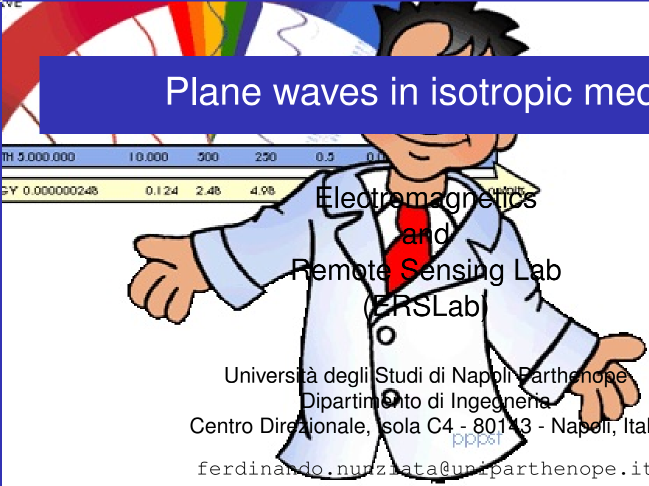
Plane waves in isotropic media



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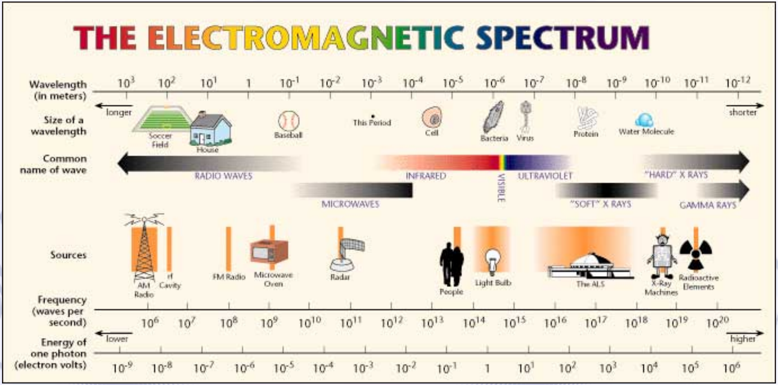
Propagation

"We have strong reason to conclude that light itself—including radiant heat and other radiation, if any—is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws."

—James C. Maxwell

Electromagnetic waves spectrum



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Helmholtz wave equation

Hermann von Helmholtz

AKA Hermann Ludwig Ferdinand von Helmholtz

Born: 31-Aug-1821
Birthplace: Potsdam, Germany
Died: 8-Sep-1894
Location of death: Charlottenburg, Berlin, Germany
Cause of death: unspecified

Gender: Male
Race or Ethnicity: White
Sexual orientation: Straight
Occupation: Physicist

Nationality: Germany
Executive summary: Law of Conservation of Energy

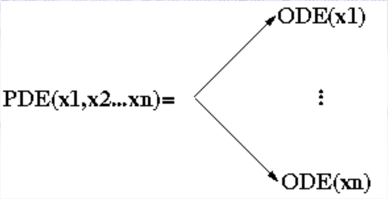
German philosopher and man of science, born on the 31st of August 1821 at Potsdam, near Berlin. His father, Ferdinand, was a teacher of philology and philosophy in the gymnasium, while his mother was a Hanoverian lady, a lineal descendant of the great Quaker William Penn. Delicate in early life, Helmholtz became by habit a student, and his father at the same time directed his thoughts to natural phenomena. He soon showed mathematical powers, but these were not fostered by the careful training mathematicians usually receive, and it may be said that in after years his attention was directed to the higher mathematics mainly by force of circumstances.

$$\boxed{\nabla^2 E - k^2 E = 0}$$

From a PDE to ODEs

One of the most used approaches to solve Partial Differential Equations (PDEs) in mathematical physics is the so-called method of *Separation of Variables* (SV).

It basically consists of breaking a given PDE in a set of *Ordinary Differential Equations* (ODEs), which can be solved separately from one another, by isolating each independent variable in a separate equation.



SV method

SV method is applicable only under restrictive assumptions:

- The PDE must be separable. The set S of solutions obtained by SV method needs to be a complete set of solutions. This means that S is dense enough to allow one writing any PDE solution as a linear combination of solutions belonging to S. A given PDE is typically separable only in few reference frames.
- The boundary conditions must be separable. Any differential equation must satisfy suitable boundary conditions (BC). BCs are themselves separable if the boundary is a coordinate surface (or a set of coordinate surfaces) in one of the reference frames where the PDE is separable.

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The em problem

The PDE which governs both radiation and propagation phenomena is the so-called *Helmholtz* equation. It is a second-order elliptic PDE.

The following electromagnetic (em) problem is defined:

1. Domain: 3D space/ω.
2. Medium: linear, isotropic, homogeneous and lossy.
3. Sources: no imposed currents ($\mathbf{J}_o = 0$).
4. BCs: Sommerfeld conditions for the field at infinity.

Uniqueness theorem ensures (ω exterior problem) that, once the above mentioned requirements are known, Maxwell's equations have a unique solution in the given domain.

The em problem

The em field satisfying Maxwell's equation under the previously stated requirements may be calculated solving Helmholtz equation for **E** or **H** through the SV method.

The Helmholtz equation to be solved is given by:

$$\nabla^2 \mathbf{E} - k_\varepsilon^2 \mathbf{E} = 0 \quad , \tag{1}$$

where:

$$k_\varepsilon^2 = -\omega^2 \mu \varepsilon_c = -\omega^2 \mu \left(\varepsilon - j \frac{\sigma}{\omega} \right) \quad . \tag{2}$$

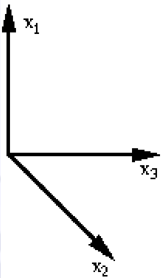
SV method: Helmholtz equation

3D Helmholtz equation (1) is separable only in a few number of coordinate systems which can be derived from the *orthogonal ellipsoidal coordinate* system:

1. Orthogonal Cartesian.
2. Circular cylindrical.
3. Elliptical cylindrical.
4. Parabolic cylindrical.
5. Rotation parabolic.
6. Paraboidal.
7. Spherical.
8. Prolate spheroidal.
9. Oblate spheroidal.
10. Conical.

SV method: Helmholtz equation

A Cartesian orthogonal coordinate system (x_1, x_2, x_3) is hereinafter adopted:



Eq.(1) can be written by components:

$$\nabla^2 E_i \equiv \frac{\partial^2 E_i}{\partial x_1^2} + \frac{\partial^2 E_i}{\partial x_2^2} + \frac{\partial^2 E_i}{\partial x_3^2} = k_\varepsilon^2 E_i \quad . \quad (3)$$

The three scalar equations are independent of each other, hence, the linearity of the medium allows, without loss of generality, considering:

$$\mathbf{E} = E\hat{x}_1 \quad . \quad (4)$$

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SV method: Helmholtz equation

The SV method consists of making the following ansatz:

$$E(x_1, x_2, x_3) = f_1(x_1) f_2(x_2) f_3(x_3) \quad , \tag{5}$$

which leads to:

$$\frac{f_1''}{f_1} + \frac{f_2''}{f_2} + \frac{f_3''}{f_3} = k_\varepsilon^2 \quad , \tag{6}$$

where f_j'' denotes the second derivative of f_j.

SV method: Helmholtz equation

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For a fixed ω, k_ε^2 is constant and, therefore, (6) can be satisfied if and only if:

$$\frac{f_i''}{f_i} = S_i^2 \quad i = 1, 2, 3 \quad , \tag{7}$$

with the following separation condition:

$$S_1^2 + S_2^2 + S_3^2 = k_\varepsilon^2 \quad . \tag{8}$$

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SV method: Helmholtz equation

SV method leads to the following three ODEs:

$$\frac{f_i''}{f_i} = S_i^2 \quad i = 1, 2, 3 \quad , \tag{9}$$

whose general integral can be written as follows:

$$f_i = F_{1i} e^{-S_i x_i} + F_{2i} e^{S_i x_i} \quad i = 1, 2, 3 \quad . \tag{10}$$

where F_{1i} and F_{2i} are arbitrary complex constants.

- The separation equation (8) deals with S_i^2. It does not tell anything about $S_i = \pm\sqrt{S_i^2}$.

SV method: Helmholtz equation

Accordingly, there is no loss of generality in the following formula:

$$\begin{aligned} E(x_1, x_2, x_3) &= f_1(x_1) f_2(x_2) f_3(x_3) \\ &= E_o e^{-(S_1 x_1 + S_2 x_2 + S_3 x_3)} \\ &= E_o e^{-\mathbf{S} \cdot \mathbf{r}} \quad , \end{aligned} \tag{11}$$

Propagation vector

$$\mathbf{S} = \sum_i S_i \hat{x}_i \quad , \quad \mathbf{r} = \sum_i x_i \hat{x}_i$$

are the propagation vector and the position vector, respectively.

Electric and magnetic fields

It must be noted that $\mathbf{S} = \mathbf{a} + j\mathbf{k} = \alpha\hat{a} + \beta\hat{k}$, where $\mathbf{a}$ is called attenuation vector, $\mathbf{k}$ phase vector, α attenuation constant, β phase constant is such that:

$$\mathbf{S} \cdot \mathbf{S} = k_\varepsilon^2 = -\omega^2 \mu \varepsilon_c \quad .$$

Note that, since $\mathbf{S}$ is a complex vector: $\mathbf{S} \cdot \mathbf{S} \neq \mathbf{S} \cdot \mathbf{S}^* = |\mathbf{S}|^2$.

$$\mathbf{E} = \mathbf{E_o} e^{-\mathbf{S} \cdot \mathbf{r}} \quad , \tag{12}$$

where:

$$\mathbf{E_o} = E_o \hat{x}_1 \quad .$$

Electric and magnetic fields

From Maxwell's equations, it follows that:

$$\mathbf{H} = -\frac{\nabla \times \mathbf{E}_o e^{-\mathbf{S}\cdot\mathbf{r}}}{j\omega\mu} \quad . \tag{13}$$

By invoking the vector identity:

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A} \quad ,$$

where f and **A** are a scalar and a vector function of space coordinates, (13) becomes:

$$\begin{aligned} -\frac{E_o}{j\omega\mu}\nabla e^{-\mathbf{S}\cdot\mathbf{r}} \times \hat{x}_1 &= \frac{\mathbf{S} \times E_o e^{-\mathbf{S}\cdot\mathbf{r}}\hat{x}_1}{j\omega\mu} \\ &= \mathbf{H_o} e^{-\mathbf{S}\cdot\mathbf{r}} \end{aligned} \tag{14}$$

Remarks

From (14) it follows that:

$$\mathbf{H} = \frac{\mathbf{S} \times \mathbf{E}}{j\omega\mu} \quad . \tag{15}$$

- This term provides a relationship between **E** and **H** which further confirms that (15) is always true, despite the restrictive hypothesis of linear polarization previously made, see eq.(4).
- Under the (unnecessary) hypothesis that all the components of **E** share the same propagation vector **S**, the general solution for **E** is given by:

$$\mathbf{E} = \mathbf{E_o} e^{-\mathbf{S}\cdot\mathbf{r}} \quad . \tag{16}$$

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The general solution for the em field is given by:

$$\mathbf{E} = \mathbf{E_o} e^{-\mathbf{S}\cdot\mathbf{r}} \tag{17}$$

$$\mathbf{H} = \frac{\mathbf{S} \times \mathbf{E_o}}{j\omega\mu} e^{-\mathbf{S}\cdot\mathbf{r}} \tag{18}$$

These equations are referred to a generic reference frame which can be changed in a completely arbitrary way.

Remarks

The solution provided by (17)-(18) is physically untenable:

- In general, it does not satisfy Sommerfeld conditions and, therefore, the uniqueness theorem (not even in the case of lossy medium).
- It carries on an infinite power.

Nevertheless, the solution (17)-(18) is:

- Perfectly legitimate as a mathematical solution of Maxwell's equations.
- A fundamental brick in building up a physically-consistent em field.

Remarks

Inserting (18) in Maxwell's equation $\nabla \times \mathbf{H} = j\omega\varepsilon_c \mathbf{E}$:

$$\begin{aligned}
\mathbf{E} &= \frac{\nabla \times \mathbf{H}}{j\omega\varepsilon_c} = \frac{1}{j\omega\varepsilon_c} \nabla \times \frac{\left(\mathbf{S} \times \mathbf{E_o} e^{-\mathbf{S}\cdot\mathbf{r}}\right)}{j\omega\mu} \\
&= \frac{1}{j\omega\varepsilon_c} \frac{\mathbf{S}}{j\omega\mu} \times (\nabla \times \mathbf{E}) = \frac{1}{j\omega\varepsilon_c} \frac{\mathbf{S}}{j\omega\mu} \times -j\omega\mu\mathbf{H} \\
&= j\frac{\mathbf{S} \times \mathbf{H}}{\omega\varepsilon_c} = -\frac{\mathbf{S} \times \mathbf{E} \times \mathbf{S}}{\omega^2\varepsilon_c\mu} \qquad (19)
\end{aligned}$$

In the same way:

$$\begin{aligned}
\mathbf{H} &= -\frac{\nabla \times \mathbf{E}}{j\omega\mu} = \frac{\nabla \times}{j\omega\mu} \frac{\mathbf{S} \times \mathbf{E} \times \mathbf{S}}{\omega^2\varepsilon_c\mu} = -\frac{j\omega\mu}{j\omega\mu} \frac{\mathbf{S} \times \mathbf{H} \times \mathbf{S}}{\omega^2\varepsilon_c\mu} \\
&= -\frac{\mathbf{S} \times \mathbf{H} \times \mathbf{S}}{\omega^2\varepsilon_c\mu} \qquad (20)
\end{aligned}$$

Remarks

From (19)-(20) it follows that:

$$\begin{aligned} \mathbf{E} \cdot \mathbf{S} &= 0 \\ \mathbf{H} \cdot \mathbf{S} &= 0 \end{aligned} \tag{21}$$

Complex vectors

By similarities with vectors defined in a real space, one may ERRONEOUSLY think that (21) implies that **E**, **H** and **S** are mutually orthogonal.

This is actually true only for linearly polarized uniform plane waves!!

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Inserting $\mathbf{S} = \mathbf{a} + j\mathbf{k}$ in (17)-(18) it follows that:

$$\mathbf{E} = \mathbf{E_o} e^{-(\mathbf{a}+j\mathbf{k})\cdot\mathbf{r}} = \mathbf{E_o} e^{-\mathbf{a}\cdot\mathbf{r}} e^{-j\mathbf{k}\cdot\mathbf{r}} \quad . \tag{22}$$

The following *loci* can be defined:

$\mathbf{a} \cdot \mathbf{r} = const$ - Equi-amplitude planes

- It implies $|\mathbf{E}| = const$ and $|\mathbf{H}| = const$.
- These *loci* are given by planes orthogonal to the attenuation vector and are generally called equi-amplitude or constant amplitude planes.

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$\mathbf{k} \cdot \mathbf{r} = const$ - Equi-phase planes

- It implies $\angle E_i = const$ and $\angle H_i = const$.
- These *loci* are given by planes orthogonal to the phase vector and are generally called equi-phase or constant phase planes.

Since the equi-phase surfaces are generally called "wavefronts" and, in this case, they are planes; such solutions of Maxwell's equations are called: Plane waves.

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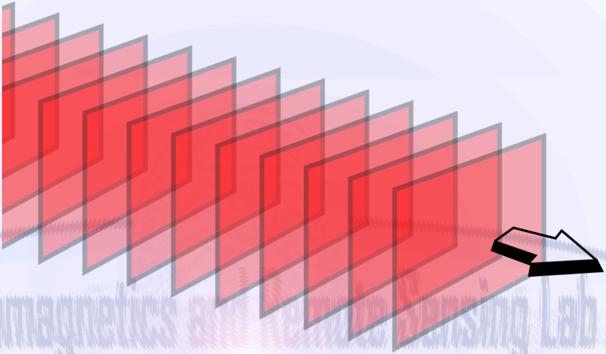
Plane wave wavefronts



At the very root propagation is just the motion of wavefronts as the time goes!

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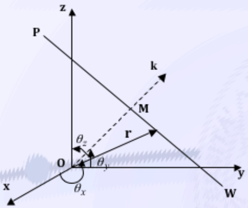
Remarks on plane waves



The plane perpendicular to the vector **k** is seen from its side appearing as a line P-W. The dot product $\mathbf{k} \cdot \mathbf{r}$ is the projection of the radial vector **r** along the normal to the plane and will have the constant value *OM* for all points on the plane.

The equation $\mathbf{k} \cdot \mathbf{r} = const$ is the characteristic property of a plane perpendicular to the direction of propagation **k**.

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Plane waves solution

- The sought solution $\mathbf{E}, \mathbf{H}$ must satisfy the divergence equation $\nabla \cdot \mathbf{E} = 0$, hence:

$$\begin{aligned} \nabla \cdot (\mathbf{E}_o e^{-\mathbf{S}\cdot\mathbf{r}}) &= e^{-\mathbf{S}\cdot\mathbf{r}} \nabla \cdot \mathbf{E}_o + \mathbf{E}_o \cdot \nabla e^{-\mathbf{S}\cdot\mathbf{r}} \\ &= -\mathbf{E}_o \cdot \mathbf{S} e^{-\mathbf{S}\cdot\mathbf{r}} \\ &= -\mathbf{E}_o \cdot (\mathbf{a} + j\mathbf{k}) e^{-\mathbf{S}\cdot\mathbf{r}} \end{aligned} \tag{23}$$

- The last equation implies that:

$$\mathbf{E}_o \cdot \hat{a} = 0 \tag{24}$$

$$\mathbf{E}_o \cdot \hat{k} = 0 \tag{25}$$

Orthogonality

Eqs.(24-25) mean that, although in general $\mathbf{E}_o$ is not orthogonal to $\mathbf{S}$, it is indeed orthogonal to each of the $\mathbf{S}$ components.

E and H fields

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- Phasor domain

$$\mathbf{E} = \mathbf{E}_o e^{-\mathbf{S}\cdot\mathbf{r}} = \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{26}$$

$$\mathbf{H} = \frac{\mathbf{S}\times\mathbf{E}}{j\omega\mu} = \frac{\mathbf{a}+j\mathbf{k}}{j\omega\mu}\times\mathbf{E}_o e^{-\mathbf{S}\cdot\mathbf{r}} = \frac{\mathbf{k}-j\mathbf{a}}{\omega\mu}\times\mathbf{E}_o e^{-\mathbf{S}\cdot\mathbf{r}} \tag{27}$$

- Time domain

$$\mathbf{e}(\mathbf{r},t) = \Re\left\{\mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} e^{-j\mathbf{k}\cdot\mathbf{r}} e^{j\omega t}\right\} = \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \tag{28}$$

$$\mathbf{h}(\mathbf{r},t) = \Re\left\{\frac{\mathbf{k}-j\mathbf{a}}{\omega\mu}\times\mathbf{E}_o e^{-\mathbf{S}\cdot\mathbf{r}} e^{j\omega t}\right\} = \frac{\mathbf{k}\times\mathbf{E}_o}{\omega\mu} cos(\omega t - \mathbf{k}\cdot\mathbf{r}) + \frac{\mathbf{a}\times\mathbf{E}_o}{\omega\mu} e^{-\mathbf{a}\cdot\mathbf{r}} sin(\omega t - \mathbf{k}\cdot\mathbf{r}) \tag{29}$$

The fields propagate along with the $\hat{k}$ direction while attenuate along with the $\hat{a}$ direction.

Poynting vector

- The Poynting vector in the phasor domain:

$$\mathbf{P} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\mathbf{E} \times \left[\frac{\mathbf{k} - j\mathbf{a}}{\omega\mu} \times \mathbf{E}\right]^* =$$

$$\frac{1}{2}\mathbf{E} \times \left[\frac{\mathbf{k} + j\mathbf{a}}{\omega\mu} \times \mathbf{E}^*\right] =$$

$$\frac{(\mathbf{k} + j\mathbf{a})\mathbf{E} \cdot \mathbf{E}^*}{2\omega\mu} - \frac{\mathbf{E}^*[(\mathbf{k} + j\mathbf{a}) \cdot \mathbf{E}]}{2\omega\mu} = \frac{(\mathbf{k} + j\mathbf{a})}{2\omega\mu}\mathbf{E} \cdot \mathbf{E}^* \qquad (30)$$

It can be noted that active and reactive powers are directed along with $\hat{k}$ and $\hat{a}$ directions, respectively.

Poynting vector

Plane wave solution is physically untenable

P depends on space coordinates only through the exponential factor $e^{-2\mathbf{a}\cdot\mathbf{r}}$:

- This implies that the flux of **P** through any plane in space is infinite.

This is physically untenable.

To determine the direction of **P** it is convenient to analyze separately the cases of uniform, evanescent and dissociated waves.

Wave impedance

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- A wave impedance along with the generic $\hat{n}$ direction can be defined as follows:

$$\eta(\hat{n})\mathbf{H} \times \hat{n} = \hat{n} \times \mathbf{E} \times \hat{n} \tag{31}$$

- This implies that:

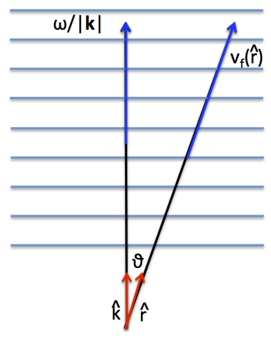
$$\eta(\hat{n}) = \frac{\hat{n} \times \mathbf{E} \times \hat{n}}{\mathbf{H} \times \hat{n}} \tag{32}$$

Wave impedance

It is basically the ratio between fields' components belonging to the plane orthogonal to a given direction $\hat{n}$.

Phase velocity

Eq.(28) is a traveling wave and the factor $cos(\mathbf{k}\cdot\mathbf{r}-\omega t)$ describes an ondulatory motion.



The ondulatory motion can be analyzed by looking at points with constant phase:

$$d(\mathbf{k}\cdot\mathbf{r}-\omega t)=0$$

$$\mathbf{k}\cdot\hat{r}dr-\omega dt$$

$$v_f(\hat{r})=\frac{dr}{dt}=\frac{\omega}{\mathbf{k}\cdot\hat{r}}=\frac{\omega}{|\mathbf{k}|cos\vartheta} \quad (33)$$

Plane waves classification

Plane waves can be classified according to the relationship between the attenuation and phase vectors. It must be noted that:

$$k_\varepsilon = \alpha + j\beta = \sqrt{k_\varepsilon^2} = \sqrt{-\omega^2 \mu \varepsilon_c} \quad ,$$

belongs to the first quadrant of the complex plane. Therefore, $\beta > 0$ and $\alpha \geq 0$. The latter inequality is saturated when the medium is lossless.

$$\mathbf{S} \cdot \mathbf{S} = |\mathbf{a}|^2 - |\mathbf{k}|^2 + 2j\mathbf{a} \cdot \mathbf{k} = k_\varepsilon^2 = -\omega^2 \mu \left(\varepsilon - j\frac{\sigma}{\omega} \right) \quad .$$

Separating real and imaginary parts:

$$|\mathbf{a}|^2 - |\mathbf{k}|^2 = -\omega^2 \mu \varepsilon \quad (34)$$

$$2\mathbf{a} \cdot \mathbf{k} = \omega \mu \sigma \quad (35)$$

Plane waves classification

From (34) it follows that $|\mathbf{k}|^2 > |\mathbf{a}|^2$ and, therefore:

$$|\mathbf{k}| > 0 \quad , \tag{36}$$

Traveling solution

According to (36), the solutions of Maxwell's equations can never have a constant phase in the region where they are defined.

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Uniform plane wave

The meaning of (35) depends on the fact that the medium is lossless ($\sigma = 0$) or lossy ($\sigma \neq 0$).

- $\sigma = 0 \Longrightarrow \mathbf{a} \cdot \mathbf{k} = 0$. This is satisfied in either of the two following cases:

  1. $\mathbf{a} = 0$.
     This implies that $|\mathbf{E}| = const$ and $|\mathbf{H}| = const$ hold for the whole 3D space. Therefore, any plane is a equi-amplitude plane.

Uniform plane waves

Generally, a convention is adopted which makes equi-amplitude planes coincident with the equi-phase ones.Such a wave is called uniform plane wave.

E and H fields

- Phasor domain

$$\mathbf{E} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \quad (37)$$

$$\mathbf{H} = \frac{\mathbf{k}\times\mathbf{E}}{\omega\mu} \quad (38)$$

- Time domain

$$\mathbf{e}(\mathbf{r},t) = \mathbf{E}_o cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \quad (39)$$

$$\mathbf{h}(\mathbf{r},t) = \frac{\mathbf{k}\times\mathbf{E}_o}{\omega\mu} cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \quad (40)$$

The fields propagate along with the $\hat{k}$ direction.

Wave parameters

- The Poynting vector in the phasor domain:

$$\mathbf{P} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\frac{|\mathbf{E}_o|^2}{\omega\mu}\mathbf{k} \tag{41}$$

Active power is directed along with $\hat{k}$

- The wave impedance along the $\hat{k}$ direction is given by:

$$\eta = \frac{\hat{k} \times \mathbf{E} \times \hat{k}}{\mathbf{H} \times \hat{k}} = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \frac{\omega\mu}{|\mathbf{k}|} = \sqrt{\frac{\mu}{\epsilon}} \tag{42}$$

Wave parameters

- When $\mu = \mu_o$, η can be also written in a completely equivalent way:

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_o \epsilon_o}{\epsilon \epsilon_o}} = \eta_o \frac{\epsilon_o}{\epsilon} = \frac{\eta_o}{\sqrt{\epsilon_r}} = \frac{\eta_o}{n} \qquad (43)$$

where η_o and n are the free space impedance and the refractive index, respectively.

- The phase velocity, according to (34), is given by:

$$|\mathbf{k}| = \omega \sqrt{\mu \epsilon} = \beta = \frac{2\pi}{\lambda}, \quad v_f = \frac{1}{\sqrt{\mu \epsilon}}$$

Note that in the vacuum $v_f = c$.

Standing waves

- Let us consider two uniform plane waves that propagate in the same medium along the same path in opposite directions:

$$\mathbf{E}_1 = \mathbf{E}_{01} e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{44}$$

$$\mathbf{E}_2 = \mathbf{E}_{02} e^{+j\mathbf{k}\cdot\mathbf{r}} \tag{45}$$

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_{01} e^{-j\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}_{02} e^{+j\mathbf{k}\cdot\mathbf{r}} \tag{46}$$

- The total field in the time domain is given by:

$$\mathbf{e}(\mathbf{r}, t) = \Re\{\mathbf{E} e^{j\omega t}\} \tag{47}$$

- By adding and subtracting the factor $\mathbf{E}_{02} e^{-j\mathbf{k}\cdot\mathbf{r}}$ one obtains:

$$\mathbf{E} = \mathbf{E}_{01} e^{-j\mathbf{k}\cdot\mathbf{r}} - \mathbf{E}_{02} e^{-j\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}_{02} e^{+j\mathbf{k}\cdot\mathbf{r}} + \mathbf{E}_{02} e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{48}$$

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Standing waves

- Eq. (48) can be rewritten as follows:

$$\begin{aligned} \mathbf{E} = & \left(\mathbf{E}_{01} - \mathbf{E}_{02}\right) e^{-j\mathbf{k}\cdot\mathbf{r}} + 2\mathbf{E}_{02}\frac{e^{j\mathbf{k}\cdot\mathbf{r}} + e^{-j\mathbf{k}\cdot\mathbf{r}}}{2} \\ = & \left(\mathbf{E}_{01} - \mathbf{E}_{02}\right) e^{-j\mathbf{k}\cdot\mathbf{r}} + 2\mathbf{E}_{02} cos(\mathbf{k}\cdot\mathbf{r}) \end{aligned} \tag{49}$$

- Hence, the field in the time domain is given by:

$$\mathbf{e}(\mathbf{r}, t) = \Re\{\left(\mathbf{E}_{01} - \mathbf{E}_{02}\right) e^{-j\mathbf{k}\cdot\mathbf{r}} e^{j\omega t} + 2\mathbf{E}_{02} cos(\mathbf{k}\cdot\mathbf{r}) e^{j\omega t}\} \tag{50}$$

Partially standing wave

$$\mathbf{e}(\mathbf{r}, t) = \left(\mathbf{E}_{01} - \mathbf{E}_{02}\right) cos(\omega t - \mathbf{k}\cdot\mathbf{r}) + 2\mathbf{E}_{02} cos(\mathbf{k}\cdot\mathbf{r}) cos(\omega t) \tag{51}$$

Partially standing wave

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- The first term in eq. (51) is a traveling wave.
- The second term is a standing wave.
- The total field results in a partially standing wave.

The standing wave terms is such that

- Spatial and temporal variations are no longer linearly mixed in the argument of a cosine function.
- The time evolution is in synchronous throughout the whole space.

This term describes an oscillation whose amplitude varies in space according to a $cos(\mathbf{k} \cdot \mathbf{r})$ rule.

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- The standing wave oscillation reaches its maximum at points (aka crests) that satisfy:

$$\mathbf{k} \cdot \mathbf{r} = m\pi \quad m = 0, \pm 1, \ldots \tag{52}$$

- The minimum is reached at points (aka nodes) that satisfy:

$$\mathbf{k} \cdot \mathbf{r} = (2m+1)\frac{\pi}{2} \quad m = 0, \pm 1, \ldots \tag{53}$$

Planes

Eqs.(52-53) define two families of planes perpendicular to **k** that would have been constant-phase planes for the two traveling waves the partially standing wave consists of.

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Evanescent wave

2. $\mathbf{a} \neq 0$ and $\mathbf{a} \perp \mathbf{k}$.
It follows that:

$$|\mathbf{k}| > |\mathbf{a}| \quad ,$$

Evanescent wave

Therefore, equi-phase planes are orthogonal to equi-amplitude ones. This implies that this wave attenuates while propagating in a lossless medium.

Such a wave is called evanescent plane wave.

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Evanescent wave

- $|\mathbf{k}| \geq \beta$, hence according to (34):

$$v_f < \frac{1}{\sqrt{\mu\epsilon}}$$

The evanescent wave in a lossless medium is also called “slow wave”

E and H fields

$$\mathbf{E} = \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{54}$$

$$\mathbf{H} = \frac{\mathbf{k} - j\mathbf{a}}{\omega\mu} \times \mathbf{E} \tag{55}$$

Orthogonality

$\mathbf{H}$ has one component orthogonal to the plane that consists of $\mathbf{k}$ and $\mathbf{E}_o$: while the other is directed along $\hat{k}$.

$$\mathbf{e}(\mathbf{r}, t) = \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \tag{56}$$

$$\mathbf{h}(\mathbf{r}, t) = \frac{\mathbf{k} \times \mathbf{E}_o}{\omega\mu} \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} cos(\omega t - \mathbf{k}\cdot\mathbf{r}) + \frac{\mathbf{a} \times \mathbf{E}_o}{\omega\mu} e^{-\mathbf{a}\cdot\mathbf{r}} sin(\omega t - \mathbf{k}\cdot\mathbf{r}) \tag{57}$$

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- The Poynting vector in the phasor domain:

$$\mathbf{P} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\frac{|\mathbf{E}_o|^2}{\omega\mu} e^{-2\mathbf{a}\cdot\mathbf{r}}(\mathbf{k} + j\mathbf{a}) \tag{58}$$

Active power is directed along with $\hat{k}$; while reactive power along $\hat{a}$

- The wave impedance is

$$\eta = \omega\mu \frac{\hat{n} \times \mathbf{E} \times \hat{n}}{(\mathbf{k} - j\mathbf{a}) \times \mathbf{E} \times \hat{n}} \tag{59}$$

- Along the $\hat{k}$ and $\hat{a}$ directions:

$$\eta(\hat{k}) = \frac{\omega\mu}{\beta} \quad \eta(\hat{a}) = j\frac{\omega\mu}{\alpha} \tag{60}$$

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- $\sigma \neq 0 \implies \mathbf{a} \cdot \mathbf{k} > 0$. This means that:

Dissociated plane wave

- $|\mathbf{a}| \neq 0$. The wave attenuates while propagating in a lossy media.
- The angle between **a** and **k** is smaller than $\pi/2$.

Such a wave is called dissociated plane wave.

Uniform plane wave

It must be explicitly pointed out that, in the special case where **a** and **k** are parallel, the wave is still called uniform plane wave.

Dissociated wave

- $|\mathbf{k}| \geq \beta$, hence according to (34):

$$v_f < \frac{1}{\sqrt{\mu\epsilon}}$$

The dissociated wave is also called “slow wave”

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$$\mathbf{E} = \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} e^{-j\mathbf{k}\cdot\mathbf{r}} \quad (61)$$

$$\mathbf{H} = \frac{\mathbf{k} - j\mathbf{a}}{\omega\mu} \times \mathbf{E} \quad (62)$$

Orthogonality

$\mathbf{a} \times \mathbf{E}_o$ is no longer directed along $\hat{k}$ as in the evanescent case.

Wave parameters

The expressions of **E** and **H** fields in the time domain, Poynting vector and wave impedance are the same of the evanescent wave.

Wave parameters

- The Poynting vector in the phasor domain:

$$\mathbf{P} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\frac{|\mathbf{E}_o|^2}{\omega\mu} e^{-2\mathbf{a}\cdot\mathbf{r}}(\mathbf{k} + j\mathbf{a}) \qquad (63)$$

Power flow

In this case, since $\hat{k}$ and $\hat{a}$ are neither parallel nor orthogonal both active and reactive powers flow in each space direction.

- The wave impedance is

$$\eta = \omega\mu \frac{\hat{n} \times \mathbf{E} \times \hat{n}}{(\mathbf{k} - j\mathbf{a}) \times \mathbf{E} \times \hat{n}} \qquad (64)$$

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E and **H** fields

$$\mathbf{E} = \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_o e^{-(\alpha + j\beta)\hat{k}\cdot\mathbf{r}} \quad (65)$$

$$\mathbf{H} = \frac{\mathbf{k} - j\mathbf{a}}{\omega\mu} \times \mathbf{E} = \frac{(\beta - j\alpha)\hat{k} \times \mathbf{E}}{\omega\mu} \quad (66)$$

$$\mathbf{e}(\mathbf{r}, t) = \mathbf{E}_o e^{-\mathbf{a}\cdot\mathbf{r}} cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \quad (67)$$

$$\mathbf{h}(\mathbf{r}, t) = \beta \frac{\hat{k} \times \mathbf{E}_o}{\omega\mu} e^{-\mathbf{a}\cdot\mathbf{r}} cos(\omega t - \mathbf{k}\cdot\mathbf{r}) + \alpha \frac{\hat{k} \times \mathbf{E}_o}{\omega\mu} e^{-\mathbf{a}\cdot\mathbf{r}} sin(\omega t - \mathbf{k}\cdot\mathbf{r}) \quad (68)$$

Wave parameters

- The Poynting vector in the phasor domain:

$$\mathbf{P} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\frac{|\mathbf{E}_o|^2}{\omega\mu} e^{-2\mathbf{a}\cdot\mathbf{r}}(\mathbf{k} + j\mathbf{a}) = \frac{1}{2}\frac{|\mathbf{E}_o|^2}{\omega\mu} e^{-2\mathbf{a}\cdot\mathbf{r}}(\beta + j\alpha)\hat{k} \tag{69}$$

Active and reactive powers are directed along with $\hat{k}$

- The wave impedance is

$$\eta(\hat{k}) = \omega\mu \frac{\hat{k} \times \mathbf{E} \times \hat{k}}{(\beta - j\alpha)\hat{k} \times \mathbf{E} \times \hat{k}} = \frac{\omega\mu}{\beta - j\alpha} \tag{70}$$

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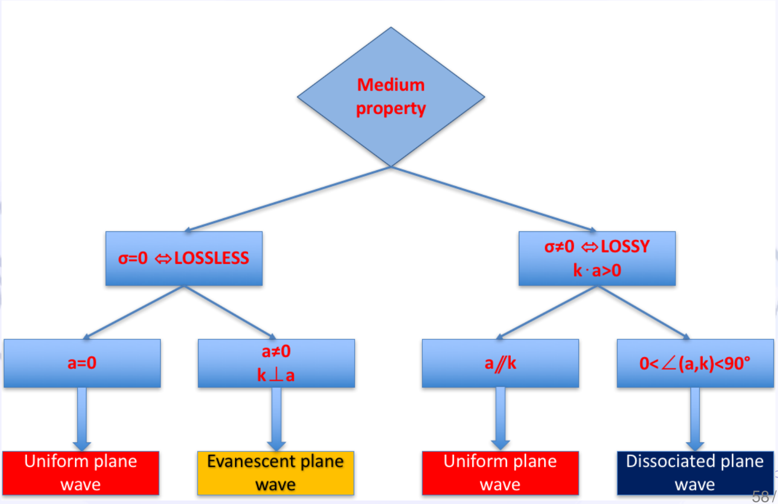
In a nutshell

The classification depends on medium parameters only



Remarks on orthogonality

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It can be shown that **E**, **H** and **S** are mutually orthogonal if and only if the following conditions are satisfied:

1. The wave is linearly polarized.
2. **a** and **k** are parallel (including also the special case $\mathbf{a} = 0$).

This means that, both in a lossless and in a lossy medium, the three above mentioned vectors are mutually orthogonal only for:

linearly polarized uniform plane waves

Note that orthogonality between the complex vectors **E**, **H** and **S** should not be confused with orthogonality between instantaneous time-harmonic vectors. The latter are of course mutually orthogonal!!!

Linearly polarized uniform plane waves

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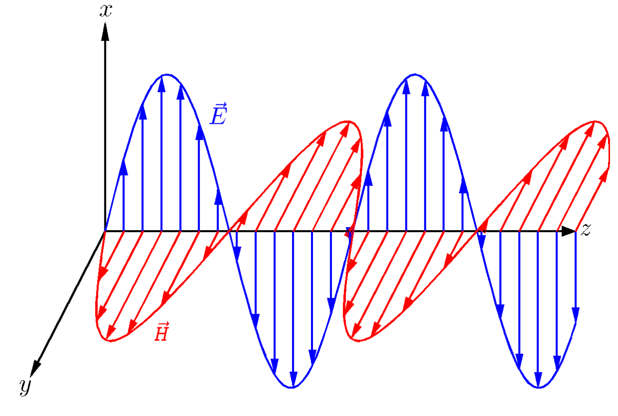
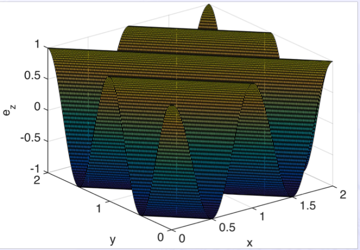
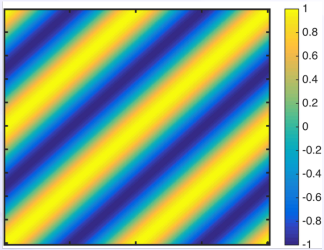
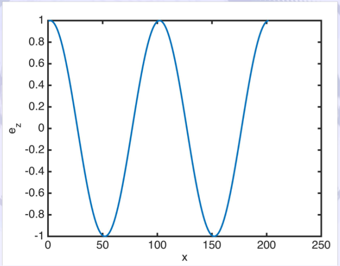


Do it yourself - Plane wave lossless case

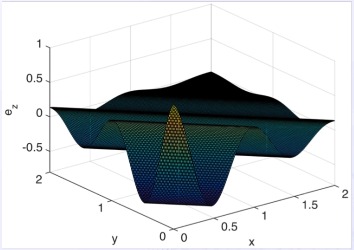


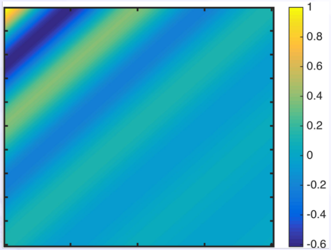


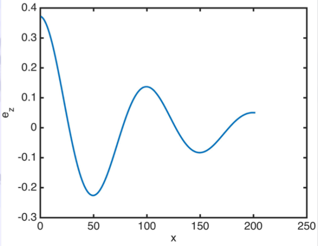


Eq. (**??**); $x = y = 0 : 0.01 : 2$; $\lambda = 1$; $t = 0$

Do it yourself - Plane wave lossy case







Eq. (**??**); $x = y = 0 : 0.01 : 2$; $\lambda = 1$; $t = 0$; $\alpha = 1$

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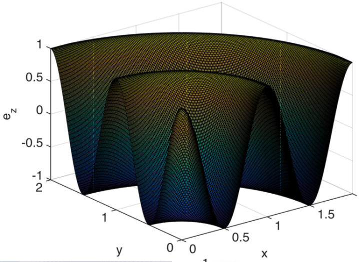
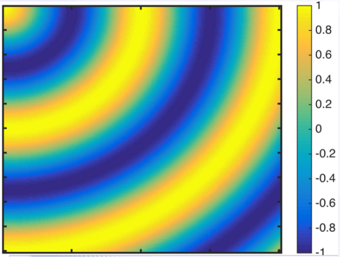
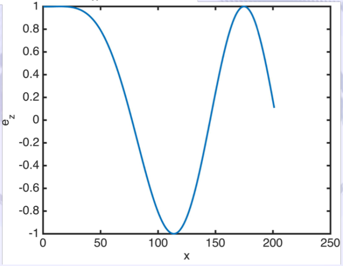
Do it yourself - Spherical wave lossless case

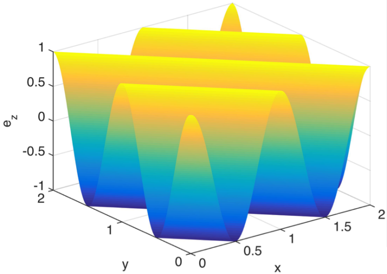






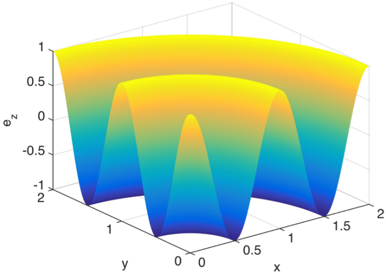
Spherical wave: e^{-jkr} with $r = \sqrt{x^2 + y^2}$

Do it yourself - 2D time evolution



Plane wave

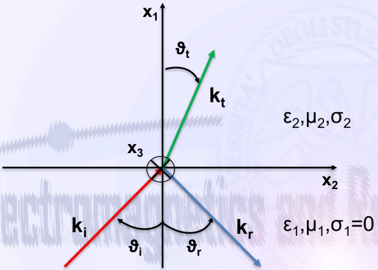
Spherical wave



$x = y = -1 : 0.01 : 1$, $\lambda = 1$m, $t = linspace(0, 60e - 9, 100)$

The em problem

We aim at analyzing the behavior of em waves at a sharp discontinuity in the medium parameters.



- A plane surface ($x_1 = 0$) separates the whole 3D space into two homogeneous media.
- Medium 1 is lossless.
- The incident wave is a TEM wave whose $\mathbf{k}_i$ lies in the plane $x_3 = 0$.

This implies that:

$$\mathbf{k}_i \cdot \hat{x}_3 = 0 \tag{71}$$

and, therefore:

$$\mathbf{k}_i = |\mathbf{k}_i|(\hat{x}_1 + \hat{x}_2) \tag{72}$$

Reflection and refraction

According to physical intuition (grounded on experimental evidence) one can test whether continuity conditions are satisfied by superimposing an incident and reflected wave in region 1, and assuming a transmitted wave in region 2.

region 1

$$\mathbf{E}_i = \mathbf{E}_{oi} e^{-j\mathbf{k}_i \cdot \mathbf{r}} \qquad \mathbf{E}_r = \mathbf{E}_{or} e^{-\mathbf{S}_r \cdot \mathbf{r}} \tag{73}$$

$$\mathbf{H}_i = \frac{\hat{k}_i \times \mathbf{E}_{oi}}{\eta_1} e^{-j\mathbf{k}_i \cdot \mathbf{r}} \qquad \mathbf{H}_r = \frac{\mathbf{S}_r \times \mathbf{E}_{or}}{j\omega\mu_1} e^{-\mathbf{S}_r \cdot \mathbf{r}}$$

Reflection and refraction

region 2

$$
\begin{aligned}
\mathbf{E}_t &= \mathbf{E}_{ot} e^{-\mathbf{S}_t \cdot \mathbf{r}} \\
\mathbf{H}_t &= \frac{\mathbf{S}_r \times \mathbf{E}_{ot}}{j\omega\mu_2} e^{-\mathbf{S}_t \cdot \mathbf{r}}
\end{aligned} \tag{74}
$$

- All those waves need to satisfy, over the whole $x_1 = 0$ plane, continuity conditions (that are often referred as Fresnel's equations):

$$
\begin{aligned}
\hat{x}_1 \times (\mathbf{E}_i + \mathbf{E}_r) &= \hat{x}_1 \times \mathbf{E}_t \\
\hat{x}_1 \times (\mathbf{H}_i + \mathbf{H}_r) &= \hat{x}_1 \times \mathbf{H}_t
\end{aligned} \tag{75}
$$

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Geometrical relationship of the three waves

- Eq.(75) is satisfied when the argument of the exponential functions (73-74) is the same for any value of:

$$\mathbf{r} = x_2\hat{x}_2 + x_3\hat{x}_3 \tag{76}$$

- Hence:

$$j\mathbf{k}_i \cdot \mathbf{r} = \mathbf{S}_r \cdot \mathbf{r} = \mathbf{S}_t \cdot \mathbf{r} \tag{77}$$

To characterize the geometrical relationships prevailing among the three waves, eq.(77) must be discussed separately for the incident-reflected and incident-transmitted waves.

Reflected wave

$$\mathbf{S}_r = \mathbf{a}_r + j\mathbf{k}_r \qquad \text{with} \qquad \mathbf{k}_r = |\mathbf{k}_r|(\hat{x}_1 + \hat{x}_2 + \hat{x}_3). \tag{78}$$

- Hence, eq.(77) is verified when:

$$\mathbf{a}_r \cdot \mathbf{r} = 0 \tag{79}$$
$$\mathbf{k}_r \cdot \mathbf{r} = \mathbf{k}_i \cdot \mathbf{r} \tag{80}$$

- The reflected wave is a uniform plane wave: $|\mathbf{k}_i| = |\mathbf{k}_r|$.
- $\mathbf{k}_i \cdot \hat{x}_3 = 0$ implies that $\mathbf{k}_r \cdot \hat{x}_3 = 0$, hence the reflected wave lies in the plane identified by $\mathbf{k}_i$ and the normal to the discontinuity (x_1 axis): the incidence plane.
- Eq.(80) implies that:

$$\mathbf{k}_r \cdot \hat{x}_2 = \mathbf{k}_i \cdot \hat{x}_2 \rightarrow \vartheta_r = \vartheta_i \tag{81}$$

The incidence angle is equal to the reflection angle

Transmitted wave

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$$\mathbf{a}_t \cdot \mathbf{r} = 0 \tag{82}$$

$$\mathbf{k}_t \cdot \mathbf{r} = \mathbf{k}_i \cdot \mathbf{r} \tag{83}$$

- Eq.(83) implies that $\mathbf{k}_t \cdot \hat{x}_3 = 0$. Hence, $\mathbf{k}_t$ lies in the incidence plane.
- Defining the angle between x_1 axis and $\mathbf{k}_t$ as the transmission angle ϑ_t, eq.(83) can be written as follows:

$$\sin\vartheta_t = \frac{\beta_1}{|\mathbf{k}_t|}\sin\vartheta_i \tag{84}$$

Lossless and lossy media

- To fully characterize eq.(84), $|\mathbf{k}_t|$ must be determined. Two cases must be distinguished: lossless and lossy media.

Transmitted wave: lossless medium

$\mathbf{a}_t \cdot \mathbf{r} = 0$ is verified in two cases:

1. $\mathbf{a}_t = 0$.

- In this case the transmitted wave is a uniform plane wave that is termed as refracted wave.
- $|\mathbf{k}_t| = \beta_2 = \omega\sqrt{\mu_2 \epsilon_2}$.
- Eq.(84) becomes:

Snell's law (aka Descartes' law)

$$\sin\vartheta_t = \frac{\beta_1}{\beta_2}\sin\vartheta_i \Leftrightarrow n_2 \sin\vartheta_t = n_1 \sin\vartheta_i \tag{85}$$

where n_i is the refractive index of the $i-$th medium.

Transmitted wave: lossless medium

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2. $\mathbf{a}_t \neq 0$, $\mathbf{a}_t$ perpendicular to the $x_1 = 0$ plane

- In this case an evanescent plane wave is in place.
- $\mathbf{k}_t$ is orthogonal to $\mathbf{a}_t$ and, therefore, it is parallel to $\hat{x}_2$. This implies that $\sin\vartheta_t = 1$.
- $|\mathbf{k}_t| > \beta_2$.

Evanescent plane wave

- Eq.(84) admits an evanescent wave when the following inequality is satisfied by a real value of ϑ_i:

$$\sin\vartheta_i > \frac{\beta_2}{\beta_1} = \frac{n_2}{n_1} \tag{86}$$

- This equation can be verified only when $n_2 < n_1$

Transmitted wave: lossy medium

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Dissociated plane wave

- The transmitted wave still needs to satisfy eq.(84), hence it is a dissociated wave.
- Constant-amplitude planes are parallel to $x_1 = 0$.
- Constant-phase planes are orthogonal to the direction defined by eq.(84).
- $|\mathbf{k}_t|$ is a function of ϑ_i

$$\sin\vartheta_t = \frac{n_1}{n_2}\sin\vartheta_i \, \frac{1}{2}\left[1 + \left(1 + \frac{\sigma_2^2}{\omega^2\epsilon_2^2\cos^2\vartheta_t}\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}} \tag{87}$$

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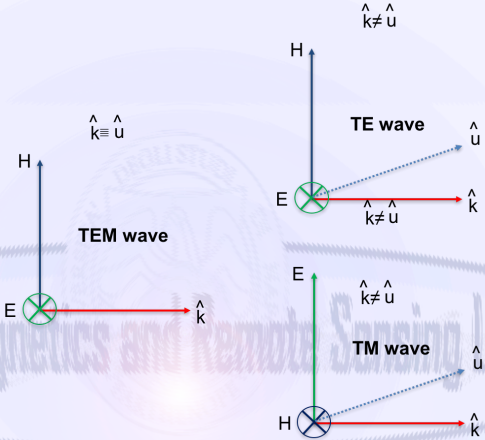
TE and TM waves

Once the geometry of the reflection problem has been characterized, the reflected and transmitted waves must be expressed as function of the incident wave.

- The simplest approach consists of expanding the incident wave into two linearly-polarized (i.e. both **E** and **H** are real vectors) independent polarizations.
- Reflection and refraction are studied separately for those two waves and then results are superimposed.
- TE wave refers to a unit vector $\hat{u}$ lying in the plane defined by **H** and **k**.
- TM wave refers to a unit vector $\hat{u}$ lying in the plane defined by **E** and **k**.

TE and TM waves



Note that “TE” and “TM” polarizations are aka “horizontal” and “vertical” polarizations or “parallel” and “perpendicular” polarizations.

Fresnel formulas

The relationship between transmitted/reflected wave and the incident wave is governed by transmission (τ)/reflection (ρ) coefficient. In case incident and transmitted TEM waves and lossless media, i.e. $\sigma_1 = \sigma_2 = 0$:

Fresnel formulas - TE/TM case

$$\rho_{TM} = \frac{\eta_2 \cos\vartheta_t - \eta_1 \cos\vartheta_i}{\eta_2 \cos\vartheta_t + \eta_1 \cos\vartheta_i} \quad (88)$$

$$\tau_{TM} = \frac{2\eta_2 \cos\vartheta_t}{\eta_2 \cos\vartheta_t + \eta_1 \cos\vartheta_i} \quad (89)$$

$$\rho_{TE} = \frac{\eta_2 \cos\vartheta_i - \eta_1 \cos\vartheta_t}{\eta_2 \cos\vartheta_i + \eta_1 \cos\vartheta_t} \quad (90)$$

$$\tau_{TE} = \frac{2\eta_2 \sec\vartheta_t}{\eta_2 \sec\vartheta_t + \eta_1 \sec\vartheta_i} \quad (91)$$

Fresnel formulas

ERSLab
F. Nunziata
Motivation
SV method
Helmholtz eq.
Plane waves
Plane waves
Classification
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Evanescent waves
Lossy medium
Reflection and refraction
Reflection an transmission coefficients
Appendix
For Further Reading

- Note that $\tau = 1 + \rho$.
- Using the formulas of wave impedance for a TM an TE wave:

$$Z_{TM} = \eta \cos \vartheta \qquad Z_{TE} = \frac{\eta}{\cos \vartheta} \tag{92}$$

- one can express reflection coefficients (90) and (88) in a unified way:

$$\rho = \frac{Z_2(\hat{x}_1) - Z_1(\hat{x}_1)}{Z_2(\hat{x}_1) + Z_1(\hat{x}_1)} \tag{93}$$

- where $Z_m(\hat{x}_1)$, with $m = 1, 2$, is the *TM* or *TE* wave impedance in the *m*-th medium in the direction orthogonal to the plane that separates the two media.

For Further Reading I

- C.G. Someda. *Electromagnetic Waves*
  CRC press - Taylor & Francis, Boca Raton, FL, 2006.
- P. Bassi and C. Zaniboni. *Introduzione ai campi elettromagnetici*
  Bononia University press, Bologna, Italy, 2016.

For further reading

'O tell me, when along the line
From my full heart the message flows,
What currents are induced in thine?
One click from thee will end my woes'.
Through many an Ohm the Weber flew,
And clicked the answer back to me,
'I am thy Farad, staunch and true,
Charged to a Volt with love for thee'.