Electromagnetic problems

Electromagnetics
and
Remote Sensing Lab
(ERSLab)

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Outline

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The em problem

Given a volume V, bounded by a regular surface S with outwarding normal $\hat{n}$ which contains imposed density currents and/or charges

one has to find solutions of Maxwell's equations that:

- are everywhere continuous in V but across abrupt discontinuities of either ϵ or μ or σ;
- satisfy continuity conditions across the abrupt discontinuities

EM problem

The above-mentioned conditions imply that "candidate" solutions should be functions belonging to the $\mathcal{L}^2(A)$ space, i.e. the Hilbert space of square integrable real and complex functions $f(\xi)$:

$$\int_A |f(\xi)|^2 d\xi < \infty \tag{1}$$

where the integration is understood in the Lebesgue sense.

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Goals

It aims at identifying the minimum set of conditions that guarantee the uniqueness of the solution.
It has a number of applications:

- It guarantees the same solution to a uniquely defined em problem; no matter what method is used to search for it.
- It establishes a one-to-one mapping between the field and the source. One can determine the source from its field.

Comments

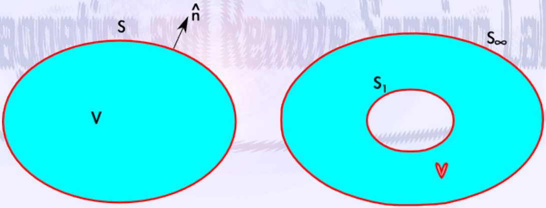
Minimum set of conditions

Note that the uniqueness theorem identifies the minimum set of conditions to ensure a unique solution. However, often Maxwell's equations can be actually solved only starting from a set larger than this minimum.
Hence, in this case, the uniqueness theorem allows saying that the extra-information cannot be independent of the minimum set.

Uniqueness theorem

Four cases must be distinguished:

Domain / Problem	Time	Frequency
Interior		
Exterior		



Interior problem

Exterior problem

Uniqueness theorem

Interior problem - Time domain

The solution $\mathbf{e}, \mathbf{h}$ is unique once initial conditions (ICs), boundary conditions (BCs) and sources are known:

- IC: $\mathbf{e}(\mathbf{r}, t)$ and $\mathbf{h}(\mathbf{r}, t)$ are known at $t = t_o$ and $\forall \mathbf{r} \in V$.
- BC: $\forall t \geq t_o$ either $\hat{n} \times \mathbf{e}$ or $\hat{n} \times \mathbf{h}$ are known on S.
- Source terms are given $\forall t \geq t_o$ and $\forall \mathbf{r} \in V$.

Remarks

Note that IC corresponds to the one associated with the Cauchy problem, where both the value of the function and its derivative must be specified. With respect to Maxwell's equations only the field's value need to be specified since the condition for the derivative is provided directly by the Maxwell's equations.

Uniqueness theorem

Interior problem - Frequency domain

The solution **E**, **H** is unique once BC and sources are known and the medium is lossy:

- BC: $\hat{n} \times \mathbf{E}$ or $\hat{n} \times \mathbf{H}$ are known on S.
- Source terms are given $\forall \mathbf{r} \in V$.

Remarks

- Note that only BCs are needed since fields are time-harmonic.
- For a lossless medium ($\epsilon_2 = \mu_2 = \sigma = 0$); fields not sustained by any source can exist in V and are called “resonant fields”.

Remarks

Remarks

It should be explicitly pointed out that assuming a lossless medium often simplifies Maxwell's equations remarkably. Hence, this assumption is very often made and can be reconciliated with the uniqueness theorem considering that lossless solution is the limit of the solution for the lossy case. In fact, uniqueness theorem requires a non-vanishing loss without specifying a minimum value for it.

Uniqueness theorem

Exterior problem - Time domain

The solution **e**, **h** is unique once ICs, BCs and the sources are known:

- Source terms are given $\forall t \geq t_o$ and $\forall \mathbf{r} \in V$.
- IC: $\mathbf{e}(\mathbf{r}, t)$ and $\mathbf{h}(\mathbf{r}, t)$ are known at $t = t_o$ and $\forall \mathbf{r} \in V$.
- BC:
  - $\forall t \geq t_o$ either $\hat{n} \times \mathbf{e}$ or $\hat{n} \times \mathbf{h}$ are known on S_1.
  - Sommerfeld's radiation conditions:

$$\lim_{r \to \infty} r|\mathbf{e}| = \lim_{r \to \infty} r|\mathbf{h}| = 0$$

or

$$\lim_{r \to \infty} r\left(\mathbf{e} - \eta \mathbf{h} \times \hat{n}\right) = 0$$

with η being the intrinsic medium impedance.

Uniqueness theorem

Exterior problem - Frequency domain

The solution $\mathbf{E}, \mathbf{H}$ is unique once BC and sources are known:

- BC:
  - $\hat{n} \times \mathbf{E}$ or $\hat{n} \times \mathbf{H}$ are known on S_1.
  - $\mathbf{E} = \eta \mathbf{H} \times \hat{n}$ at S_∞.
- Source terms are given $\forall \mathbf{r} \in V$.

Remarks

Note that the second BC is due to the fact that, in the time-harmonic regime, fields exist from $-\infty$ to $+\infty$, hence one cannot say that the field vanishes at S_∞.

Outline

Classification of em problems

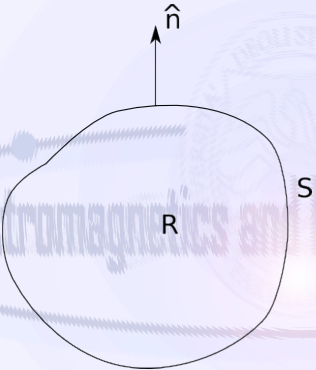
EM problems can be classified according to three items that are not independent of each other and are inherently connected with the uniqueness theorem:

- EM problems
  - Solution region.
  - Equation that describes the problem
  - BCs.

Outline

Solution region

Given a volume *R* which contains a medium (ϵ, μ, σ) and bounded by a surface *S* whose outward normal is $\hat{n}$:



The solution region *R* is:

- Interior or inner, closed and bounded, if *R* is bounded by a closed surface *S*.
- Exterior or outer, open and unbounded, if part or all of *S* is at infinity.

An em problem can be also classified according to the electrical properties of the medium, i.e. ϵ, μ, σ.

Outline

Equation that describes the em problem

The equation that describes the em problem can be

- Differential.
- Integral.
- Integro-differential.

Compact form

Linear em problems can be cast in a compact form:

$$L\phi = g \tag{2}$$

where L is a linear operator (integral, differential, integro-differential), ϕ is the unknown function and g is a source term.

Differential equations

A generic 2nd order partial differential equation (PDE) can be written as

$$a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + d\frac{\partial\phi}{\partial x} + e\frac{\partial\phi}{\partial y} + f\phi = g \qquad (3)$$

or equivalently:

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi = g \qquad (4)$$

Coefficients

The coefficients $(a \dots f)$ depend on x and y.
If they depend on ϕ a non-linear pde is to be solved; otherwise the pde is linear.

Differential equations

According to the compact form $L\phi = g$:

$$L = a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f$$

Homogeneous vs inhomogeneous pde

- Note that $L\phi = g$ represents:
  - a homogeneous pde if $g = 0$;
  - an inhomogeneous pde if $g \neq 0$.

Classification of pdes

A linear 2nd order pde can be classified according to

$$\Delta = b^2 - 4ac$$

PDE	Δ
elliptic	< 0
hyperbolic	> 0
parabolic	$= 0$

This classification relies on the fact that the quadratic equation:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

represents an ellipse, an hyperbola or a parabola if its discriminant Δ is negative, positive or zero, respectively.

Ex: elliptical pde

It is generally associated with steady-state phenomena, i.e. boundary-value problems.
No time dependence.
It generally models interior problems.

- Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = g(x, y)$$

Ex: hyperbolic pde

It generally models propagation problems where both BCs and initial conditions are due.
The solution region is generally open.

- Wave equation (scalar):

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{v_f^2} \frac{\partial^2 \phi}{\partial t^2}$$

Helmholtz's equation

Helmholtz's equation is an elliptic pde (not hyperbolic) since it represents steady-state phenomena.

Parabolic pde

It is generally associated with problems where the unknown function varies slowly with respect to the random motions that generate the variation.
It is similar to the hyperbolic pde.
Both BCs and initial conditions are due.

- Heat equation:

$$\frac{\partial^2 \phi}{\partial x^2} = k \frac{\partial \phi}{\partial t}$$

Initial conditions

Note that since the pde is only first order in time, only one initial condition needs to be specified.

Deterministic vs eigenvalue problem

- The problem $L\phi = g$ is termed as deterministic problem, since the unknown can be determined directly.
- Another class of problems, described by the functional form:

$$L\phi = \lambda\phi$$

is called eigenproblem or generalized eigenproblem:

$$L\phi = \lambda M\phi$$

where M is a linear operator and λ's are the eigenvalues.

The eigenproblem is such that only some λ_n values are admissible and, associated with these values, there are the corresponding eigenfunctions ϕ_n.

Outline

BCs

They state the conditions the unknown function ϕ must satisfy on the surface S that bounds the solution region R. BCs can be classified as:

- Dirichlet:

$$\phi(\mathbf{r}) = \begin{cases} 0 & \text{homogeneous} \\ p(\mathbf{r}) & \text{inhomogeneous} \end{cases}$$

- Neumann:

$$\frac{\partial \phi(\mathbf{r})}{\partial n} = \nabla \phi \cdot \hat{n} = \begin{cases} 0 & \text{homogeneous} \\ q(\mathbf{r}) & \text{inhomogeneous} \end{cases}$$

- Mixed:

$$\frac{\partial \phi(\mathbf{r})}{\partial n} + h(\mathbf{r})\phi(\mathbf{r}) = \begin{cases} 0 & \text{homogeneous} \\ w(\mathbf{r}) & \text{inhomogeneous} \end{cases}$$

with $\mathbf{r} \in S$.

For further reading

- C.G. Someda, Electromagnetic waves, Chapman & Hall, UK, 1998
- J.-M. Jin, Theory and computation of electromagnetic fields, Wiley, Canada, 2010
- M.N.O. Sadiku, Numerical techniques in electromagnetics, CRC Press, Boca Raton, FL, 2009