Waveguides

Electromagnetics
and
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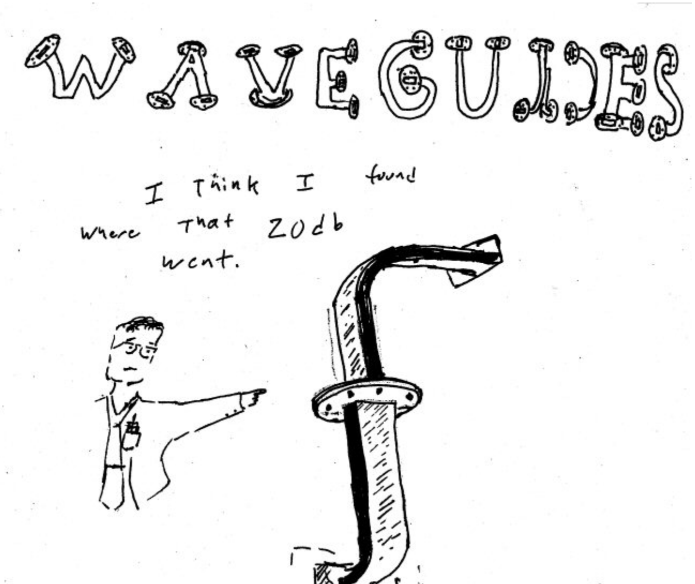
Outline

A waveguide is a structure that guides waves, such as electromagnetic waves or sound waves.

Waveguide for em waves



Guided propagation

It means conveying the energy carried on an em wave along a given path.

Medium

There are no physical reasons to let the energy carried by an electromagnetic wave travel along a given direction in a homogeneous medium.

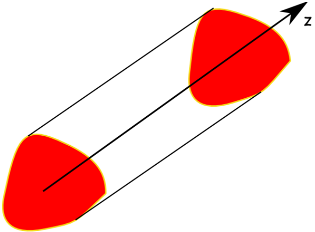
An inhomogeneous medium is needed.

The inhomogeneities commonly exploited are obtained:

- Combining dielectrics and conductors, e.g. conducting wall waveguides.
- Combining different dielectrics, i.e. optical fiber.

Cylindrical structure

A region filled with a homogeneous medium and surrounded by walls (whose nature is not specified at this stage) is considered.



- The structure is supposed to extend to infinity along the direction z and to have a constant cross-section along the z direction.
- Such a structure is called "Cylindrical structure in the z direction".

We are interested in the field configurations supported by this structure.

Outline

TE-TM decomposition

In a simply-connected region with no sources and filled by a homogeneous linear isotropic medium, any em field can be expressed as the sum of two fields that, with respect to a given direction $\hat{a}$, are:

TE-TM

- Transverse electric (TE)
- Transverse magnetic (TM)

$$(\mathbf{E}, \mathbf{H}) = (\mathbf{E}_{TE}, \mathbf{H}_{TE}) + (\mathbf{E}_{TM}, \mathbf{H}_{TM})$$

$$\mathbf{E}_{TE} \cdot \hat{a} = 0 \quad , \quad \mathbf{H}_{TM} \cdot \hat{a} = 0 \tag{1}$$

TE and TM fields can be obtained from a scalar function of spatial coordinates satisfying the scalar Helmholtz equation.

TE-TM decomposition

The simplest way to demonstrate TE-TM decomposition theorem is assuming that:

TE-TM hypothesis

- The Transverse em (TEM) solution is neglected.
- The em field can be written as:

$$\mathbf{E}(x,y,z) = [\bar{\mathcal{E}}(x,y) + \hat{z}\mathcal{E}_z(x,y)]e^{-\gamma z} \quad (2)$$

$$\mathbf{H}(x,y,z) = [\bar{\mathcal{H}}(x,y) + \hat{z}\mathcal{H}_z(x,y)]e^{-\gamma z} \quad . \quad (3)$$

$\bar{\mathcal{E}}(x,y)$ and $\bar{\mathcal{H}}(x,y)$ represent the transverse $(\hat{x},\hat{y})$ electric and magnetic field components; while $\mathcal{E}_z(x,y)$ and $\mathcal{H}_z(x,y)$ are the longitudinal components.

Hereinafter, TE and TM are understood with respect to the z direction.

- By applying the symbolic determinant notation to Maxwell's equations:

$$\begin{aligned} \nabla \times \bar{\mathcal{E}} &= -j\omega\mu\bar{\mathcal{H}} \\ \nabla \times \bar{\mathcal{H}} &= j\omega\varepsilon_c\bar{\mathcal{E}} \end{aligned} \qquad \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathcal{E}_x & \mathcal{E}_y & \mathcal{E}_z \end{pmatrix}$$

Neglecting the z components (since not relevant), the following linear system of four equations in four unknowns is obtained:

$$\begin{aligned}
\frac{\partial \mathcal{E}_z}{\partial y} + \gamma \mathcal{E}_y &= -j\omega\mu \mathcal{H}_x \\
-\gamma \mathcal{E}_x - \frac{\partial \mathcal{E}_z}{\partial x} &= -j\omega\mu \mathcal{H}_y \\
\frac{\partial \mathcal{H}_z}{\partial y} + \gamma \mathcal{H}_y &= j\omega\varepsilon_c \mathcal{E}_x \\
-\gamma \mathcal{H}_x - \frac{\partial \mathcal{H}_z}{\partial x} &= j\omega\varepsilon_c \mathcal{E}_y
\end{aligned} \tag{4}$$

TE-TM decomposition

The algebraic system (4) can be solved for the four unknowns:

$$
\begin{aligned}
\mathcal{E}_x &= -\frac{1}{\gamma^2 - k_\varepsilon^2}\left(\gamma\frac{\partial \mathcal{E}_z}{\partial x} + j\omega\mu\frac{\partial \mathcal{H}_z}{\partial y}\right)\\
\mathcal{E}_y &= -\frac{1}{\gamma^2 - k_\varepsilon^2}\left(\gamma\frac{\partial \mathcal{E}_z}{\partial y} - j\omega\mu\frac{\partial \mathcal{H}_z}{\partial x}\right)\\
\mathcal{H}_x &= -\frac{1}{\gamma^2 - k_\varepsilon^2}\left(-j\omega\varepsilon_c\frac{\partial \mathcal{E}_z}{\partial y} + \gamma\frac{\partial \mathcal{H}_z}{\partial x}\right)\\
\mathcal{H}_y &= -\frac{1}{\gamma^2 - k_\varepsilon^2}\left(j\omega\varepsilon_c\frac{\partial \mathcal{E}_z}{\partial x} + \gamma\frac{\partial \mathcal{H}_z}{\partial y}\right) \quad ,
\end{aligned}
\tag{5}
$$

where, $k_\varepsilon^2 = -\omega^2\mu\varepsilon_c$.

TE-TM decomposition

The solution (5) proves the first part of the TE-TM decomposition theorem:

TE-TM field components

- TE field consists of $\mathcal{H}_z$ and the four components on the right-hand side of the right-hand term in (5).
- TM field consists of $\mathcal{E}_z$ and the four components on the left-hand side of the right-hand term in (5).

TE-TM decomposition

It must be explicitly pointed out that the above field solutions require $\gamma^2 \neq k_\varepsilon^2$.

- When $\gamma^2 = k_\varepsilon^2$, it can be shown that a non-trivial solution can be obtained only for $\mathcal{E}_z = \mathcal{H}_z = 0$, i.e. a TEM field.

Remarks on TEM mode

The loss of generality due to the assumption of $\gamma^2 \neq k_\varepsilon^2$, i.e. neglecting TEM mode, is not harmful because of the physical and technological peculiarities of the TEM mode which calls for non-simply connected structures.

TE-TM decomposition

The second part of the TE-TM decomposition theorem, i.e. the existence of scalar functions that fully determine the TE and TM components, can be easily proved by looking at (5):

Scalar field components

- TE $\Longrightarrow \mathcal{H}_z$.
- TM $\Longrightarrow \mathcal{E}_z$.

The two scalar functions straightforwardly satisfy the scalar Helmholtz equation.

They are the component along z of **H** and **E** which, in a homogeneous medium, satisfy homogeneous Helmholtz equation.

Helmholtz equation

The homogeneous Helmholtz equation to be satisfied by $\mathcal{F} = \{\mathcal{E}_z(x, y), \mathcal{H}_z(x, y)\}$ can be obtained by defining the transverse Laplacian ∇_t^2 as follows:

$$\nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2} = \nabla^2 - \gamma^2 \quad . \tag{6}$$

Accordingly,

$$\nabla^2 \mathcal{F} - k_\varepsilon^2 \mathcal{F} = 0 \quad , \tag{7}$$

becomes:

$$\nabla_t^2 \mathcal{F} = -\chi^2 \mathcal{F} \quad , \tag{8}$$

where:

$$\chi^2 = (\gamma^2 - k_\varepsilon^2) \neq 0 \quad . \tag{9}$$

Remarks on the eigenvalue equation

- Eq.(8) is an eigenvalue equation.
- The Helmholtz equation discussed when dealing with plane waves is also an eigenvalue equation.
- In that case, however, the eigenvalue $\chi^2 = -k_\varepsilon^2$ is unique: it is a given number once the frequency and the medium are known.
- The eigenvalue of eq.(8) is unknown.
- When suitable boundary conditions are satisfied, eq.(8) admits an infinity of discrete solutions.
- Each eigenvalue corresponds to a value of γ that, according to (2), plays the role of propagation constant along with z direction.

Outline

TEM field configuration

- It can be shown that a field resulting from the eigenvalue $\chi^2 = 0$ can be non-identically zero only if $\mathcal{E}_z = \mathcal{H}_z = 0$, i.e. it is a:

  Transverse electro-magnetic (TEM) field

- For a TEM field: $\bar{\mathcal{E}} = \eta \bar{\mathcal{H}} \times \hat{a}_z$. With η being the intrinsic impedance of the homogeneous medium filling the cylindrical structure.

Uniform plane wave

This implies that the propagation constant and the wave impedance are those of a uniform plane wave propagating in an unbounded medium with the same parameters of the medium filling the cylindrical structure.

TEM field configuration

- It can be noted that, when $\chi^2 = 0$, eq.(8) becomes:

$$\nabla_t^2 \mathcal{F} = 0 \tag{10}$$

- This implies that the functions of the transverse coordinates satisfy the 2D Laplace equation.
- When the cylindrical structure consists of an ideal conductor, boundary conditions imply that the field is identically zero everywhere.

TEM peculiarity

Accordingly, TEM field cannot exist if the cross-section is a simply connected domain.

Boundary conditions

BCs

Although it has been proved that any em field in a homogeneous cylindrical structure can be decomposed in a TE and TM field (and in some cases a TEM field), only once BCs are specified one can say whether these fields may actually exist in a given structure independently of the others.

The simple case of a cylindrical structure shielded by an ideal electric conductor is considered.

It must be explicitly pointed out that, by introducing simple refinements, this unrealistic model may fit interesting realistic scenarios.

Ideal conducting walls

It can be shown that both TE and TM transmission modes are compatible with the BCs which come from ideal electric conductor walls:

$$\begin{aligned} \mathcal{E}_z &= 0 \\ \frac{\partial \mathcal{H}_z}{\partial n} &= 0 \end{aligned} \tag{11}$$

TE and TM modes

TE and TM can exist in a cylindrical structure with ideal conducting walls.

BCs together with the eigenvalue equation (8) make guided propagation in cylindrical structures with ideal conducting walls a scalar problem, from a mathematical viewpoint.

Outline

Modes properties

Hereinafter, the cylindrical structure is assumed to be strictly lossless:

Hypothesis

- The dielectric is lossless.
- The walls consist of ideal conductors.

As a matter of fact, $k_\varepsilon^2 = -\omega^2\mu\varepsilon$ is real and negative.

Eigenvalues and cut-off frequency

- Using the 2D form of the Green identity it can be shown that all the eigenvalues, χ^2, of (9) are real and positives.
- For each mode (labeled as "n") there is a cut-off frequency, ω_{cn}, which is such that the longitudinal propagation constant (2) γ vanishes:

$$\begin{aligned} \chi_n^2 &= -(k_\varepsilon^2 - \gamma_n^2) = \omega^2 \varepsilon\mu + \gamma_n^2 \\ 0 = \gamma_n^2 &= \chi_n^2 - \omega^2 \varepsilon\mu \\ \omega_{cn} &= \frac{\chi_n}{\sqrt{\varepsilon\mu}} \end{aligned} \tag{12}$$

Note that since all the eigenvalues are real and positives, ω_{cn} is real and positive.

Cut-off of a transmission mode

The propagation properties on the *n*-th transmission mode depends on $\gamma_n = \alpha_n + j\beta_n$. Two cases must be distinguished:

$\omega > \omega_{cn}$

- $\gamma_n^2 < 0$;
- $\gamma_n^2 = \omega^2\varepsilon\mu - \omega_{cn}^2\varepsilon\mu$;
- $\gamma_n = j\beta_n = j\omega\sqrt{\varepsilon\mu}\sqrt{1 - (\frac{\omega_{cn}}{\omega})^2}$

The transmission mode propagates, i.e. its phase changes vs *z*

$\omega < \omega_{cn}$

- $\gamma_n^2 > 0$;
- $\gamma_n^2 = \omega_{cn}^2\varepsilon\mu - \omega^2\varepsilon\mu$;
- $\gamma_n = \alpha_n = \omega\sqrt{\varepsilon\mu}\sqrt{(\frac{\omega_{cn}}{\omega})^2 - 1}$

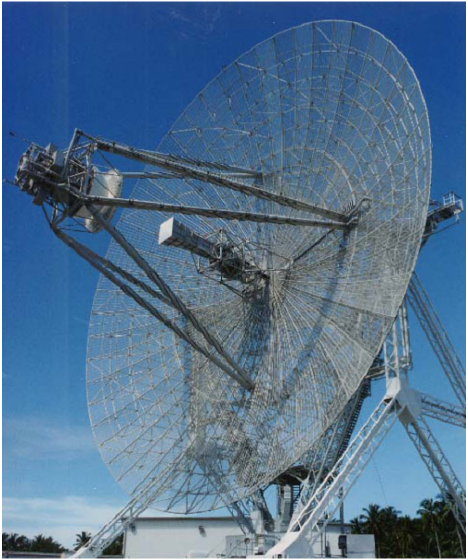
The mode does not propagate, i.e. evanescent wave

Cut-off of a transmission mode: Mw oven

The metallic mesh-like screen in the front door of the microwave oven prevents transmission beyond the boundary of the screen.
The maximum permissible size of mesh opening which retains this behaviour is frequency dependent, so in a domestic microwave oven you have what you see:

a filter which stops microwaves but permits visible light to pass.

Cut-off of a transmission mode: radar antennas



Phase and group velocities

For any propagating mode, i.e. $\omega > \omega_{cn}$, the longitudinal phase and group velocities are given by:

- Phase velocity tends to infinity for $\omega \to \omega_{cn}$ while it decreases monotonically towards c as ω increases.

$$v_{fn} = \frac{\omega}{\beta_n} = c\frac{1}{\sqrt{1-(\frac{\omega_{cn}}{\omega})^2}} \tag{13}$$

- Group velocity is 0 at cut-off while it tends to c for $\omega \to \infty$.

$$v_{gn} = \frac{1}{\frac{\partial \beta_n}{\partial \omega}} = c\sqrt{1-\left(\frac{\omega_{cn}}{\omega}\right)^2} \tag{14}$$

Note that this kind of guiding structures are such that:

$v_{fn}v_{gn} = c^2$

For a fixed ω, looking at (13) and (14), one can note that:

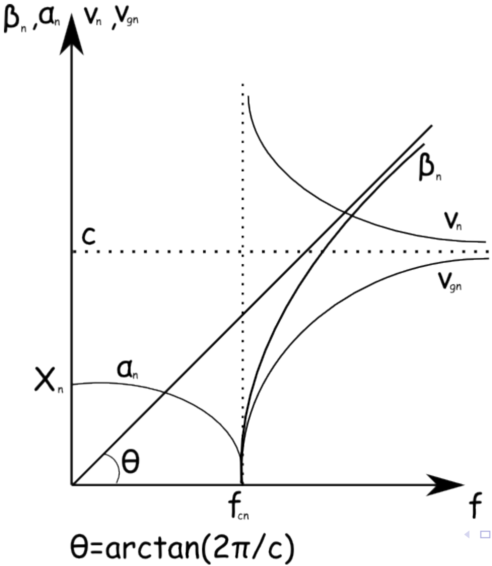
- Modes having different cut-off frequencies (i.e. different eigenvalues) call for different group velocities (14) and different phase velocities too (13).

Multipath

A modulating signal is distorted when several modes propagate together.

This is due to the fact that the group delay from the waveguide input to its output has as many values as the propagating modes.

Brillouin diagram



It depicts attenuation constant, phase constant, phase velocity and group velocity vs frequency for a generic mode.

The fundamental mode

It is the mode that corresponds to the lowest eigenvalue χ_0^2

- If it is not degenerate, it is the ONLY mode that can propagate alone in the cylindrical structure.

To avoid multipath

The fundamental mode can propagate alone in the guiding structure in the frequency range that goes from $\chi_o c$ up to the smallest among the cut-off frequencies of the other modes.

This is the unique frequency range that allows avoiding multipath problems!

In a non-simply connected structure the fundamental mode is the TEM mode whose cut-off frequency is $\omega_{cTEM} = 0$

Mode orthogonality

Orthogonality relationships can be proved:

- Between TE modes characterized by different eigenvalues.
- Between TM modes characterized by different eigenvalues.

For degenerate modes Schmidt procedure applies.

- Between TE and TM modes.

Orthogonality

Mode orthogonality is an extension to an infinite-dimensional vector space (the space of solution of (8)) of a well-known property which applies for eigenvalue problems of finite dimension:

eigenvectors belonging to different eigenvalues are orthogonal

Mode orthogonality

Orthogonality relationships have profound implications:

- In a cylindrical waveguide the total power flowing through the cross-section is the sum of the power associated to each individual mode.
- A field **E**, **H** in a cylindrical waveguide can be expressed as a series of modes and the expansion coefficients are given by an orthogonal projection:

$$\int_S \mathbf{E} \cdot \mathbf{E}_i^* dS \quad , \quad \int_S \mathbf{H} \cdot \mathbf{H}_i^* dS$$

- Completeness...

On the completeness of the solution

TE-TM decomposition

1. TE-TM decomposition theorem showed that in a homogeneous cylindrical structure any em field can be decomposed (neglecting the TEM field) in a TE and TM component.
2. BCs relevant to ideal conducting walls are satisfied by each TE and TM mode.

Remarks

Although the first result is always true; this is not the case for the second one. An important counter-example is provided by cylindrical structures with non-ideal conducting walls, where hybrid modes, in addition to TE and TM ones, play a dominant role.

On the completeness of the solution

When dealing with PDE boundary conditions and completeness are strictly connected.

- In general, the weaker the BCs are, the larger is the set of solutions in order to be complete.

Plane wave expansion

It is useful to recall that the basis to expand a monochromatic field carrying on a finite power consists of a continuous infinity of elements.

In this case, the only requirement is the field be modulus-square integrable. Hence, only a continuous set could be complete.

On the completeness of the solution

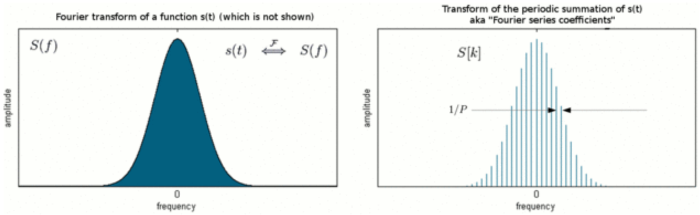
In the cylindrical structure we have considered, BCs hold on a closed surface which is all at finite distance from any point where the field is defined. Hence, a stronger BC applies.

- To analyze the effects of those stronger BCs on the complete solutions set, an analogy with signals in time domain and their Fourier analysis can be considered.

Fourier series vs Fourier integral

- Aperiodic signals (i.e. boundary conditions at infinity) $\rightarrow$ Fourier integral, i.e. continuous spectrum.
- Periodic signals (i.e. boundary conditions at finite or field being non-zero only over a finite spatial interval) $\rightarrow$ Fourier series, i.e. discrete spectrum.

On the completeness of the solution



As a matter of fact, a sufficient condition for the complete set of solutions to become discrete is that:
the field needs to be non-zero only over a finite spatial interval.

Accordingly, when a cylindrical structure with ideal conducting wall is considered:

A discrete spectrum is obtained

Remarks on the completeness

Can an infinity of discrete elements be considered a complete set for the above mentioned problem ?

2D problem

The waveguide problem is a 2D problem:

In two-dimensions, a discrete infinity can be built up in an infinite number of possible ways.
This means that the cardinality of a set is not enough to assure completeness.

Remarks on the completeness

If the PDE is separable, the completeness of the solution is a problem which can be analyzed in a simpler way.

Separable PDE

In this case, a complete set of solutions of the PDE consists of all the products of a countable infinity of linearly independent solutions of the first-coordinate ODE multiplied by a countable infinity of linearly independent solutions of the second-coordinate ODE.

Remarks on the completeness

For positive real eigenvalues (8) is separable only in four types of orthogonal coordinate systems:

1. Rectangular Cartesian.
2. Circular polar.
3. Parabolic.
4. Elliptical.

Remarks on the completeness

If a generic cylindrical structure is assumed, it is always possible describing it by using one of the formerly mentioned coordinate systems.

BSc and separability

The chosen reference frame is actually useful only when BCs are imposed on coordinate lines. When this is not the case (8) can be still decomposed in two ODEs. Their solutions, however, do not satisfy BCs on an individual basis. This means that each mode is not expressed as a product of two functions but as the product of two series of functions.

For Further Reading I

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