n}$ is obtained when $m = 1$ and $n = 0$.
- Hence, $TE_{1,0}$ is the Fundamental mode.
- The cut-off frequency of the $TE_{1,0}$ mode is given by:

$$\omega_c = \frac{c\pi}{a} \tag{20}$$

**Remarks**

Eq.(20) shows that at $\omega_c$ the distance between the short walls is $\lambda/2$.

Hence, at lower frequencies, only evanescent waves can satisfy BCs!

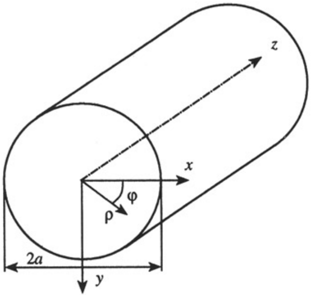
# Circular waveguides

Circular waveguides are generally used in communication systems, in specific areas of radar and as rotating joints of the mechanical point of the antennas rotation.

- Like other waveguides constructed from a single, enclosed conductor, the circular waveguide supports TE and TM modes.
- These modes have a cutoff frequency, below which electromagnetic energy is severely attenuated.
- Circular waveguide's round cross section makes it easy to machine, and it is often used to feed conical horns.

# Coordinate system

Circular waveguides are used in specific areas of radar and communications systems, such as rotating joints used at the mechanical point where the antennas rotate




- Let us consider a cylindrical coordinate system $(r, \varphi, z)$
- The origin is at the center of the cross-section.
- $z$ axis is the axis of the cylinder.
- The radius of the circular-cylinder conductor is $a$.

# Outline

# Separation of variables

Eq.(1) is to be solved in cylindrical coordinates. Hence, $\mathcal{F} = \{\mathcal{E}_z(r,\varphi), \mathcal{H}_z(r,\varphi)\}$.

- The differential operator $\nabla_t^2$ can be written as:

$$\nabla_t^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \tag{21}$$

- Hence, eq.(1) becomes:

$$\frac{1}{r}\frac{\partial \mathcal{F}}{\partial r} + \frac{\partial^2 \mathcal{F}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \mathcal{F}}{\partial \varphi^2} + \chi_m^2 \mathcal{F} = 0 \tag{22}$$

# Separation of variables



- Let $\mathcal{F}(r,\varphi) = R(r)\phi(\varphi)$, eq.(22) becomes (dividing by $\frac{R\phi}{r^2} \neq 0$):

$$\frac{r}{R}R' + \frac{r^2}{R}R'' + r^2\chi_m^2 = -\frac{1}{\phi}\phi'' \tag{23}$$

- The left-hand-side term depends on the $r$ coordinate only; while the term on the right-hand-side depends on $\varphi$ only. Hence, the equality can be satisfied if and only if both terms are equal to a constant $\nu^2$.

$$\begin{cases} \phi'' + \nu^2\phi & = 0 \\ \frac{r^2}{R}R'' + \frac{r}{R}R' + r\chi_n^2 & = \nu^2 \end{cases} \tag{24}$$

# Separation of variables



## ODEs

$$\phi'' + \nu^2 \phi = 0 \tag{25}$$

$$R'' + \frac{1}{R} R' + R\left(\chi_m^2 - \frac{\nu^2}{r^2}\right) = 0 \tag{26}$$

Eq.(25) is an harmonic equation whose general integral is given by:

$$\phi(\varphi) = a_1 e^{-j\nu\varphi} + a_2 e^{j\nu\varphi} \quad \text{or} \quad b_1 \cos\nu\varphi + j b_2 \sin\nu\varphi \tag{27}$$

# Separation of variables

## Solution of the harmonic equation

The two equivalent forms of the solution (25) admit different interpretations:

- The trigonometric form can be interpreted as the superposition of two polarization states corresponding to $b_1 = 0$ and $b_2 = 0$, respectively.
- The exponential form can be interpreted as the superposition of two traveling waves whose polarizations are clockwise ($a_1 = 0$) and counter-clockwise ($a_2 = 0$).

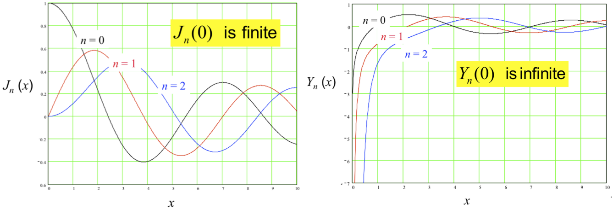
Note that to obtain a physically meaningful solution $\phi(\varphi)$ must be a single-valued function: $\phi(\varphi + 2\pi n) = \phi(\varphi)$ with $n$ being an integer. This implies that $\nu$ must be an integer. Hereinafter, $\nu$ is replaced by $n$.

# Separation of variables

Neglecting the TEM mode, i.e. $\chi_m^2 \neq 0$, eq.(26) is the Bessel function of integer order $n$, whose general integral:

$$R(r) = c_1 J_n(\chi_m r) + c_2 Y_n(\chi_m r) \tag{28}$$

with $J_n(\cdot)$ and $Y_n(\cdot)$ being the first- and second-kind Bessel function of order $n$, respectively.




Note that, since $Y_n \rightarrow -\infty$ when the argument tends to zero, $Y_n$ must be disregarded; hence, $c_2 = 0$.

# Outline

# BCs - TE

The boundary conditions on the metallic walls are different for TE and TM modes. When dealing with TE, the following BC applies for the separated variable:

$$\left.\frac{dR}{dr}\right|_{r=a} = 0 \rightarrow \chi_m c_1 J_n'(\chi_m a) = 0 \tag{29}$$

where $a$ is the radius of the cylindrical wall and $J_n'(\cdot)$ stands for the first derivative of the Bessel function with respect to its argument, whose zeros are given by:

| $k$ | $J_0'(x)$ | $J_1'(x)$ | $J_2'(x)$ | $J_3'(x)$ | $J_4'(x)$ | $J_5'(x)$ |
|---|---|---|---|---|---|---|
| 1 | 3.8317 | 1.8412 | 3.0542 | 4.2012 | 5.3175 | 6.4156 |
| 2 | 7.0156 | 5.3314 | 6.7061 | 8.0152 | 9.2824 | 10.5199 |
| 3 | 10.1735 | 8.5363 | 9.9695 | 11.3459 | 12.6819 | 13.9872 |
| 4 | 13.3237 | 11.7060 | 13.1704 | 14.5858 | 15.9641 | 17.3128 |
| 5 | 16.4706 | 14.8636 | 16.3475 | 17.7887 | 19.1960 | 20.5755 |

# BCs - TE

Denoting by $p'_{n,m}$ the $m$-th zero of $J'_n$:

- The eigenvalue of the TE$_{n,m}$ mode is:

$$\chi_{n,m} = \frac{p'_{n,m}}{a} \tag{30}$$

- The cut-off frequency is:

$$\omega_{cn,m} = \frac{p'_{n,m}}{a\sqrt{\mu\epsilon}} \tag{31}$$

- The longitudinal field component $\mathcal{H}_z$ is:

$$\mathcal{H}_z = HJ_n\left(p'_{n,m}\frac{r}{a}\right)\begin{cases}\cos n\varphi \\ \sin n\varphi\end{cases} \tag{32}$$

## Orthogonal degenerate modes

It can be noted that, according to (32), for $n \neq 0$ the longitudinal field component consists of two orthogonal modes that must be considered as a pair of degenerate modes.

# BCs - TM

When dealing with TM, the following BC applies for the separated variable:

$$R(a) = 0 \rightarrow c_1 J_n(\chi_m a) = 0 \tag{33}$$

The zeros of $J_n(\cdot)$ are given by:

| $k$ | $J_0(x)$ | $J_1(x)$ | $J_2(x)$ | $J_3(x)$ | $J_4(x)$ | $J_5(x)$ |
|---|---|---|---|---|---|---|
| 1 | 2.4048 | 3.8317 | 5.1356 | 6.3802 | 7.5883 | 8.7715 |
| 2 | 5.5201 | 7.0156 | 8.4172 | 9.7610 | 11.0647 | 12.3386 |
| 3 | 8.6537 | 10.1735 | 11.6198 | 13.0152 | 14.3725 | 15.7002 |
| 4 | 11.7915 | 13.3237 | 14.7960 | 16.2235 | 17.6160 | 18.9801 |
| 5 | 14.9309 | 16.4706 | 17.9598 | 19.4094 | 20.8269 | 22.2178 |

# BCs - TM



Denoting by $p_{n,m}$ the $m$-th zero of $J_n$:

- The eigenvalue of the TM$_{n,m}$ mode is:

$$\chi_{n,m} = \frac{p_{n,m}}{a} \tag{34}$$

- The cut-off frequency is:

$$\omega_{cn,m} = \frac{p_{n,m}}{a\sqrt{\mu\epsilon}} \tag{35}$$

- The longitudinal field component $\mathcal{E}_z$ is:

$$\mathcal{E}_z = EJ_n\left(p_{n,m}\frac{r}{a}\right)\left\{\begin{array}{c}\cos n\varphi\\ \sin n\varphi\end{array}\right. \tag{36}$$

Even in this case a pair of degenerate modes is obtained for $n \neq 0$.

# Fundamental mode





- All the zeros of $J_n(\cdot)$ and $J'_n(\cdot)$ are irrational numbers.
- The smallest zero between $p_{n,m}$ and $p'_{n,m}$ is $p'_{1,1} \approx 1.84$.

## Fundamental mode

- The fundamental mode of the circular waveguide is the $TE_{1,1}$ mode.
- Its cut-off frequency is given by:

$$\omega_{c1,1} = \frac{p'_{1,1}}{a\sqrt{\mu\epsilon}} \tag{37}$$

# For Further Reading I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