Formulario di Teoria dei Fenomeni Aleatori

Formule trigonometriche

$\sin(\alpha+\beta)$	$\sin(\alpha)\,\cos(\beta)+\cos(\alpha)\,\sin(\beta)$
$\sin(\alpha-\beta)$	$\sin(\alpha)\,\cos(\beta)-\cos(\alpha)\,\sin(\beta)$
$\cos(\alpha+\beta)$	$\cos(\alpha)\,\cos(\beta)-\sin(\alpha)\,\sin(\beta)$
$\cos(\alpha-\beta)$	$\cos(\alpha)\,\cos(\beta)+\sin(\alpha)\,\sin(\beta)$
$\tan(\alpha+\beta)$	$\frac{\tan(\alpha)+\tan(\beta)}{1-\tan\alpha\,\tan\beta}$
$\tan(\alpha-\beta)$	$\frac{\tan(\alpha)-\tan(\beta)}{1+\tan\alpha\,\tan\beta}$
$\sin(2\,\alpha)$	$2\,\sin(\alpha)\,\cos(\beta)$
$\cos(2\,\alpha)$	$\cos^2(\alpha)-\sin^2(\alpha)$
$\cos(2\,\alpha)$	$\cos^2(\alpha)-\sin^2(\alpha)$
$\tan(2\,\alpha)$	$\frac{2\,\tan(\alpha)}{1-\tan^2\alpha}$
$\sin(\alpha/2)$	$\pm\sqrt{\frac{1-\cos(\alpha)}{2}}$
$\cos(\alpha/2)$	$\pm\sqrt{\frac{1+\cos(\alpha)}{2}}$
$\tan(\alpha/2)$	$\frac{1-\cos(\alpha)}{\sin(\alpha)}=\frac{\sin(\alpha)}{1+\cos(\alpha)}$

Proprietà della Trasformata tempo-continua di Fourier e trasformate notevoli

1. Simmetria hermitiana della trasformata:

$$x(t)\ \text{reale} \quad \Longleftrightarrow \quad X^*(f)=X(-f)\,.$$

2. Valore nell'origine:

$$X(0)=\int_{-\infty}^{+\infty} x(t)\,\mathrm{d}t\,, \qquad x(0)=\int_{-\infty}^{+\infty} X(f)\,\mathrm{d}f\,.$$

3. **Dualità:**

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f)\,,$$
$$X(t) \stackrel{\mathcal{F}}{\longleftrightarrow} x(-f)\,.$$

4. **Simmetria:**

$$x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(-f)\,,$$
$$x^*(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(-f)\,,$$
$$x^*(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(f)\,.$$

5. **Traslazione nel tempo:**

$$y(t) = x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f) = X(f)\,e^{-j2\pi f t_0}\,, \quad \forall t_0 \in \mathbb{R}\,.$$

6. **Traslazione in frequenza:**

$$y(t) = x(t)\,e^{j2\pi f_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f) = X(f-f_0)\,, \quad \forall f_0 \in \mathbb{R}\,.$$

7. **Modulazione:**

$$y(t) = x(t)\,\cos(2\pi f_0 t + \varphi_0) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f) = \frac{1}{2}\,e^{j\varphi_0}X(f-f_0) + \frac{1}{2}e^{-j\varphi_0}X(f+f_0), \quad \forall f_0, \varphi_0 \in \mathbb{R}.$$

8. **Cambiamento di scala:**

$$y(t) = x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f) = \frac{1}{|a|}X\left(\frac{f}{a}\right)\,, \quad \forall a \in \mathbb{R} - \{0\}\,.$$

9. **Derivazione nel tempo:**

$$y(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\,x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f) = (j2\pi f)^k X(f)\,, \quad \forall k \in \mathbb{N}\,.$$

10. **Derivazione in frequenza:**

$$(-t)^k\,x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{(j2\pi)^k}\frac{\mathrm{d}^k}{\mathrm{d}f^k}X(f)\,, \quad \forall k \in \mathbb{N}\,.$$

11. **Integrazione:**

$$y(t) = \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f) = \frac{1}{2}\,X(0)\,\delta(f) + \frac{X(f)}{j2\pi f}\,.$$

12. **Convoluzione:**

$$z(t) = x(t) * y(t) = \int_{-\infty}^{+\infty} x(\tau)\,y(t-\tau)\,\mathrm{d}\tau \stackrel{\mathcal{F}}{\longleftrightarrow} Z(f) = X(f)\,Y(f)\,.$$

13. Prodotto:

$$z(t) = x(t)\,y(t) \overset{\text{FS}}{\longleftrightarrow} Z(f) = X(f) * Y(f) = \int_{-\infty}^{+\infty} X(\lambda)\,Y(f-\lambda)\,\mathrm{d}\lambda\,.$$

14. Parseval:

$$\mathcal{E}_x = \int_{-\infty}^{+\infty} |x(t)|^2\,\mathrm{d}t = \int_{-\infty}^{+\infty} |X(f)|^2\,\mathrm{d}f, \qquad \mathcal{E}_{xy} = \int_{-\infty}^{+\infty} x(t)\,y^*(t)\mathrm{d}t = \int_{-\infty}^{+\infty} X(f)\,Y^*(f)\,\mathrm{d}f.$$

15. Replicazione/campionamento:

$$y(t) = \sum_{k=-\infty}^{+\infty} x(t-kT_0) \overset{\mathcal{F}}{\longleftrightarrow} Y(f) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_0}\right)\delta\left(f-\frac{k}{T_0}\right)\,, \quad \forall T_0 \in \mathbb{R}_+\,,$$

$$y(t) = \sum_{k=-\infty}^{+\infty} x(kT_0)\,\delta(t-kT_0) \overset{\mathcal{F}}{\longleftrightarrow} Y(f) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X\left(f-\frac{k}{T_0}\right)\,, \quad \forall T_0 \in \mathbb{R}_+\,.$$

16. Formule di Poisson:

$$\sum_{k=-\infty}^{+\infty} x(t-kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_0}\right) e^{j2\pi\frac{k}{T_0}t}\,, \quad \forall T_0 \in \mathbb{R}_+ \qquad \text{(prima formula di Poisson)}\,,$$

$$\sum_{k=-\infty}^{+\infty} x(kT_0) e^{-j2\pi\frac{k}{T_0}t} = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X\left(f-\frac{k}{T_0}\right)\,, \quad \forall T_0 \in \mathbb{R}_+ \qquad \text{(seconda formula di Poisson)}\,.$$

17. Trasformata di Fourier di un segnale periodico e campionamento in frequenza:

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k\, e^{j2\pi\frac{k}{T_0}t} \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \sum_{k=-\infty}^{+\infty} X_k\,\delta\left(f-\frac{k}{T_0}\right)$$

$$x(t) = \mathrm{rep}_{T_0}[x_{\mathrm{g}}(t)] \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_{\mathrm{g}}\left(\frac{k}{T_0}\right)\delta\left(f-\frac{k}{T_0}\right)$$

$$X_k = \frac{1}{T_0} X_{\mathrm{g}}\left(\frac{k}{T_0}\right)$$

18. **Trasformate notevoli:**

$x(t)$	$X(f)$
$\delta(t)$	1
1	$\delta(f)$
$\mathrm{u}(t)$	$\frac{1}{2}\,\delta(f) + \frac{1}{j2\pi f}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{t}$	$-j\pi \mathrm{sgn}(f)$
$\mathrm{rect}(t)$	$\mathrm{sinc}(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}(t)$	$\mathrm{rect}(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$
$e^{-at}\,\mathrm{u}(t)$, $a \in \mathbb{R}_+$	$\frac{1}{a + j2\pi f}$
$t\,e^{-at}\mathrm{u}(t)$, $a \in \mathbb{R}_+$	$\frac{1}{(a + j2\pi f)^2}$
$e^{-a\|t\|}$, $a \in \mathbb{R}_+$	$\frac{2a}{a^2 + (2\pi f)^2}$
$e^{j2\pi f_0 t}$	$\delta(f - f_0)$
$\cos(2\pi f_0 t + \varphi_0)$	$\frac{1}{2}\,e^{j\varphi_0}\,\delta(f - f_0) + \frac{1}{2}\,e^{-j\varphi_0}\,\delta(f + f_0)$
$\sin(2\pi f_0 t + \varphi_0)$	$\frac{1}{2j}\,e^{j\varphi_0}\,\delta(f - f_0) - \frac{1}{2j}\,e^{-j\varphi_0}\,\delta(f + f_0)$
$\sum_{k=-\infty}^{+\infty} \delta(t - kT_0)$, $T_0 \in \mathbb{R}_+$	$\frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_0}\right)$

Proprietà della Trasformata tempo-discreta di Fourier e trasformate notevoli

1. **Simmetria hermitiana della trasformata:**

$$x(n) \text{ reale} \quad \Longleftrightarrow \quad X^*(\nu) = X(-\nu)\,.$$

2. **Valore nell'origine:**

$$X(0) = \sum_{n=-\infty}^{+\infty} x(n)\,, \qquad x(0) = \int_{-1/2}^{1/2} X(\nu)\,\mathrm{d}\nu\,.$$

3. **Simmetria:**

$$\begin{aligned} x(-n) &\overset{\mathcal{F}}{\longleftrightarrow} X(-\nu)\,, \\ x^*(n) &\overset{\mathcal{F}}{\longleftrightarrow} X^*(-\nu)\,, \\ x^*(-n) &\overset{\mathcal{F}}{\longleftrightarrow} X^*(\nu)\,. \end{aligned}$$

4. Traslazione nel tempo:

$$y(n) = x(n - n_0) \overset{\mathcal{F}}{\longleftrightarrow} Y(\nu) = X(\nu)\, e^{-j2\pi\nu n_0}\,, \quad \forall n_0 \in \mathbb{Z}\,.$$

5. Traslazione in frequenza:

$$y(n) = x(n)\, e^{j2\pi\nu_0 n} \overset{\mathcal{F}}{\longleftrightarrow} Y(\nu) = X(\nu - \nu_0)\,, \quad \forall \nu_0 \in \mathbb{R}\,.$$

6. Modulazione:

$$y(n) = x(n)\, \cos(2\pi\nu_0 n + \varphi_0) \overset{\mathcal{F}}{\longleftrightarrow} Y(\nu) = \frac{1}{2}\, e^{j\varphi_0} X(\nu - \nu_0) + \frac{1}{2} e^{-j\varphi_0} X(\nu + \nu_0), \quad \forall \nu_0, \varphi_0 \in \mathbb{R}.$$

7. Espansione:

$$y(n) = x\left[\frac{n}{L}\right] \overset{\mathcal{F}}{\longleftrightarrow} Y(\nu) = X(L\nu)\,, \quad \forall L \in \mathbb{N}\,.$$

8. Decimazione:

$$y(n) = x(nM) \overset{\mathcal{F}}{\longleftrightarrow} Y(\nu) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\nu - k}{M}\right)\,, \quad \forall M \in \mathbb{N}\,.$$

9. Differenza:

$$\nabla_k[x(n)] = \nabla_1 \left\{\nabla_{k-1}[x(n)]\right\} \overset{\mathcal{F}}{\longleftrightarrow} Y(\nu) = \left(1 - e^{-j2\pi\nu}\right)^k X(\nu)\,, \quad \forall k \in \mathbb{N}\,.$$

10. Derivazione in frequenza:

$$(-n)^k\, x(n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(j2\pi)^k} \frac{\mathrm{d}^k}{\mathrm{d}\nu^k} X(\nu)\,, \quad \forall k \in \mathbb{N}\,.$$

11. Somma:

$$y(n) = \sum_{k=-\infty}^{n} x(k) \overset{\mathcal{F}}{\longleftrightarrow} Y(\nu) = \frac{1}{2}\, X(0)\, \widetilde{\delta}(\nu) + \frac{X(\nu)}{1 - e^{-j2\pi\nu}}\,.$$

12. Convoluzione:

$$z(n) = x(n) * y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, y(n - k) \overset{\mathcal{F}}{\longleftrightarrow} Z(\nu) = X(\nu)\, Y(\nu)\,.$$

13. Prodotto:

$$z(n) = x(n)\, y(n) \overset{\mathcal{F}}{\longleftrightarrow} Z(\nu) = X(\nu) * Y(\nu) = \int_{-1/2}^{1/2} X(\lambda)\, Y(\nu - \lambda)\, \mathrm{d}\lambda\,.$$

14. Parseval:

$$\mathcal{E}_x = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \int_{-1/2}^{1/2} |X(\nu)|^2\, \mathrm{d}\nu, \qquad \mathcal{E}_{xy} = \sum_{n=-\infty}^{+\infty} x(n)\, y^*(n) = \int_{-1/2}^{1/2} X(\nu)\, Y^*(\nu)\, \mathrm{d}\nu.$$

15. Replicazione/campionamento:

$$y(n) = \sum_{k=-\infty}^{+\infty} x(n - kN_0) \xleftrightarrow{\mathcal{F}} Y(\nu) = \frac{1}{N_0} \sum_{k=0}^{N_0-1} X\left(\frac{k}{N_0}\right) \widetilde{\delta}\left(\nu - \frac{k}{N_0}\right) , \quad \forall N_0 \in \mathbb{N} ,$$

$$y(n) = \sum_{k=-\infty}^{+\infty} x(kN_0)\, \delta(n - kN_0) \xleftrightarrow{\mathcal{F}} Y(\nu) = \frac{1}{N_0} \sum_{k=0}^{N_0-1} X\left(\nu - \frac{k}{N_0}\right) , \quad \forall N_0 \in \mathbb{N} .$$

16. Formule di Poisson:

$$\sum_{k=-\infty}^{+\infty} x(n - kN_0) = \frac{1}{N_0} \sum_{k=0}^{N_0-1} X\left(\frac{k}{N_0}\right) e^{j2\pi \frac{k}{N_0} n} , \quad \forall N_0 \in \mathbb{N} ,$$

$$\sum_{k=-\infty}^{+\infty} x(kN_0) e^{-j2\pi \frac{k}{N_0} n} = \frac{1}{N_0} \sum_{k=0}^{N_0-1} X\left(\nu - \frac{k}{N_0}\right) , \quad \forall N_0 \in \mathbb{N} .$$

17. Trasformata di Fourier di un segnale periodico e campionamento in frequenza:

$$x(n) = \frac{1}{N_0} \sum_{k=0}^{N_0-1} X(k)\, e^{j2\pi \frac{k}{N_0} n} \xleftrightarrow{\mathcal{F}} X(\nu) = \frac{1}{N_0} \sum_{k=0}^{N_0-1} X(k)\, \widetilde{\delta}\left(\nu - \frac{k}{N_0}\right)$$

$$x(n) = \mathrm{rep}_{N_0}[x_{\mathrm{g}}(n)] \xleftrightarrow{\mathcal{F}} X(\nu) = \frac{1}{N_0} \sum_{k=0}^{N_0-1} X_{\mathrm{g}}\left(\frac{k}{N_0}\right) \widetilde{\delta}\left(\nu - \frac{k}{N_0}\right)$$

$$X(k) = X_{\mathrm{g}}\left(\frac{k}{N_0}\right)$$

18. Trasformate notevoli:

$x(n)$	$X(\nu)$
$\delta(n)$	1
1	$\widetilde{\delta}(\nu)$
$\mathrm{u}(n)$	$\frac{1}{2}\,\widetilde{\delta}(\nu) + \frac{1}{1-e^{-j2\pi\nu}}$
$\mathrm{sgn}(n)$	$\frac{2}{1-e^{-j2\pi\nu}}$
$\mathcal{R}_N(n)$	$\mathcal{D}_N(\nu)$
$\mathcal{B}_{2N}(n)$	$\frac{1}{N}\,\mathcal{D}_N^2(\nu)\,e^{-j2\pi\nu}$
$\mathrm{sinc}(2\nu_c n)$, $0 < \nu_c < \frac{1}{2}$	$\frac{1}{2\nu_c}\mathrm{rep}_1\left[\mathrm{rect}\left(\frac{\nu}{2\nu_c}\right)\right]$
$\mathrm{sinc}^2(2\nu_c n)$, $0 < \nu_c < \frac{1}{2}$	$\frac{1}{2\nu_c}\mathrm{rep}_1\left[\Lambda\left(\frac{\nu}{2\nu_c}\right)\right]$
$a^n\mathrm{u}(n)$, $\|a\| < 1$	$\frac{1}{1-a\,e^{-j2\pi\nu}}$
$(n+1)\,a^n\mathrm{u}(n)$, $\|a\| < 1$	$\frac{1}{(1-a\,e^{-j2\pi\nu})^2}$
$a^{\|n\|}$, $\|a\| < 1$	$\frac{1-a^2}{1+a^2-2a\,\cos(2\pi\nu)}$
$e^{j2\pi\nu_0 n}$	$\widetilde{\delta}(\nu-\nu_0)$
$\cos(2\pi\nu_0 n+\varphi_0)$	$\frac{1}{2}\,e^{j\varphi_0}\,\widetilde{\delta}(\nu-\nu_0) + \frac{1}{2}\,e^{-j\varphi_0}\,\widetilde{\delta}(\nu+\nu_0)$
$\sin(2\pi\nu_0 n+\varphi_0)$	$\frac{1}{2j}\,e^{j\varphi_0}\,\widetilde{\delta}(\nu-\nu_0) - \frac{1}{2j}\,e^{-j\varphi_0}\,\widetilde{\delta}(\nu+\nu_0)$
$\sum_{k=-\infty}^{+\infty}\delta(n-kN_0)$, $N_0\in\mathbb{N}$	$\frac{1}{N_0}\sum_{k=0}^{N_0-1}\widetilde{\delta}\left(\nu-\frac{k}{N_0}\right) = \frac{1}{N_0}\sum_{k=-\infty}^{+\infty}\delta\left(\nu-\frac{k}{N_0}\right)$

Variabili aleatorie discrete notevoli

Nome della distribuzione	DF	Media	Varianza	Momenti e momenti centrali $\mu_n \triangleq E(X^n)$; $\sigma_n \triangleq E[(X-\mu)^n]$
Uniforme discreta	$p(k) = \frac{1}{N}$ $k = 1, 2, \ldots, N$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\mu_3 = \frac{N(N+1)^2}{4}$ $\mu_4 = \frac{(N+1)(2N+1)(3N^2+3N-1)}{30}$
Bernoulli $X \sim \mathrm{Bern}(p)$	$p(k) = \begin{cases} q\,, & k=0 \\ p\,, & k=1 \end{cases}$ $p \in [0,1]$, $q = 1-p$	p	pq	$\mu_n = p, \forall n \in \mathbb{N}$
Binomiale $X \sim \mathrm{B}(n,p)$	$p(k) = \binom{n}{k} p^k\, q^{n-k}$ $k = 0, 1, \ldots, n$, $p \in [0,1]$, $q = 1-p$	np	npq	$\sigma_3 = npq(q-p)$ $\sigma_4 = 3n^2p^2q^2 + npq(1-6pq)$
Binomiale negativa $X \sim \mathrm{NB}(r,p)$	$p(k) = \binom{r+k-1}{k} p^r q^k$ $k \in \mathbb{N}_0$, $r \in \mathbb{N}$, $p \in [0,1]$, $q = 1-p$	$\frac{r}{p}$	$\frac{rq}{p^2}$	$\sigma_3 = \frac{r(q+q^2)}{p^3}$ $\sigma_4 = \frac{r(q+(3r+4)q^2+q^3}{p^4}$
Geometrica $X \sim \mathrm{Geom}(p)$	$p(k) = p\, q^{k-1}$ $k \in \mathbb{N}$, $p \in [0,1]$, $q = 1-p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\sigma_3 = \frac{q+q^2}{p^2}$ $\sigma_4 = \frac{q+7q^2+q^3}{p^4}$
Poisson $X \sim \mathrm{Poiss}(\lambda)$	$p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ $k \in \mathbb{N}_0$, $\lambda > 0$	λ	λ	$\sigma_3 = \lambda$ $\sigma_4 = \lambda + 3\lambda^2$

Variabili aleatorie continue notevoli

Nome della distribuzione	pdf	Media	Varianza	Momenti e momenti centrali $\mu_n \triangleq E(X^n)$; $\sigma_n \triangleq E[(X-\mu)^n]$
Uniforme $X \sim \mathrm{U}(a,b)$	$f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{altrove} \end{cases}$ $-\infty < a < b < +\infty$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\sigma_n = \begin{cases} 0, & n \text{ dispari} \\ \frac{(b-a)^n}{2^n(n+1)}, & n \text{ pari} \end{cases}$
Gaussiana o normale $X \sim \mathrm{N}(\mu,\sigma)$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\mu \in \mathbb{R}$, $\sigma > 0$,	μ	σ^2	$\sigma_n = \begin{cases} 0, & n \text{ dispari} \\ \sigma^n (n-1)!!, & n \text{ pari} \end{cases}$
Esponenziale $X \sim \mathrm{Exp}(\lambda)$	$f(x) = \lambda\, e^{-\lambda x}\, \mathrm{u}(x)$ $\lambda > 0$,	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\mu_n = \frac{n!}{\lambda^n}$
Laplace $X \sim \mathrm{Lap}(\lambda)$	$f(x) = \frac{\lambda}{2}\, e^{-\lambda\|x\|}$ $\lambda > 0$,	0	$\frac{2}{\lambda^2}$	$\sigma_n = \begin{cases} 0, & n \text{ dispari} \\ \frac{n!}{\lambda^n}, & n \text{ pari} \end{cases}$
Rayleigh $X \sim \mathrm{Rayleigh}(b)$	$f(x) = \frac{2x}{b}\, e^{-\frac{x^2}{b}}\, \mathrm{u}(x)$ $b > 0$,	$\sqrt{\frac{\pi b}{4}}$	$b\left(1 - \frac{\pi}{4}\right)$	$\mu_n = b^{n/2}\, \frac{n}{2} \Gamma\left(\frac{n}{2}\right)$ $\Gamma(x)$ funzione gamma euleriana