ERSLab

F. Nunziata

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A brief introduction on em theory

Electromagnetics
and
Remote Sensing Lab
(ERSLab)

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Electromagnetic (EM) theory forms a chapter of mathematical physics which can be organized as an axiomatic theory.
All the fundamental concepts, as well as many notions of technical interest, can be deduced from a small set of postulates.

Generally speaking, each theory consists of three key steps:

1. Set of entities aimed at describing the phenomena of interest.
2. Set of mathematical equations aimed at describing the evolution of the entities.
3. Relationship between the equations and the physics.

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EM theory

EM theory can be regarded as the study of fields; i.e. vector functions whose magnitude and direction vary as function of their position in space, produced by electric charges at rest or in motion:

Static vs dynamic

- Electrostatic fields are usually produced by static electric charges.
- Magnetostatic fields are due to the motion of electric charges with uniform velocity (direct current).
- Time varying fields are usually due to accelerated charges or time-varying currents.

Entities

All the entities, in general, depend of both space $\mathbf{r}$ and time t and, in the MKSΩ or Giorgi system of units, they are given by:

- $\mathbf{e}(\mathbf{r}, t)$ $(\frac{V}{m})$ electric field;
- $\mathbf{h}(\mathbf{r}, t)$ $(\frac{A}{m})$ magnetic field;
- $\mathbf{d}(\mathbf{r}, t)$ $(\frac{C}{m^2})$ electric induction;
- $\mathbf{b}(\mathbf{r}, t)$ $(\frac{Wb}{m^2})$ or (T) magnetic induction;
- $\rho(\mathbf{r}, t)$ $(\frac{C}{m^3})$ electric charge density;
- $\mathbf{j}(\mathbf{r}, t)$ $(\frac{A}{m^2})$ current density;

Macroscopic laws

The theory to be presented here deals only with macroscopic scale phenomena; i.e. those phenomena where consequences of the discrete nature of the electric charge are irrelevant.

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Key founding fathers

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Faraday's law of em induction

It links $\mathbf{e}(\mathbf{r},t)$ and $\mathbf{b}(\mathbf{r},t)$ through an integral relationship between the flux of $\mathbf{b}$ and the circulation of $\mathbf{e}$

$$\oint_C \mathbf{e}(\mathbf{r},t)\cdot\hat{c}\,\mathrm{d}C = -\frac{d}{dt}\iint_S \mathbf{b}(\mathbf{r},t)\cdot\hat{n}\,\mathrm{d}S \qquad (1)$$

Physical meaning

The circulation of the electric field intensity around any closed path C equals the time-rate change of the flux of the magnetic induction through a surface that has C at the edge.

Maxwell-Ampère law

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It links $\mathbf{h}(\mathbf{r},t)$, $\mathbf{j}(\mathbf{r},t)$ and the displacement current density $\frac{\partial \mathbf{d}(\mathbf{r},t)}{\partial t}$ $\left(\frac{A}{m^2}\right)$. Note that, at the very root, the displacement current was the fundamental Maxwell's contribute.

$$\oint_C \mathbf{h}(\mathbf{r},t)\cdot\hat{c}\,\mathrm{d}C = \iint_S \left(\frac{\partial \mathbf{d}(\mathbf{r},t)}{\partial t} + \mathbf{j}(\mathbf{r},t)\right)\cdot\hat{n}\,\mathrm{d}S \qquad (2)$$

Physical meaning

The circulation of the magnetic field intensity around any closed path C equals the flux of the electric current density through a surface that has C at the edge plus the time-rate change of the flux of the electric induction through a surface that has C at the edge.

Gauss' laws

- Gauss' law for electric induction:

$$\oiint_S \mathbf{d}(\mathbf{r}, t) \cdot \hat{n}\, \mathrm{d}S = \iiint_\tau \rho(\mathbf{r}, t) \mathrm{d}\tau \tag{3}$$

- Gauss' law for magnetic induction:

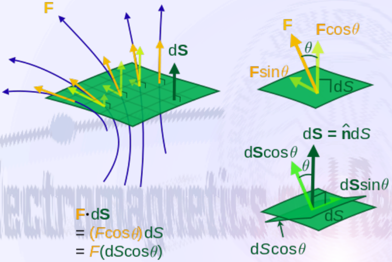
$$\oiint_S \mathbf{b}(\mathbf{r}, t) \cdot \hat{n}\, \mathrm{d}S = 0 \tag{4}$$

Physical meanings

The flux of the electric induction through any closed surface equals the net charge inside the volume enclosed by the surface. The flux of the magnetic induction through any closed surface is zero.

Remarks on flux and circulation

The flux is defined as the rate of flow of an entity per unit area



Flux of a vector field represents how much of the field is going through a given surface. It is usually defined with respect to a given surface and depends on how much the field is perpendicular to the surface.

If a field has a circulation along a given path, that means the field will have net flow that adds together along the given path.

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Relationship with physics

The link between em and mechanics is given by the Lorentz force equation

$$\mathbf{f} = q(\mathbf{e} + \mathbf{v} \times \mathbf{b}) \tag{5}$$

where **f** (N/m) is the force experienced by a particle with charge q moving at velocity **v** (m/s) in an em field.

Comments

It can be considered as a "definition" of the electric field intensity and the magnetic induction.

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Maxwell's equations in integral form

In summary, the following four equations are referred to as Maxwell's equations in integral form.

$$\oint_C \mathbf{e}(\mathbf{r},t)\cdot\hat{c}\,\mathrm{d}C = -\frac{d}{dt}\iint_S \mathbf{b}(\mathbf{r},t)\cdot\hat{n}\,\mathrm{d}S \qquad (6)$$

$$\oint_C \mathbf{h}(\mathbf{r},t)\cdot\hat{c}\,\mathrm{d}C = \iint_S \left(\frac{\partial\mathbf{d}(\mathbf{r},t)}{\partial t} + \mathbf{j}(\mathbf{r},t)\right)\cdot\hat{n}\,\mathrm{d}S \qquad (7)$$

$$\oint\!\!\!\oint_S \mathbf{d}(\mathbf{r},t)\cdot\hat{n}\,\mathrm{d}S = \iiint_\tau \rho(\mathbf{r},t)\mathrm{d}\tau \qquad (8)$$

$$\oint\!\!\!\oint_S \mathbf{b}(\mathbf{r},t)\cdot\hat{n}\,\mathrm{d}S = 0 \qquad (9)$$

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Divergence equations

The Maxwell's equations in integral form can be rewritten in a "local" form using Stokes and Gauss theorems, under the hypothesis that scalar and vector fields are regular, i.e. they are continuous to all the orders implied in the calculations.

Using Gauss theorem, eq.(3)-(4) can be written a follows:

$$\nabla \cdot \mathbf{d}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \tag{10}$$

$$\nabla \cdot \mathbf{b}(\mathbf{r}, t) = 0 \tag{11}$$

Magnetic charges

Note that **b** is a solenoidal vector; hence, no free magnetic charges exist.

Curl equation

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Invoking Stokes theorem, eq.(1)-(2) can be written as follows:

$$\nabla \times \mathbf{e}(\mathbf{r}, t) = -\frac{\partial \mathbf{b}(\mathbf{r}, t)}{\partial t} \tag{12}$$

$$\nabla \times \mathbf{h}(\mathbf{r}, t) = \frac{\partial \mathbf{d}(\mathbf{r}, t)}{\partial t} + \mathbf{j}(\mathbf{r}, t) \tag{13}$$

Maxwell's equations

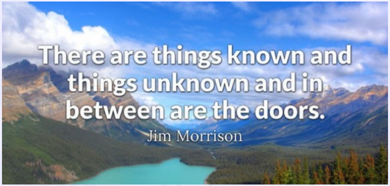
In summary, Maxwell's equations in differential or local form are given by:

$$
\begin{aligned}
\nabla \times \mathbf{e}(\mathbf{r}, t) &= -\frac{\partial \mathbf{b}(\mathbf{r}, t)}{\partial t} \\
\nabla \times \mathbf{h}(\mathbf{r}, t) &= \frac{\partial \mathbf{d}(\mathbf{r}, t)}{\partial t} + \mathbf{j}(\mathbf{r}, t) + \mathbf{j}_o(\mathbf{r}, t) \\
\nabla \cdot \mathbf{d}(\mathbf{r}, t) &= \rho(\mathbf{r}, t) \\
\nabla \cdot \mathbf{b}(\mathbf{r}, t) &= 0
\end{aligned}
$$

They are first-order coupled differential equations relating the vector field quantities to each other.

- Note that the total current **j** is partitioned into the sum of a convection (or conduction current) **j** plus an imposed current $\mathbf{j}_o$. The latter is a source term.
- The term em field refers to a pair of vector functions **e**, **h** that satisfy Maxwell's equations.

Remarks on imposed currents



Known terms

We consider “known terms” the physical quantities whose distributions can either “guessed” or experimentally determined easily.

Usually, electric current density satisfies this requirement since it is defined within a region in the space that includes the “sources”

Constitutive relationships

To solve Maxwell's equations, the medium must be accounted for. This can be done by introducing 3 equations termed as constitutive relationships.

For a simple medium, i.e. linear, homogeneous, isotropic and time-invariant, they are given by:

$$\mathbf{d}(\mathbf{r}, t) = \epsilon \mathbf{e}(\mathbf{r}, t) \tag{14}$$

$$\mathbf{b}(\mathbf{r}, t) = \mu \mathbf{h}(\mathbf{r}, t) \tag{15}$$

$$\mathbf{j}(\mathbf{r}, t) = \sigma \mathbf{e}(\mathbf{r}, t) \tag{16}$$

where ϵ is the electric permittivity (F/m), μ is the magnetic permeability (H/m) and σ is the conductivity (S/m). Note that for a simple medium ϵ, μ, σ are constants.

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Vacuum

The simplest medium, i.e. the vacuum, is characterized by the following constitutive relationships:

$$\mathbf{d}(\mathbf{r}, t) = \epsilon_o \mathbf{e}(\mathbf{r}, t) \tag{17}$$

$$\mathbf{b}(\mathbf{r}, t) = \mu_o \mathbf{h}(\mathbf{r}, t) \tag{18}$$

$$\mathbf{j}(\mathbf{r}, t) = \sigma \mathbf{e}(\mathbf{r}, t) = 0 \tag{19}$$

where:

- $\epsilon_o \approx 8.85 \times 10^{-12}$ is called vacuum permittivity, permittivity of free space or electric constant.
- $\mu_o \approx 1.25 \times 10^{-6}$ is called the vacuum permeability, permeability of free space, or magnetic constant.

Free space

Free space is a good approximation of vacuum.

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Classification of media

The medium where the field exists is characterized by its constitutive parameters: ϵ, μ and σ.

The medium is said to be:

- linear: ϵ, μ and σ are independent of **e** and **h**;
- homogeneous: ϵ, μ and σ are not function of space variables;
- time-invariant: ϵ, μ and σ are not function of time variables;
- isotropic: ϵ, μ and σ are independent of direction (they are scalar quantities).

Boundary conditions

Maxwell's equations in the differential form are valid at any point in a continuous medium.

- They cannot be applied to discontinuous fields that may occur at interfaces between different media.
- Maxwell's equations in integral form can be applied to find the relations between the fields on the two sides of an interface.
- Such relations are known as Boundary Conditions (BCs) or continuity conditions.

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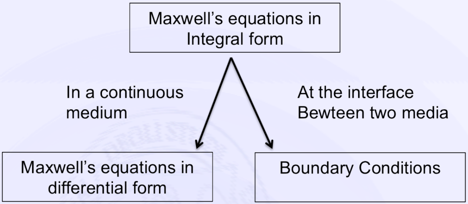
Boundary conditions



Free space

An infinitely large (unbounded) homogeneous medium, characterized by constant ϵ and μ, is often referred to as free space.

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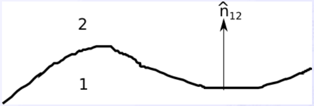
Boundary conditions

At the interface separating two different media the field satisfies the following BCs:



- Medium 1: ϵ_1, μ_1, σ_1.
- Medium 2: ϵ_2, μ_2, σ_2.

$$(\mathbf{e}_1 - \mathbf{e}_2) \times \hat{n}_{12} = 0 \quad (20)$$
$$(\mathbf{h}_1 - \mathbf{h}_2) \times \hat{n}_{12} = \mathbf{j}_s \quad (21)$$
$$(\mathbf{d}_1 - \mathbf{d}_2) \cdot \hat{n}_{12} = \rho_s \quad (22)$$
$$(\mathbf{b}_1 - \mathbf{b}_2) \cdot \hat{n}_{12} = 0 \quad (23)$$

where $\mathbf{j}_s$ $(\frac{A}{m})$ and ρ_s $(\frac{C}{m^2})$ are surface currents and surface free charges, respectively.

Maxwell's equations in simple media

Maxwell's equations in simple media can be rewritten as follows

$$
\begin{aligned}
\nabla \times \mathbf{e}(\mathbf{r}, t) &= -\mu \frac{\partial \mathbf{h}(\mathbf{r}, t)}{\partial t} \\
\nabla \times \mathbf{h}(\mathbf{r}, t) &= \epsilon \frac{\partial \mathbf{e}(\mathbf{r}, t)}{\partial t} + \sigma \mathbf{e}(\mathbf{r}, t) + \mathbf{j}_o(\mathbf{r}, t) \\
\nabla \cdot \mathbf{e}(\mathbf{r}, t) &= \frac{\rho(\mathbf{r}, t)}{\epsilon} \\
\nabla \cdot \mathbf{h}(\mathbf{r}, t) &= 0
\end{aligned}
$$

Notation

Note that hereinafter, to simplify the notation, the time and space dependence of scalar and vector field functions is omitted.

The wave equation

Maxwell's equations are solved in a simple medium and a source-free solution region is considered, i.e. $\mathbf{j}_o = 0$, $\rho = 0$. Moreover, the medium is considered to be an ideal dielectric, i.e. $\sigma = 0$.

- Since Maxwell's equations are coupled, to decouple them a second-order differential equation is obtained:

$$\nabla \times \nabla \times \mathbf{e} = -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{h}) = -\mu\epsilon \frac{\partial^2 \mathbf{e}}{\partial t^2}$$

Using the vector identity: $\nabla \times \nabla \times \mathbf{c} = \nabla\nabla \cdot \mathbf{c} - \nabla^2 \mathbf{c}$:

$$\nabla\nabla \cdot \mathbf{e} - \nabla^2 \mathbf{e} = -\mu\epsilon \frac{\partial^2 \mathbf{e}}{\partial t^2}$$

Since $\rho = 0$, $\nabla \cdot \mathbf{e} = 0$, hence:

$$\nabla^2 \mathbf{e} - \mu\epsilon \frac{\partial^2 \mathbf{e}}{\partial t^2} = 0 \qquad (24)$$

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Eq.(24) is the time-dependent vector wave equation (aka D'Alembert's vector equation). It is a second-order partial differential equation (PDE) which contains the **e** field only.

- Analogously, one can obtain the wave equation for the **h** field:

$$\nabla^2 \mathbf{h} - \mu\epsilon \frac{\partial^2 \mathbf{h}}{\partial t^2} = 0 \qquad (25)$$

Solutions of D'Alembert's equation

Note that the solutions of D'Alembert's equation are referred to as waves or wave functions and they can have quite different physical dimensions and meanings.

The wave equation

- It must be explicitly pointed out that, since Maxwell's equations are first-order PDEs, only a linear combination of the solutions of the wave equation (2nd order differential equation) will be solution for the Maxwell's equations.
- Spurious solutions are filtered out using divergence equations (10-11).
- Em wave is often taken as synonymous with em field, in the fast time-varying regime. However, it must be explicitly pointed out that wave equation can be derived from Maxwell's equations under certain assumptions.

Wave functions

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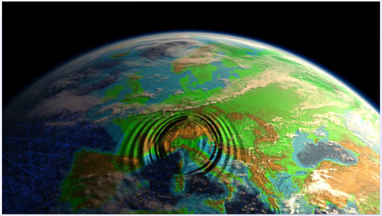

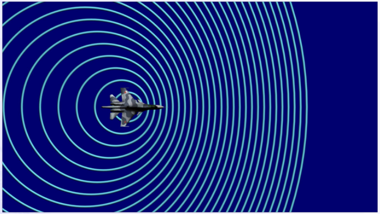

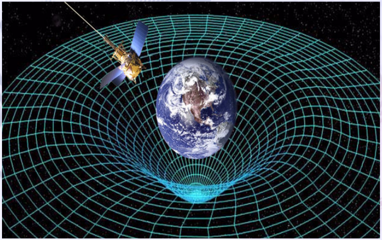

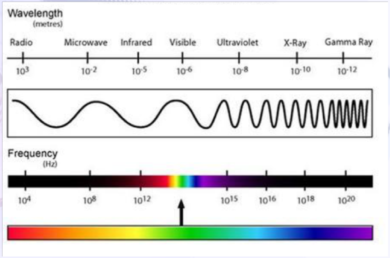

Electromagnetic waves

- Eq.(24) is satisfied by any regular function of the following type:

$$\mathbf{e}(\mathbf{r}, t) = \mathbf{e}(\mathbf{r} - \hat{\mathbf{r}} v_f t) \qquad (26)$$

- $v_f = \frac{1}{\sqrt{\mu\epsilon}}$ has the dimension of a velocity and is called phase velocity.
- In the vacuum, $v_f = \frac{1}{\sqrt{\mu_o \epsilon_o}} = c \approx 3 \cdot 10^8 \text{ ms}^{-1}$ is the speed of light.

Propagation

Eq.(26) describes a propagation phenomenon, i.e.; a function that travels unchanged in the direction $\hat{r}$ with velocity v_f

The scalar wave equation

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e and **h** are vector fields

- $\mathbf{e} = (e_x, e_y, e_z)$;
- $\mathbf{h} = (h_x, h_y, h_z)$.

Each component should satisfy the scalar wave equation:

$$\nabla^2 \psi - \frac{1}{v_f^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{27}$$

Potentials

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To solve practical problems (in particular radiation problems) it is often convenient using auxiliary functions:

- the scalar electric potential, v;
- the vector magnetic potential, **a**.

Helmholtz's partition theorem

At the very root the potentials rely on the fact that a given vector is completely specified once its irrotational and solenoidal parts are specified.

Vector potential

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The vector potential **a** is defined by:

$$\mathbf{b} = \nabla \times \mathbf{a} \tag{28}$$

By substituting (28) in the Maxwell's equation:

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t}$$

one obtains:

$$\begin{aligned} \nabla \times \mathbf{e} &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{a}) \\ \nabla \times \left(\mathbf{e} + \frac{\partial \mathbf{a}}{\partial t}\right) &= 0 \end{aligned} \tag{29}$$

Vector potential

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Eq.(29) represents an irrotational field. Hence it can be written as the gradient of a scalar function v:

$$\mathbf{e} + \frac{\partial \mathbf{a}}{\partial t} = -\nabla v$$

Hence:

$$\mathbf{e} = -\nabla v - \frac{\partial \mathbf{a}}{\partial t} \tag{30}$$

The role of auxiliary functions

If one knows the potential functions **a** and v, the em field can be obtained using (30) and (28)

Wave equation

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The wave equation for **a** and v can be obtained starting from Maxwell's equations and considering a simple medium:

$$\nabla \times \mathbf{h} = \epsilon \frac{\partial \mathbf{e}}{\partial t} + \mathbf{j}_o$$

Using (28) and (30) one obtains:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{a} &= \epsilon\mu \frac{\partial \mathbf{e}}{\partial t} + \mu \mathbf{j}_o = \mu\epsilon \left(\frac{\partial}{\partial t} \left(-\nabla v - \frac{\partial \mathbf{a}}{\partial t} \right) \right) + \mu \mathbf{j}_o \\ &= -\mu\epsilon \nabla \frac{\partial v}{\partial t} - \mu\epsilon \frac{\partial^2 \mathbf{a}}{\partial t^2} + \mu \mathbf{j}_o \end{aligned}$$

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Wave equation

Using the equality $\nabla \times \nabla \times \mathbf{a} = \nabla\nabla \cdot \mathbf{a} - \nabla^2 \mathbf{a}$:

$$\nabla\nabla \cdot \mathbf{a} - \nabla^2 \mathbf{a} + \mu\epsilon\nabla\frac{\partial v}{\partial t} + \mu\epsilon\frac{\partial^2 \mathbf{a}}{\partial t^2} = \mu\mathbf{j}_o$$

$$\nabla^2 \mathbf{a} - \mu\epsilon\frac{\partial^2 \mathbf{a}}{\partial t^2} = \nabla\left(\nabla \cdot \mathbf{a} + \mu\epsilon\frac{\partial v}{\partial t}\right) - \mu\mathbf{j}_o \quad (31)$$

According to the Helmholtz's partition theorem, to completely specify a well-behaved vector field its curl and divergence are due.
Up to now the curl of **a** has been specified; hence a degree of freedom is still available to fix its divergence.

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Hence, the divergence can be chosen to simplify (31):

$$\nabla \cdot \mathbf{a} = -\mu\epsilon \frac{\partial v}{\partial t} \tag{32}$$

Eq.(32) is called Lorentz's gauge. Hence, eq.(31) can be rewritten as:

$$\nabla^2 \mathbf{a} - \mu\epsilon \frac{\partial^2 \mathbf{a}}{\partial t^2} = -\mu \mathbf{j}_o \tag{33}$$

This is the inhomogeneous wave equation for the vector potential **a**

Wave equation

To derive the wave equation for the scalar potential v the divergence equation must be considered:

$$\nabla \cdot \mathbf{e} = \frac{\rho}{\epsilon}$$

Since $\mathbf{e} = -\nabla v - \frac{\partial \mathbf{a}}{\partial t}$:

$$\nabla \cdot \left(-\nabla v - \frac{\partial \mathbf{a}}{\partial t} \right) = \frac{\rho}{\epsilon}$$

$$\nabla^2 v + \frac{\partial}{\partial t} \nabla \cdot \mathbf{a} = -\frac{\rho}{\epsilon}$$

Using the Lorentz's gauge:

$$\nabla^2 v - \mu\epsilon \frac{\partial^2 v}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{34}$$

This is the inhomogeneous wave equation for v

Retarded potential

Integral solutions of the two wave equations (33) and (34) are the so-called retarded potentials

$$\mathbf{a} = \int_\tau \frac{\mu[\mathbf{j}]}{4\pi R} \mathrm{d}\tau$$

$$v = \int_\tau \frac{[\rho]}{4\pi\epsilon R} \mathrm{d}\tau$$

Retarded potentials

They are called retarded potentials because $[\mathbf{j}]$ and $[\rho]$, i.e. the source terms, are specified at a time $\frac{R}{\sqrt{\mu\epsilon}}$ earlier than the time $\mathbf{a}$ and v are being determined.

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Time-harmonic regime

Maxwell's equations in simple media form a linear system; hence no generality is lost by considering the "monochromatic" or "steady-state" regime, in which all the quantities are simply periodic in time, i.e. time-harmonic.

Fourier's theorem

Note that by Fourier's theorem, any linear field of arbitrary time-dependence can be synthesized from the knowledge of the monochromatic field.

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Phasors

In the time-harmonic regime, $\mathbf{e}(\mathbf{r},t)$, $\mathbf{h}(\mathbf{r},t)$, $\mathbf{d}(\mathbf{r},t)$, $\mathbf{b}(\mathbf{r},t)$, $\mathbf{j}(\mathbf{r},t)$ and $\rho(\mathbf{r},t)$, vary sinusoidally in time with an angular frequency ω.

The one-to-one mapping between the set of time-harmonic vectors in $\mathbb{R}^3$ and the complex-vector space $\mathbb{C}^3$ can be exploited (Steinmetz's representation):
Let $f(\mathbf{r},t) = a(\mathbf{r})\cos(\omega t + \phi(\mathbf{r}))$ a scalar time-harmonic field whose angular frequency ω is fixed. According to Euler's formula:

$$a(\mathbf{r})e^{j(\omega t+\phi(\mathbf{r}))} = a(\mathbf{r})cos(\omega t + \phi(\mathbf{r})) + ja(\mathbf{r})sin(\omega t + \phi(\mathbf{r}))$$

Hence:

$$f(\mathbf{r},t) = a(\mathbf{r})\cos(\omega t + \phi(\mathbf{r})) = \Re\{a(\mathbf{r})e^{j\phi(\mathbf{r})}e^{j\omega t}\}$$

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Phasors

- Hence:

$$f(\mathbf{r},t) = \Re\{\dot{F}(\mathbf{r})e^{j\omega t}\}$$

- where:

$$\dot{F}(\mathbf{r}) = a(\mathbf{r})e^{j\phi(\mathbf{r})} \quad (35)$$

it is called phasor and it is a complex number characterized by a one-to-one relationship with a time-harmonic signal of angular frequency ω.

- When a vector field is considered $\mathbf{f}(\mathbf{r},t)$:

$$\mathbf{f}(\mathbf{r},t) = \mathbf{a}(\mathbf{r})\cos(\omega t + \phi(\mathbf{r})) = \Re\{\mathbf{a}(\mathbf{r})e^{j\phi(\mathbf{r})}e^{j\omega t}\}$$

-

$$\dot{\mathbf{F}}(\mathbf{r}) = \mathbf{a}(\mathbf{r})e^{j\phi(\mathbf{r})} = a_x(\mathbf{r})e^{j\phi_x(\mathbf{r})}\hat{x} + a_y(\mathbf{r})e^{j\phi_y(\mathbf{r})}\hat{y} + a_z(\mathbf{r})e^{j\phi_z(\mathbf{r})}\hat{z} \quad (36)$$

is the generalized phasor associated with the vectorial time-harmonic field.

Phasors

- Note that to reduce the system to the monochromatic state, the $e^{j\omega t}$ time dependence is adopted, which implies that the following Fourier transforms pair is understood:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\mathrm{d}t \tag{37}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}\mathrm{d}\omega \tag{38}$$

- Note that phasors are indicated using dotted capital letters. The only exception is the charge density scalar field function.
- Phasors have the same physical dimension of the un-transformed field functions.
- Phasors depend on the spatial coordinate only.

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Steady state Maxwell's equations

Maxwell's equations in sinusoidal steady-state

$$\nabla \times \dot{\mathbf{E}}(\mathbf{r}) = -j\omega\dot{\mathbf{B}}(\mathbf{r}) \tag{39}$$

$$\nabla \times \dot{\mathbf{H}}(\mathbf{r}) = j\omega\dot{\mathbf{D}}(\mathbf{r}) + \dot{\mathbf{J}}(\mathbf{r}) \tag{40}$$

$$\nabla \cdot \dot{\mathbf{D}}(\mathbf{r}) = \dot{\rho}(\mathbf{r}) \tag{41}$$

$$\nabla \cdot \dot{\mathbf{B}}(\mathbf{r}) = 0 \tag{42}$$

Assuming a simple medium, the constitutive relationships are given by:

$$\dot{\mathbf{D}}(\mathbf{r}) = \epsilon\dot{\mathbf{E}}(\mathbf{r})$$

$$\dot{\mathbf{B}}(\mathbf{r}) = \mu\dot{\mathbf{H}}(\mathbf{r})$$

$$\dot{\mathbf{J}}(\mathbf{r}) = \sigma\dot{\mathbf{E}}(\mathbf{r})$$

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Remarks

- Note that phasors have the same dimensions of the un-transformed fields.
- Note that ϵ, μ, σ are constants since a simple medium is considered.
- In general ϵ, μ, σ are phasors.
- Hereinafter, to simplify the notation, the dot symbol which indicates phasors is omitted without ambiguity.

Potentials

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Given a time-harmonic source defined by an electric density $\mathbf{J}_o(\mathbf{r})$, the em field generated by this source in the free space satisfies Maxwell's equations:

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu\mathbf{H}(\mathbf{r}) \tag{43}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\epsilon\mathbf{E}(\mathbf{r}) + \mathbf{J}_o(\mathbf{r}) \tag{44}$$

$$\nabla \cdot \epsilon\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r}) \tag{45}$$

$$\nabla \cdot \mu\mathbf{H}(\mathbf{r}) = 0 \tag{46}$$

Coupling

It can be noted that the electric and magnetic fields are coupled in these equations. Moreover, the degree of coupling depends on the frequency.

Static case

When the frequency approaches 0, the static case is achieved.

- The electrostatic field produced by electric charges is governed by:

$$\nabla \times \mathbf{E} = 0 \quad , \quad \nabla \cdot \epsilon \mathbf{E} = \rho \tag{47}$$

- The magnetostatic field produced by electric currents is governed by:

$$\nabla \times \mathbf{H} = \mathbf{J}_o \quad , \quad \nabla \cdot \mu \mathbf{H} = 0 \tag{48}$$

Note that, to simplify the notation, the phasors' space dependence is omitted.

Electrostatic field

To solve eq.(47), i.e. two first-order PDEs, for a single unknown vector function **E**, it must be noted that:

- **E** is an irrotational vector function and, therefore, it can be expressed as the gradient of a scalar function V which is called electric scalar potential:

$$\nabla \times \mathbf{E} = 0 \rightarrow \mathbf{E} = -\nabla V \tag{49}$$

- Considering eq.(45), one obtains:

$$-\nabla \cdot \epsilon \nabla V = \rho \tag{50}$$

- It is a second-order PDE that, in a homogeneous medium, becomes:

$$\nabla^2 V = -\frac{\rho}{\epsilon} \tag{51}$$

- It is called Poisson's equation.

Magnetostatic field

To solve eq.(48), i.e. two first-order PDEs, for a single unknown vector function **H**, it must be noted that eq.(46) implies that:

- $\mathbf{B} = \mu\mathbf{H}$ is a solenoidal vector function; hence it can be expressed as the curl of a vector function:

$$\mathbf{B} = \mu\mathbf{H} = \nabla \times \mathbf{A} \tag{52}$$

- The vector function **A** is called magnetic vector potential.
- Substituting eq.(52) in $\nabla \times \mathbf{H} = \mathbf{J}_o$ one obtains:

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{A}\right) = \mathbf{J}_o \tag{53}$$

Helmholtz's partition theorem

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Eq.(53) is a second-order PDE that, for a homogeneous medium, becomes:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A} &= \mu \mathbf{J}_o \\ \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} &= \mu \mathbf{J}_o \end{aligned} \tag{54}$$

- Since **A** is a vector field, according to the Helmholtz's partition theorem, to completely specify **A** its curl and divergence are due.
- This is obvious if one consider eq.(53). In fact, it can be easily proven that this equation is satisfied by **A** but also by $\mathbf{A} + \nabla f$ (Note that $\nabla \times \nabla f = 0$).

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Helmholtz's partition theorem

Since up to now only the curl of **A** has been specified through eq.(52), to uniquely determine **A** its divergence must be specified.

- With the intent to simplify eq.(54), one may set the divergence of **A** to zero (Coulomb gauge condition):

$$\nabla \cdot \mathbf{A} = 0 \tag{55}$$

- Hence, eq.(54) becomes:

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}_o \tag{56}$$

- It is a vector Poisson's equation

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Electrodynamics fields

Considering the Maxwell's equation (43)-(46)

- Eq.(46) implies that:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{57}$$

Hence:

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega \nabla \times \mathbf{A} \\ \nabla \times (\mathbf{E} + j\omega \mathbf{A}) &= 0 \end{aligned} \tag{58}$$

- Eq.(58) can be satisfied introducing the electric scalar potential V:

$$\mathbf{E} + j\omega \mathbf{A} = -\nabla V \tag{59}$$

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Electrodynamics fields

As a matter of fact, **E** can be obtained once **A** and V are known:

$$\mathbf{E} = -\nabla V - j\omega \mathbf{A} \tag{60}$$

- To obtain the Helmholtz's equation for the vector potential **A**, eq.(60) is substituted into eq.(44):

$$\begin{aligned}
\frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} &= j\omega\epsilon\left(-\nabla V - j\omega \mathbf{A}\right) + \mathbf{J}_o \\
\nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} &= -j\omega\epsilon\mu\nabla V + \omega^2\epsilon\mu\mathbf{A} + \mu\mathbf{J}_o \\
\nabla^2\mathbf{A} + \omega^2\epsilon\mu\mathbf{A} &= \nabla\left(\nabla \cdot \mathbf{A} + j\omega\epsilon\mu V\right) - \mu\mathbf{J}_o
\end{aligned} \tag{61}$$

Helmholtz's equation

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As far as for the magnetostatic case, only the curl of the magnetic vector potential **A** is specified, see eq.(57). Hence, its divergence can be specified to simplify eq.(61) without affecting the field itself.

- By choosing (Lorentz gauge condition):

$$\nabla \cdot \mathbf{A} = -j\omega\epsilon\mu V \tag{62}$$

one obtains:

$$\nabla^2 \mathbf{A} - k^2 \mathbf{A} = -\mu \mathbf{J} \tag{63}$$

- It is the vector Helmholtz's equation with $k^2 = -\omega^2\mu\epsilon$ and its roots $\pm k$ define the propagation constant.

Helmholtz's equation

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Poisson's and Laplace's equations are special cases of the Helmholtz equation.

$$\nabla^2 \mathbf{A} - k^2 \mathbf{A} = G \tag{64}$$

where G is the source term.

- When $k = 0$, i.e. $\omega = 0$ static case, the Poisson's equation is achieved:

$$\nabla^2 \mathbf{A} = G$$

- When $k = G = 0$, the Laplace's equation is achieved:

$$\nabla^2 \mathbf{A} = 0$$

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Helmholtz's equation

Once **A** is known, the fields **H** and **E** can be easily obtained:

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \tag{65}$$

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla V = -j\omega\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\epsilon\mu} \tag{66}$$

Gauge condition

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It must be noted that both in the static and in the dynamic cases, the specification of the divergence of **A** is simply for a unique determination of **A** itself.

Gauge

Since **A** is an auxiliary function, its uniqueness is not important. Even if **A** is not unique, due to $\mu\mathbf{H} = \nabla \times \mathbf{A}$, **H** will be always unique!
The divergence of **A** does not affect the solution to the magnetic field **H**; hence it can be specified arbitrarily: Gauge condition.
Same comments apply for **a**.

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Why using potential functions ?

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- Helmholtz's equation was derived for the potential **A**, see eq.(63).
- However, one can derive Helmholtz's equation directly from Maxwell's equation (43)-(46):

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{E} &= -j\omega\mu\nabla \times \mathbf{H} = -j\omega\mu\left(j\omega\epsilon\mathbf{E} + \mathbf{J}_o\right) \\
&= \omega^2\epsilon\mu\mathbf{E} - j\omega\mu\mathbf{J}_o \\
\nabla\nabla \cdot \mathbf{E} - \nabla^2\mathbf{E} &= \omega^2\epsilon\mu\mathbf{E} - j\omega\mu\mathbf{J}_o \\
\nabla^2\mathbf{E} + \omega^2\epsilon\mu\mathbf{E} &= \nabla\nabla \cdot \mathbf{E} + j\omega\mu\mathbf{J}_o \qquad (67)
\end{aligned}$$

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Why using potential functions ?

- Since, for a homogeneous medium:

$$\begin{aligned} \nabla \cdot \nabla \times \mathbf{H} &= j\omega\epsilon\nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{J}_o \\ 0 &= j\omega\epsilon\nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{J}_o \\ \nabla \cdot \mathbf{E} &= -\frac{\nabla \cdot \mathbf{J}_o}{j\omega\epsilon} \end{aligned} \tag{68}$$

Comments

This equation means that, inside a homogeneous medium, the divergence of the electric field can differ from zero only either where the flow lines of the imposed current are open, or at the boundary of the medium.

Hence:

$$\nabla^2\mathbf{E} - k^2\mathbf{E} = -\frac{\nabla\nabla \cdot \mathbf{J}_o}{j\omega\epsilon} + j\omega\mu\mathbf{J}_o \tag{69}$$

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Why using potential functions ?

On the surface, eq.(69) is more complicated than eq.(63). However, subtle differences exist.

- Their left-hand side operators are exactly the same; hence solutions of the same form are expected, as actually is.
- The solution of eq.(63) in the free space is given by:

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu}{4\pi} \iiint_\tau \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} \mathrm{d}\tau \\ \mathbf{E} &= -j\omega \mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\epsilon\mu} \end{aligned} \tag{70}$$

- When eq.(69) is accounted for:

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi} \iiint_\tau \left\{ j\omega\mu\mathbf{J}(\mathbf{r}') - \frac{1}{j\omega\epsilon} \nabla\left(\nabla\cdot\mathbf{J}(\mathbf{r}')\right) \right\} \frac{e^{-jkR}}{R} \mathrm{d}\tau \tag{71}$$

Why using potential functions ?

Mathematically, the two approaches to evaluate the fields from given sources involve the same number of calculations:

- A volume integral.
- Differential operators.

Differences

The main and subtle difference is that in eq.(71) the differential operators are applied to the source function; whereas in eq.(70) these operations are applied to the vector potential.

Why using potential functions ?

For source functions that are analytic the two approaches are indeed equivalent.

- Unfortunately, in many practical cases, source functions do not have such a behavior, e.g. line current and surface current. In such cases, they need to be expressed in terms of Dirac delta functions and, hence, the generalized functions must be used to evaluate differential operators.
- The vector potential function is always analytic in $\mathbf{r}$; hence differential operators can be applied straightforwardly.

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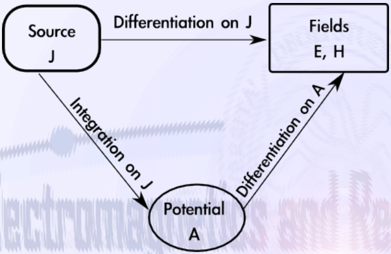

With the introduction of auxiliary potential functions the requirement on the form of the source functions is significantly relaxed, making the approach operationally interesting.

For further reading

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